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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 1
Pages: 119-136

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.555067.1124

Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 1 (2023), 119-136
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.555067.1124

On E-Proximinality and Strongly Proximinality in Complex-Valued, Bounded and Continuous Functions Space

Rasul Tourani[1] and Ali Reza Khoddami[2*]

Abstract. For a non-zero normed linear space A, we consider $C^b(K)$, the complex-valued, bounded and continuous functions space on K with $\|\cdot\|_\infty$, where $K = \overline{B_1^{(0)}}$ (the closed unit ball of A). Also for a non-zero element $\varphi \in A^*$ with $\|\varphi\| \leq 1$, we consider the space $C^{b\varphi}(K)$ as the linear space $C^b(K)$ with the new norm $\|f\|_\varphi = \|f\varphi\|_\infty$ for all $f \in C^b(K)$. Some basic properties such as, proximinality, E-proximinality, strongly proximinality and quasi Chebyshev for certain subsets of $C^b(K)$ are characterized with the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$. Some examples for illustration and for comparison between the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ on $C^b(K)$ are presented.

1. Introduction

Let X be a normed linear space and Y be a subset of X. If $x \in X$, then the distance of x from Y is denoted by $d(x, Y)$ that is

$$d(x,Y) = \inf\{\|x-y\|, \quad y \in Y\}.$$

An element $y \in Y$ is said to be a best approximation of $x \in X$ from Y if $\|x-y\| = d(x,Y)$. The set of all best approximations of $x \in X$ from Y is denoted by $P_Y(x)$. If for any $x \in X$, $P_Y(x) \neq \emptyset$, then we say that Y is proximinal in X. Also if for any $x \in X$, $P_Y(x)$ is singleton, then Y is a Chebyshev subset of X. A sequence $\{y_n\}_n \subseteq Y$ is called a minimizing sequence for $x \in X$ if $\lim_{n\to\infty} \|x-y_n\| = d(x,Y)$. One can be referred to [1], as a source concerning best approximation theory.

2020 *Mathematics Subject Classification.* 41A50, 46B28, 46H25, 46B50.

Key words and phrases. Best approximation, E-proximinal, Quasi Chebyshev, Strongly proximinal.

Received: 03 June 2022, Accepted: 19 September 2022.

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For a non-zero normed linear space A, let $K = \overline{B_1^{(0)}}$ and

$$C^b(K) = \{f : K \longrightarrow \mathbb{C}, \quad f \text{ is bounded and continuous}\},$$

be the complex-valued, bounded and continuous functions on K with the uniform norm $\|f\|_\infty = \sup\{|f(x)|, \quad x \in K\}$. Clearly $\left(C^b(K), \|\cdot\|_\infty\right)$ is a Banach space. Given $0 \neq \varphi \in A^*$ with $\|\varphi\| \leq 1$, the new norm $\|f\|_\varphi = \|f\varphi\|_\infty$, $f \in C^b(K)$, is a non-equivalent norm with $\|\cdot\|_\infty$ on $C^b(K)$ that is not complete [5, Proposition 2.2 and Corollary 2.3]. Set $C^{b\varphi}(K) = \left(C^b(K), \|\cdot\|_\varphi\right)$. So $C^{b\varphi}(K)$ is not a Banach space. Some basic properties such as weak and cyclic amenability of $C^{b\varphi}(K)$ are investigated in [4].

In this paper some basic properties such as, proximinality, strongly proximinality, E-proximinality, and quasi Chebyshev for certain subsets of $C^b(K)$ are characterized with the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$. Also some examples for illustration and for comparison between the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ on $C^b(K)$ are presented. The results of this paper can be applied as a source of examples and counterexamples in the best approximation theory.

2. Compactness with Respect to $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$

In this section, let A be a non-zero normed linear space and $0 \neq \varphi \in A^*$ be an element with $\|\varphi\| \leq 1$. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. Suppose that $\|f\|_\varphi = \|f\varphi\|_\infty$ for all $f \in C^b(K)$. Set $C^{b\varphi}(K) = \left(C^b(K), \|\cdot\|_\varphi\right)$. Clearly

$$\begin{aligned} \|f\|_\varphi &= \|f\varphi\|_\infty \\ &\leq \|f\|_\infty \|\varphi\|_\infty \\ &= \|f\|_\infty \|\varphi\|, \end{aligned}$$

for all $f \in C^b(K)$. Also $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$ are not equivalent norms. Indeed, if $f_n(x) = \dfrac{1}{1 + n|\varphi(x)|}$, then $\|f_n\|_\infty = 1$ and $\|f_n\|_\varphi = \|f_n\varphi\|_\infty = \dfrac{\|\varphi\|}{1 + n\|\varphi\|}$. So if there exists $\alpha \in \mathbb{R}^+$ such that $\|g\|_\infty \leq \alpha\|g\|_\varphi$ for all $g \in C^b(K)$, then $\|f_n\|_\infty \leq \alpha\|f_n\|_\varphi$ for all $n \in \mathbb{N}$. It follows that $1 \leq \dfrac{\alpha\|\varphi\|}{1 + n\|\varphi\|}$ which implies $1 \leq \lim\limits_{n\to\infty} \dfrac{\alpha\|\varphi\|}{1 + n\|\varphi\|}$. This shows that $1 \leq 0$ that is a contradiction.

In the following, we present several results concerning compactness of certain subsets of $C^b(K)$ with respect to $\|\cdot\|_\varphi$ and $\|\cdot\|_\infty$.

Theorem 2.1. *Let* $M \subseteq C^b(K)$. *Then* $\overline{M}^{\|\cdot\|_\infty} \subseteq \overline{M}^{\|\cdot\|_\varphi}$. *But the converse is not true in general.*

Proof. Let $f \in \overline{M}^{\|\cdot\|_\infty}$. Then there exists $\{f_n\}_n \subseteq M$ such that $f_n \overset{\|\cdot\|_\infty}{\longrightarrow} f$. Therefore $f_n\varphi \overset{\|\cdot\|_\infty}{\longrightarrow} f\varphi$. So $f_n \overset{\|\cdot\|_\varphi}{\longrightarrow} f$ and finally $f \in \overline{M}^{\|\cdot\|_\varphi}$. Let M_0 be the set defined in Theorem 4.8. Then $\overline{M_0}^{\|\cdot\|_\varphi} = C^b(K) \neq \overline{M_0}^{\|\cdot\|_\infty} = M_0$. □

Theorem 2.2. *Let* M *be a subset of* $C^b(K)$. *If* M *is compact with respect to* $\|\cdot\|_\infty$, *then* M *is compact with respect to* $\|\cdot\|_\varphi$.

Proof. Let M be compact with respect to $\|\cdot\|_\infty$ and let $\{f_n\}_n \subseteq M$ be a sequence. Since M is compact with respect to $\|\cdot\|_\infty$, exists a subsequence $\{f_{n_k}\}_k \subseteq \{f_n\}_n$ and an $f \in M$, such that $f_{n_k} \overset{\|\cdot\|_\infty}{\longrightarrow} f$. Therefore $f_{n_k}\varphi \overset{\|\cdot\|_\infty}{\longrightarrow} f\varphi$. So $f_{n_k} \overset{\|\cdot\|_\varphi}{\longrightarrow} f$. Hence, M is compact with respect to $\|\cdot\|_\varphi$. □

The following example shows that the converse of Theorem 2.2 is not the case in general.

Example 2.3. Consider

$$M = \left\{ f_n : K \longrightarrow \mathbb{C}, \quad f_n(x) = \frac{1 - |\varphi(x)|}{1 + n|\varphi(x)|} \right\} \cup \{0\}.$$

We will show that M is compact in $C^{b\varphi}(K)$, but it is not compact in $C^b(K)$.

Proof. First, we note that M is compact in $C^{b\varphi}(K)$ with respect to $\|\cdot\|_\varphi$. Indeed, let $\{g_k\}_k, k \in \mathbb{N}$, be a sequence in M. We will show that the sequence $\{g_k\}_k$ has a subsequence that converges to a point of M with respect to $\|\cdot\|_\varphi$. If the sequence $\{g_k\}_k$ has a constant subsequence $g_{k_j} = g \in M, j \in \mathbb{N}$, then g_{k_j} converges to $g \in M$ as $j \longrightarrow \infty$. Otherwise, the sequence $\{g_k\}_k$ has a subsequence $g_{k_j} = f_{n_{k_j}}, j \in \mathbb{N}$ such that $n_{k_1} < n_{k_2} < n_{k_3} < \cdots$. Hence, $f_{n_{k_j}}, j \in \mathbb{N}$ is a subsequence of the sequence $\{f_n\}_n$. According to [5, Example 2.4], since $\|f_n\|_\varphi \longrightarrow 0$ as $n \longrightarrow \infty$, $\left\|f_{n_{k_j}}\right\|_\varphi \longrightarrow 0$ as $j \longrightarrow \infty$. Therefore $\|g_{k_j}\|_\varphi \longrightarrow 0$ as $j \longrightarrow \infty$. This shows that $g_{k_j} \longrightarrow 0 \in M$ as $j \longrightarrow \infty$, providing that M is compact in $C^{b\varphi}(K)$.

Now, we shall show that M is not compact in $C^b(K)$ with respect to $\|\cdot\|_\infty$. Suppose by contradiction that M is compact in $C^b(K)$ equipped with the norm $\|\cdot\|_\infty$. Then there is a subsequence $\{f_{n_k}\}_k \subseteq \{f_n\}_n$ such that converges to a point of M as $k \longrightarrow \infty$. So $\{f_{n_k}\}_k$ is either convergent to 0 or convergent to an element like $f_m = \dfrac{1-|\varphi|}{1+m|\varphi|}$ for some $m \in \mathbb{N}$, with respect to the norm $\|\cdot\|_\infty$. In the first case $1 = f_{n_k}(0) \longrightarrow 0$ that is a contradiction. But in the second case, for any $x \in K$, we have $f_{n_k}(x) \longrightarrow f_m(x)$ as $k \longrightarrow \infty$. Since

$$\frac{1-|\varphi(x)|}{1+n_k|\varphi(x)|} \longrightarrow \frac{1-|\varphi(x)|}{1+m|\varphi(x)|},$$

for all $x \in K \setminus \ker\varphi$ as $k \longrightarrow \infty$, it is obvious that $\dfrac{1-|\varphi(x)|}{1+m|\varphi(x)|} = 0$ for all $x \in K \setminus \ker\varphi$. So $|\varphi(x)| = 1$ for all $x \in K \setminus \ker\varphi$. Now, if $e \in A$ is an element such that $\varphi(e) \neq 0$ and $x_0 \in K \cap \ker\varphi$, then

$$\frac{1}{n}\frac{e}{\|e\|} + \frac{n-1}{n}x_0 \longrightarrow x_0,$$

as $n \longrightarrow \infty$. By the continuity of φ and this fact that $\dfrac{1}{n}\dfrac{e}{\|e\|} + \dfrac{n-1}{n}x_0 \in K \setminus \ker\varphi$ for all $n \in \mathbb{N}$,

$$1 = \left|\varphi\left(\frac{1}{n}\frac{e}{\|e\|} + \frac{n-1}{n}x_0\right)\right| \longrightarrow |\varphi(x_0)|,$$

as $n \longrightarrow \infty$. Hence for any $x_0 \in K \cap \ker\varphi$, $|\varphi(x_0)| = 1$ that is a contradiction. Because, $|\varphi(0)| = 0$. Therefore, the sequence $\{f_n\}_n \subseteq M$ has no convergent subsequence to a point of M by the norm $\|\cdot\|_\infty$ and consequently M is not compact. □

Theorem 2.4. *Let 1_K be the constant function $1_K(x) = 1, x \in K$. Then*

$$\begin{aligned}\overline{\{\alpha 1_K \mid \alpha \in \mathbb{C}\}}^{\|\cdot\|_\infty} &= \{\alpha 1_K \mid \alpha \in \mathbb{C}\} \\ &= \overline{\{\alpha 1_K \mid \alpha \in \mathbb{C}\}}^{\|\cdot\|_\varphi}.\end{aligned}$$

Proof. Let $f \in \overline{\{\alpha 1_K \mid \alpha \in \mathbb{C}\}}^{\|\cdot\|_\infty}$. Then there exists a sequence $\{\alpha_n 1_K\}_n$ such that $\alpha_n 1_K \stackrel{\|\cdot\|_\infty}{\longrightarrow} f$. So for any $x \in K$, we have $\alpha_n 1_K(x) \longrightarrow f(x)$. Therefore $\alpha_n \longrightarrow f(x)$ for all $x \in K$. By the uniqueness of limit, we have $f(x) = f(0)$ for all $x \in K$. So $f = f(0)1_K \in \{\alpha 1_K \mid \alpha \in \mathbb{C}\}$. Hence $\overline{\{\alpha 1_K \mid \alpha \in \mathbb{C}\}}^{\|\cdot\|_\infty} = \{\alpha 1_K \mid \alpha \in \mathbb{C}\}$.

To prove the other part of the equality, let $f \in \overline{\{\alpha 1_K \mid \alpha \in \mathbb{C}\}}^{\|\cdot\|_\varphi}$. So there exists a sequence $\{\alpha_n 1_K\}_n$ such that $\alpha_n 1_K \stackrel{\|\cdot\|_\varphi}{\longrightarrow} f$. Equivalently, $\alpha_n\varphi \stackrel{\|\cdot\|_\infty}{\longrightarrow} f\varphi$. Therefore, for any $x \in K\setminus\ker\varphi$, $\alpha_n\varphi(x) \longrightarrow f(x)\varphi(x)$. So for any $x \in K\setminus\ker\varphi$, $\alpha_n \longrightarrow f(x)$. Hence, for a fixed element

$x_0 \in K \backslash \ker \varphi$ we have $f(x) = f(x_0)$, $x \in K \backslash \ker \varphi$. Let $e \in A$ be an element such that $\varphi(e) \neq 0$ and suppose that $x_1 \in K \cap \ker \varphi$. Then

$$\frac{1}{n} \frac{e}{\|e\|} + \frac{n-1}{n} x_1 \longrightarrow x_1$$

as $n \longrightarrow \infty$. By the continuity of f we can conclude that

$$f\left(\frac{1}{n} \frac{e}{\|e\|} + \frac{n-1}{n} x_1\right) \longrightarrow f(x_1).$$

As $\left(\frac{1}{n} \frac{e}{\|e\|} + \frac{n-1}{n} x_1\right) \in K \backslash \ker \varphi$ for all $n \in \mathbb{N}$, $f\left(\frac{1}{n} \frac{e}{\|e\|} + \frac{n-1}{n} x_1\right) = f(x_0)$ for all $n \in \mathbb{N}$ and consequently $f(x_1) = f(x_0)$. It follows that

$$f(x) = \begin{cases} f(x_0); & x \in K \backslash \ker \varphi, \\ f(x_0); & x \in K \cap \ker \varphi. \end{cases}$$

Hence, $f = f(x_0) = f(x_0) 1_K$. So $f \in \left\{\alpha 1_K \;\middle|\; \alpha \in \mathbb{C}\right\}$. □

3. The Comparison Between $P_M^{\varphi}(g)$ and $P_M^{\infty}(g)$

In this section, let A be a non-zero normed linear space and $0 \neq \varphi \in A^*$ be an element with $\|\varphi\| \leq 1$. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. Suppose that $\|f\|_{\varphi} = \|f\varphi\|_{\infty}$ for all $f \in C^b(K)$. Set $C^{b\varphi}(K) = \left(C^b(K), \|\cdot\|_{\varphi}\right)$.

In the sequel we use the following notations repeatedly for $f, g \in C^b(K)$ and $M \subseteq C^b(K)$.

$$\begin{aligned}
&d^{\infty}(f, g) = \|f - g\|_{\infty}, \\
&d^{\varphi}(f, g) = \|f - g\|_{\varphi}, \\
&d^{\infty}(f, M) = \inf\{\|f - g\|_{\infty}, \quad g \in M\}, \\
&d^{\varphi}(f, M) = \inf\left\{\|f - g\|_{\varphi}, \quad g \in M\right\}, \\
&P_M^{\infty}(f) = \{g \in M, \quad \|f - g\|_{\infty} = d^{\infty}(f, M)\}, \\
&P_M^{\varphi}(f) = \left\{g \in M, \quad \|f - g\|_{\varphi} = d^{\varphi}(f, M)\right\}.
\end{aligned}$$

The aim of this section is the presentation of some examples concerning best approximation with the norms $\|\cdot\|_{\varphi}$ and $\|\cdot\|_{\infty}$. Also, we compare $P_M^{\varphi}(f)$ with $P_M^{\infty}(f)$ for certain subsets $M \subseteq C^b(K)$ and some elements $f \in C^b(K)$.

Example 3.1. Let

$$M = \left\{f_n : K \longrightarrow \mathbb{C}, \quad f_n(x) = \frac{1}{1 + n|\varphi(x)|}, \quad n \in \mathbb{N}, x \in K\right\},$$

and $g(x) = 1_K(x) = 1, x \in K$. Then $\|g - f_n\|_\infty = 1 - \dfrac{1}{1 + n\|\varphi\|}$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} d^\infty(g, M) &= 1 - \frac{1}{1 + \|\varphi\|} \\ &= \|g - f_1\|_\infty . \end{aligned}$$

It follows that $P_M^\infty(g) = \{f_1\}$. Also

$$\begin{aligned} \|g - f_n\|_\varphi &= \|g\varphi - f_n\varphi\|_\infty \\ &= \|\varphi\| - \frac{\|\varphi\|}{1 + n\|\varphi\|}, \end{aligned}$$

for all $n \in \mathbb{N}$. So

$$\begin{aligned} d^\varphi(g, M) &= \|\varphi\| - \frac{\|\varphi\|}{1 + \|\varphi\|} \\ &= \|g - f_1\|_\varphi . \end{aligned}$$

Hence $P_M^\varphi(g) = \{f_1\}$ and $P_M^\infty(g) = P_M^\varphi(g)$. But the equality $P_M^\infty(g) = P_M^\varphi(g)$ is not the case in general. The following example shows this fact.

Example 3.2. Let

$$M = \left\{ f_n : K \longrightarrow \mathbb{C}, \quad f_n(x) = |\varphi(x)| - \frac{1}{n}\|\varphi\|, \ n \in \mathbb{N}, x \in K \right\} \cup \{|\varphi|\},$$

and $g(x) = \|\varphi\| - |\varphi(x)|, \quad x \in K$. Then

$$\begin{aligned} d^\infty(g, |\varphi|) &= \sup\{|g(x) - |\varphi(x)||, \quad x \in K\} \\ &= \sup\{|\|\varphi\| - 2|\varphi(x)||, \quad x \in K\} \\ &= \|\varphi\|, \end{aligned}$$

and

$$\begin{aligned} d^\infty(g, f_n) &= \sup\{|g(x) - f_n(x)|, \quad x \in K\} \\ &= \sup\left\{ \left| \frac{n+1}{n}\|\varphi\| - 2|\varphi(x)| \right|, \quad x \in K \right\} \\ &= \frac{n+1}{n}\|\varphi\|, \end{aligned}$$

for all $n \in \mathbb{N}$. So

$$\begin{aligned} d^\infty(g, M) &= \inf\left\{ \|\varphi\|, \frac{n+1}{n}\|\varphi\|, \quad n \in \mathbb{N} \right\} \\ &= \|\varphi\| \\ &= d^\infty(g, |\varphi|). \end{aligned}$$

It follows that $P_M^\infty(g) = \{|\varphi|\}$. Also

$$\begin{aligned} d^\infty(g\varphi, |\varphi|\varphi) &= \sup\{|g(x)\varphi(x) - |\varphi(x)|\varphi(x)|, \quad x \in K\} \\ &= \sup\{|\|\varphi\|\varphi(x) - 2|\varphi(x)|\varphi(x)|, \quad x \in K\} \\ &= \sup\left\{\left|\|\varphi\||\varphi(x)| - 2|\varphi(x)|^2\right|, \quad x \in K\right\} \\ &= \|\varphi\|^2, \end{aligned}$$

and

$$\begin{aligned} d^\infty(g\varphi, f_n\varphi) &= \sup\{|g(x)\varphi(x) - f_n(x)\varphi(x)|, \quad x \in K\} \\ &= \sup\left\{\left|\frac{n+1}{n}\|\varphi\||\varphi(x)| - 2|\varphi(x)|^2\right|, \quad x \in K\right\} \\ &= \begin{cases} \frac{1}{2}\|\varphi\|^2; & n = 1; \\ \frac{n-1}{n}\|\varphi\|^2; & n > 1. \end{cases} \end{aligned}$$

So

$$\begin{aligned} d^\varphi(g, M) &= \inf\left\{\|\varphi\|^2, \frac{1}{2}\|\varphi\|^2, \frac{n-1}{n}\|\varphi\|^2, \quad n > 1\right\} \\ &= \frac{1}{2}\|\varphi\|^2 \\ &= d^\varphi(g, f_1) \\ &= d^\varphi(g, f_2). \end{aligned}$$

It follows that $P_M^\varphi(g) = \{f_1, f_2\}$. Clearly $P_M^\infty(g) \subsetneqq P_M^\varphi(g)$ and $d^\infty(g, M) \neq d^\varphi(g, M)$. Indeed, $d^\infty(g, M) = d^\varphi(g, M)$ implies $\|\varphi\| = 2$, that is a contradiction.

Example 3.3. Let

$$M = \left\{f_n : K \longrightarrow \mathbb{C}, \quad f_n(x) = \frac{1}{1 + n|\varphi(x)|}, \quad n \in \mathbb{N}, \quad x \in K\right\},$$

and $g(x) = |\varphi(x)|, x \in K$. Then $\|g - f_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and so

$$\begin{aligned} d^\infty(g, M) &= 1 \\ &= \|g - f_n\|_\infty. \end{aligned}$$

It follows that $P_M^\infty(g) = M$. Indeed

$$\begin{aligned} \left|\frac{1}{1 + n|\varphi(x)|} - |\varphi(x)|\right| &= \left|\frac{1 - |\varphi(x)| - n|\varphi(x)|^2}{1 + n|\varphi(x)|}\right| \\ &= \left|\frac{n|\varphi(x)|^2 + |\varphi(x)| - 1}{1 + n|\varphi(x)|}\right| \end{aligned}$$

for all $x \in K$ and $n \in \mathbb{N}$. Let $h(z) = \dfrac{nz^2 + z - 1}{1 + nz}$, $z \in [0, \|\varphi\|]$. Since $h'(z) = \dfrac{n^2 z^2 + 2nz + n + 1}{(1 + nz)^2} > 0$, h is an increasing function on $[0, \|\varphi\|]$. Hence $\|g - f_n\|_\infty = \max\{|h(0)|, |h(\|\varphi\|)|\}, n \in \mathbb{N}$. But $|h(0)| = 1$ and

$$\begin{aligned} |h(\|\varphi\|)| &= \left| \frac{n\|\varphi\|^2 + \|\varphi\| - 1}{1 + n\|\varphi\|} \right| \\ &= \left| 1 - \|\varphi\| - \frac{n\|\varphi\|}{1 + n\|\varphi\|} \right|. \end{aligned}$$

One can easily check that $-1 < 1 - \|\varphi\| - \dfrac{n\|\varphi\|}{1 + n\|\varphi\|} < 1$ for all $n \in \mathbb{N}$. Therefore $|h(\|\varphi\|)| < 1$ and we have $\|g - f_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} d^\infty(g, M) &= 1 \\ &= \|g - f_n\|_\infty, \end{aligned}$$

for all $n \in \mathbb{N}$. This shows that $P_M^\infty(g) = M$.
Also

$$\begin{aligned} |f_n(x)\varphi(x) - g(x)\varphi(x)| &= |\varphi(x)| |f_n(x) - g(x)| \\ &= \left| \frac{|\varphi(x)|}{1 + n|\varphi(x)|} - |\varphi(x)|^2 \right|, \end{aligned}$$

for all $x \in K$ and $n \in \mathbb{N}$. Let $k(z) = \dfrac{z}{1 + nz} - z^2$, $0 \le z \le \|\varphi\|$. Since $k(z) = -zh(z)$ and $-1 < -h(z) = 1 - z - \dfrac{nz}{1 + nz} < 1$ for all $z \in [0, \|\varphi\|]$ and $n \in \mathbb{N}$, $|k(z)| = z|h(z)| \le z$ for all $z \in [0, \|\varphi\|]$. Since $k'(z) = \dfrac{1}{(1 + nz)^2} - 2z$, the only real root of the equation $k'(z) = 0$ is

$$z_n = \frac{\left(\frac{1}{n} + \sqrt{\frac{1}{n^2} + \frac{8}{27n^3}} \right)^{\frac{2}{3}}}{2} + \frac{\left(\frac{1}{n} - \sqrt{\frac{1}{n^2} + \frac{8}{27n^3}} \right)^{\frac{2}{3}}}{2} - \frac{2}{3n}, \qquad n \in \mathbb{N}.$$

Indeed, $k'(z) = 0$ implies $2z = \dfrac{1}{(1 + nz)^2}$. So $\sqrt{2z} = \dfrac{1}{(1 + nz)}$. Let $\sqrt{2z} = t$. It follows that $t^3 + \frac{2}{n}t - \frac{2}{n} = 0$. So

$$t_n = \left(\frac{1}{n} + \sqrt{\frac{1}{n^2} + \frac{8}{27n^3}} \right)^{\frac{1}{3}} + \left(\frac{1}{n} - \sqrt{\frac{1}{n^2} + \frac{8}{27n^3}} \right)^{\frac{1}{3}}, \qquad n \in \mathbb{N}.$$

Hence

$$z_n = \frac{\left(\frac{1}{n} + \sqrt{\frac{1}{n^2} + \frac{8}{27n^3}}\right)^{\frac{2}{3}}}{2} + \frac{\left(\frac{1}{n} - \sqrt{\frac{1}{n^2} + \frac{8}{27n^3}}\right)^{\frac{2}{3}}}{2} - \frac{2}{3n}, \qquad n \in \mathbb{N}.$$

Clearly,

$$\begin{aligned} \|f_n - g\|_{\varphi} &= \|f_n\varphi - g\varphi\|_{\infty} \\ &= \begin{cases} \max\left(|k(z_n)|, |k(\|\varphi\|)|\right); & z_n \leq \|\varphi\|, \\ |k(\|\varphi\|)|; & o.w. \end{cases} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{\|\varphi\|}{1+n\|\varphi\|} = 0$, there exists $N \in \mathbb{N}$ such that $z_n < \frac{\|\varphi\|^2}{4}$ and $\frac{\|\varphi\|}{1+n\|\varphi\|} < \frac{\|\varphi\|^2}{4}$ for all $n \geq N$. So $|k(z_n)| \leq z_n < \frac{\|\varphi\|^2}{4}$ and $\|\varphi\|^2 - \frac{\|\varphi\|}{1+n\|\varphi\|} > \frac{3\|\varphi\|^2}{4}$ for all $n \geq N$. Since $z_n < \frac{\|\varphi\|^2}{4} \leq \|\varphi\|$, $n \in \mathbb{N}$, it follows that

$$\begin{aligned} \|f_n - g\|_{\varphi} &= \max\left(|k(z_n)|, |k(\|\varphi\|)|\right) \\ &= \|\varphi\|^2 - \frac{\|\varphi\|}{1+n\|\varphi\|}, \end{aligned}$$

for all $n \geq N$. So $d^{\varphi}(g, M) \leq \|\varphi\|^2 - \frac{\|\varphi\|}{1+N\|\varphi\|}$. Hence $P_M^{\varphi}(g) \subseteq \{f_1, f_2, \ldots, f_N\}$ that implies $P_M^{\varphi}(g) \neq P_M^{\infty}(g) = M$.

4. Quasi Chebyshev, Strongly Proximinalitu and E-Proximinality of Certain Subset of $C^b(K)$

In this section, we use the same notations in the previous sections. Also quasi Chebyshev, strongly proximinality, and E-proximinality of certain subsets of $C^b(K)$ are investigated by the norms $\|\cdot\|_{\varphi}$ and $\|\cdot\|_{\infty}$.

Definition 4.1. Let X be a normed linear space and $Y \subseteq X$. Y is said to be quasi Chebyshev if $P_Y(x)$ is non-empty and compact for every $x \in X$ [6].

Definition 4.2. Let X be a normed linear space and Y be a proximinal subset of X. Then Y is said to be strongly proximinal, if for every $x \in X$ and every $\delta > 0$, there is $\epsilon > 0$ such that $d(y, P_Y(x)) < \delta$ for all $y \in P_Y(x, \epsilon)$ where $P_Y(x, \epsilon) = \{y \in Y \text{ s.t. } \|x - y\| < d(x, Y) + \epsilon\}$ [2].

The following definition that is an equivalent version of strongly proximinality is presented in [7].

Definition 4.3. Let X be a normed linear space. Then a proximinal subset $Y \subseteq X$ is said to be strongly proximinal, if for any $x \in X$, for any minimizing sequence $\{y_n\}_n \subseteq Y$ for x, there is a subsequence $\{y_{n_k}\}_k$ and a sequence $\{z_k\}_k \subseteq P_Y(x)$ such that $\lim_{k\to\infty} \|y_{n_k} - z_k\| = 0$.

Definition 4.4. Let X be a normed linear space. Then a proximinal subset $Y \subseteq X$ is said to be E-proximinal at $x \in X$ if for given $\epsilon > 0$ there exists $y \in P_Y(x)$ such that $\|y\| < \alpha(x) + \epsilon$ where

$$\alpha(x) = \inf\left\{r \geq 0 \,\middle|\, \exists \{y_n\}_n \subseteq Y \text{ s.t. } \|y_n\| \leq r, \lim_{n\to\infty} \|x - y_n\| = d(x, Y)\right\}.$$

If Y is E-proximinal at every $x \in X$, then Y is said to be E-proximinal in X [3].

Theorem 4.5. *Let $f \in C^b(K)$ and $M \subseteq C^b(K)$. Then the following assertions hold.*

1) $d^\varphi(f, M) \leq \|\varphi\|_\infty d^\infty(f, M)$,
2) $d^\varphi(f, M) = d^\infty(f\varphi, M\varphi)$,
3) $P^\infty_{M\varphi}(f\varphi) = P^\varphi_M(f)\varphi$. *In particular, M is Chebyshev in $C^{b\varphi}(K)$, if and only if $M\varphi$ is Chebyshev in $C^b(K)\varphi$.*
4) *M is quasi Chebyshev in $C^{b\varphi}(K)$ if and only if $M\varphi$ is quasi Chebyshev in $C^b(K)\varphi$.*

Proof. 1) To prove it, we have

$$\begin{aligned} d^\varphi(f, M) &= \inf\left\{\|f - g\|_\varphi, g \in M\right\} \\ &= \inf\{\|f\varphi - g\varphi\|_\infty, g \in M\} \\ &\leq \inf\{\|f - g\|_\infty \|\varphi\|_\infty, g \in M\} \\ &= \|\varphi\|_\infty \inf\{\|f - g\|_\infty, g \in M\} \\ &= \|\varphi\|_\infty d^\infty(f, M). \end{aligned}$$

2) It is clear that $d^\varphi(f, M) = \inf\{\|f\varphi - g\varphi\|_\infty, g \in M\} = d^\infty(f\varphi, M\varphi)$.
3) If $g \in M$, then

$$\begin{aligned} g\varphi \in P^\infty_{M\varphi}(f\varphi) \quad &\Leftrightarrow \quad \|f\varphi - g\varphi\|_\infty = d^\infty(f\varphi, M\varphi) \\ &\Leftrightarrow \quad \|f - g\|_\varphi = d^\varphi(f, M) \\ &\Leftrightarrow \quad g \in P^\varphi_M(f) \\ &\Leftrightarrow \quad g\varphi \in P^\varphi_M(f)\varphi. \end{aligned}$$

Clearly M is Chebyshev in $C^{b\varphi}(K)$ if and only if $M\varphi$ is Chebyshev in $C^b(K)\varphi$.

4) Let $M \subseteq C^{b\varphi}(K)$ be quasi Chebyshev. So $P_M^{\varphi}(f)$ is non-empty and compact with respect to $\|\cdot\|_{\varphi}$ for all $f \in C^b(K)$. Since $P_{M\varphi}^{\infty}(f\varphi) = P_M^{\varphi}(f)\varphi$, $P_{M\varphi}^{\infty}(f\varphi) \neq \emptyset$. Let $\{h_n\}_n \subseteq P_{M\varphi}^{\infty}(f\varphi) = P_M^{\varphi}(f)\varphi$. Then there is a $g_n \in P_M^{\varphi}(f)$ such that $h_n = g_n\varphi$, for all $n \in \mathbb{N}$. So there exists a subsequence $\{g_{n_k}\}_k \subseteq \{g_n\}_n$ and a $g_0 \in P_M^{\varphi}(f) \subseteq M$ such that $g_{n_k} \xrightarrow{\|\cdot\|_{\varphi}} g_0$ as $k \longrightarrow \infty$. Hence $g_{n_k}\varphi \xrightarrow{\|\cdot\|_{\infty}} g_0\varphi$ as $k \longrightarrow \infty$. It follows that $h_{n_k} \xrightarrow{\|\cdot\|_{\infty}} g_0\varphi \in P_{M\varphi}^{\infty}(f\varphi)$ as $k \longrightarrow \infty$. Hence $P_{M\varphi}^{\infty}(f\varphi)$ is non-empty and compact with respect to $\|\cdot\|_{\infty}$ for all $f\varphi \in C^b(K)\varphi$. So $M\varphi$ is quasi Chebyshev.

Conversely, let $M\varphi \subseteq C^b(K)\varphi$ be quasi Chebyshev with respect to $\|\cdot\|_{\infty}$. So $P_{M\varphi}^{\infty}(f\varphi)$ is non-empty and compact for all $f\varphi \in C^b(K)\varphi$. The equality $P_{M\varphi}^{\infty}(f\varphi) = P_M^{\varphi}(f)\varphi$ implies $P_M^{\varphi}(f) \neq \emptyset$ for all $f \in C^b(K)$. Let $\{h_n\}_n \subseteq P_M^{\varphi}(f) \subseteq M$ be a sequence. So $\{h_n\varphi\}_n \subseteq P_{M\varphi}^{\infty}(f\varphi)$. Since $P_{M\varphi}^{\infty}(f\varphi)$ is compact,then there exists a subsequence $\{h_{n_k}\varphi\}_k \subseteq \{h_n\varphi\}_n$ and a $h_0\varphi \in P_{M\varphi}^{\infty}(f\varphi) = P_M^{\varphi}(f)\varphi$ such that $h_{n_k}\varphi \xrightarrow{\|\cdot\|_{\infty}} h_0\varphi$ as $k \longrightarrow \infty$. Hence $h_{n_k} \xrightarrow{\|\cdot\|_{\varphi}} h_0 \in P_M^{\varphi}(f)$ as $k \longrightarrow \infty$. So $P_M^{\varphi}(f)$ is non-empty and compact with respect to $\|\cdot\|_{\varphi}$ for all $f \in C^{b\varphi}(K)$. This shows that M is quasi Chebyshev. □

Remark 4.6. Let X be a normed linear space. If $Y \subseteq X$ is strongly proximinal, then Y is E-proximinal in X. Indeed, let $x \in X$ and $\epsilon > 0$ be given. Then there exists

$$r_0 \in \left\{r \geq 0 \;\middle|\; \exists \{y_n\}_n \subseteq Y \text{ s.t. } \|y_n\| \leq r, \lim_{n\to\infty} \|x - y_n\| = d(x, Y)\right\},$$

such that $r_0 < \alpha(x) + \dfrac{\epsilon}{2}$. For r_0 there is a sequence $\{y_n\}_n$ such that $\|y_n\| \leq r_0$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to\infty} \|x - y_n\| = d(x, Y)$. Because $Y \subseteq X$ is strongly proximinal, there exists a subsequence $\{y_{n_k}\}_k \subseteq \{y_n\}_n$ and a sequence $\{z_k\}_k \subseteq P_Y(x)$ such that $\lim\limits_{k\to\infty} \|y_{n_k} - z_k\| = 0$. For $\dfrac{\epsilon}{2} > 0$ there is a $k_0 \in \mathbb{N}$ which $\left\|z_{k_0} - y_{n_{k_0}}\right\| < \dfrac{\epsilon}{2}$. As a result,

$$\begin{aligned} \|z_{k_0}\| - \left\|y_{n_{k_0}}\right\| &\leq \left\|z_{k_0} - y_{n_{k_0}}\right\| \\ &< \frac{\epsilon}{2}. \end{aligned}$$

So

$$\begin{aligned}\|z_{k_0}\| &< \frac{\epsilon}{2} + \left\|y_{n_{k_0}}\right\| \\ &\leq \frac{\epsilon}{2} + r_0 \\ &< \frac{\epsilon}{2} + \alpha(x) + \frac{\epsilon}{2}.\end{aligned}$$

Then

$$\|z_{k_0}\| < \alpha(x) + \epsilon.$$

Therefore Y is E-proximinal at x. Since x is an arbitrary element of X, we may conclude that Y is E-proximinal in X.

Theorem 4.7. *Let $X = C^b(K)$ and $M_0 = \{g \in X \mid g(0) = 0\}$. Then M_0 is a non-Chebyshev proximinal subset of X.*

Proof. Let $f \in X$ and $g \in M_0$. Since $|f(0) - g(0)| \leq \|f - g\|_\infty$, we can conclude that $|f(0)| \leq \|f - g\|_\infty$ and consequently $|f(0)| \leq d^\infty(f, M_0)$. Also since $f - f(0) \in M_0$, $|f(0)| = \|f - (f - f(0))\|_\infty \geq d^\infty(f, M_0)$. It follows that

$$\begin{aligned}d^\infty(f, M_0) &= |f(0)| \\ &= \|f - (f - f(0))\|_\infty, \quad f \in C^b(K).\end{aligned}$$

Hence $f - f(0) \in P^\infty_{M_0}(f)$ for all $f \in C^b(K)$. This shows that M_0 is a proximinal subset of X. Define $f : K \longrightarrow \mathbb{C}$ by $f(x) = 1 - \|x\|^2$. If $g_n : K \longrightarrow \mathbb{C}$ is defined by $g_n(x) = \frac{-\|x\|^2}{n}, \quad x \in K, \quad n \in \mathbb{N}$, then $g_n \in P^\infty_{M_0}(f)$ for all $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}\|f - g_n\|_\infty &= \sup\left\{\left|1 - \frac{n-1}{n}\|x\|^2\right|, \quad x \in K\right\} \\ &= 1 \\ &= |f(0)| \\ &= d^\infty(f, M_0).\end{aligned}$$

So, $\left\{g_n : K \longrightarrow \mathbb{C}, \quad g_n(x) = \frac{-\|x\|^2}{n}, \quad x \in K, \quad n \in \mathbb{N}\right\} \subseteq P^\infty_{M_0}(f)$.

This shows that M_0 is not Chebyshev. Generally, if $f \in C^b(K)$ is an element such that $\|f\|_\infty = |f(0)|$, then $g_n = \frac{f - f(0)}{n} \in P^\infty_{M_0}(f)$ for all $n \in \mathbb{N}$. Because,

$$\|f - g_n\|_\infty = \left\|f - \left(\frac{f - f(0)}{n}\right)\right\|_\infty$$

$$= \left\| \frac{n-1}{n} f + \frac{f(0)}{n} \right\|_{\infty}$$
$$= \sup \left\{ \left| \frac{n-1}{n} f(x) + \frac{f(0)}{n} \right|, \quad x \in K \right\}$$
$$= |f(0)| = d^{\infty}(f, M_0), \quad n \in \mathbb{N}.$$

□

Theorem 4.8. *Let* $M_0 = \{ g \in C^b(K) \mid g(0) = 0 \}$. *Then*

1) $d^{\varphi}(f, M_0) = 0$ *for all* $f \in C^b(K)$
2) $\overline{M_0}^{\|\cdot\|_{\varphi}} = C^b(K)$. *In particular,* M_0 *is not closed and also proximinal with respect to the norm* $\|\cdot\|_{\varphi}$.

Proof. Let $f \in C^b(K)$ and let $N = \{ h\varphi \mid h \in C^b(K) \}$. Then by [4, Theorem 4.1], since $\overline{N}^{\|\cdot\|_{\varphi}} = C^b(K)$, there exists a sequence $\{h_n\varphi\}_n \subseteq N \subseteq M_0$ such that $h_n\varphi \xrightarrow{\|\cdot\|_{\varphi}} f$ as $n \longrightarrow \infty$. It follows that

$$0 = d^{\varphi}(h_n\varphi, M_0) \longrightarrow d^{\varphi}(f, M_0),$$

as $n \longrightarrow \infty$, that implies $d^{\varphi}(f, M_0) = 0$, providing 1. Since $h_n\varphi \xrightarrow{\|\cdot\|_{\varphi}} f$ and $\{h_n\varphi\}_n \subseteq M_0$, $f \in \overline{M_0}^{\|\cdot\|_{\varphi}}$, providing 2. If f is a non-zero constant function on K, then $f \notin M_0$. So by part 2 of theorem, M_0 is not closed with respect to $\|\cdot\|_{\varphi}$. Also for each $f \in C^b(K) \backslash M_0$, $P^{\varphi}_{M_0}(f) = \emptyset$. Indeed, suppose by contradiction that $g \in P^{\varphi}_{M_0}(f)$. Then $0 = d^{\varphi}(f, M_0) = \|f - g\|_{\varphi}$. Hence $f = g \in M_0$ that is a contradiction. So $M_0 \subseteq C^{b\varphi}(K)$ is not proximinal. □

Proposition 4.9. *Let* $M_0 = \{ g \in C^b(K), \quad g(0) = 0 \}, \quad g \in M_0$ *and* $f \in C^b(K)$. *Then* $g \in P^{\infty}_{M_0}(f)$ *if and only if the following conditions hold for all* $x \in K$:

1) $\mathrm{Re}f(x) - |f(0)| \leq \mathrm{Re}g(x) \leq \mathrm{Re}f(x) + |f(0)|$
2)

$$-\sqrt{|f(0)|^2 - (\mathrm{Re}g(x) - \mathrm{Re}f(x))^2} \leq \mathrm{Im}g(x) - \mathrm{Im}f(x) \leq \sqrt{|f(0)|^2 - (\mathrm{Re}g(x) - \mathrm{Re}f(x))^2}$$

Proof. Suppose that parts 1 and 2 are established. So

$$|\mathrm{Im}g(x) - \mathrm{Im}f(x)| \leq \sqrt{|f(0)|^2 - (\mathrm{Re}g(x) - \mathrm{Re}f(x))^2}, \quad x \in K.$$

Hence

$$(\mathrm{Im}g(x) - \mathrm{Im}f(x))^2 \leq |f(0)|^2 - (\mathrm{Re}g(x) - \mathrm{Re}f(x))^2, \quad x \in K.$$

So

$$(\mathrm{Re}g(x)-\mathrm{Re}f(x))^2+(\mathrm{Im}g(x)-\mathrm{Im}f(x))^2\leq|f(0)|^2,\qquad x\in K.$$

That implies,

$$\begin{aligned}(\mathrm{Re}g(x))^2+(\mathrm{Im}g(x))^2+(\mathrm{Re}f(x))^2+(\mathrm{Im}f(x))^2-2\mathrm{Re}g(x)\,\mathrm{Re}f(x)\\ -2\mathrm{Im}g(x)\,\mathrm{Im}f(x)\leq|f(0)|^2,\qquad x\in K.\end{aligned}$$

It follows that

$$|g(x)|^2+|f(x)|^2-\left(g\overline{f}+\overline{g}f\right)(x)\leq|f(0)|^2,\qquad x\in K.$$

So

$$(g(x)-f(x))\left(\overline{g}(x)-\overline{f}(x)\right)\leq|f(0)|^2$$

for all $x\in K$. Hence $|g(x)-f(x)|^2\leq|f(0)|^2$ for all $x\in K$. This shows that

$$\begin{aligned}\|f-g\|_\infty&\leq|f(0)|\\&=d^\infty(f,M_0).\end{aligned}$$

Since

$$\begin{aligned}\|f-g\|_\infty&\geq|f(0)-g(0)|\\&=|f(0)|\\&=d^\infty(f,M_0),\end{aligned}$$

$d^\infty(f,M_0)=\|f-g\|_\infty$ that implies, $g\in P^\infty_{M_0}(f)$.

Conversely, if $g\in P^\infty_{M_0}(f)$, then $\|f-g\|_\infty=|f(0)|$. Since $|g(x)-f(x)|\leq|f(0)|$ for all $x\in K$, $(g(x)-f(x))\left(\overline{g}(x)-\overline{f}(x)\right)\leq|f(0)|^2$. Hence a similar calculation reveals that

$$|\mathrm{Im}g(x)-\mathrm{Im}f(x)|\leq\sqrt{|f(0)|^2-(\mathrm{Re}g(x)-\mathrm{Re}f(x))^2},$$

for all $x\in K$. So

$$|f(0)|^2-(\mathrm{Re}g(x)-\mathrm{Re}f(x))^2\geq0,$$

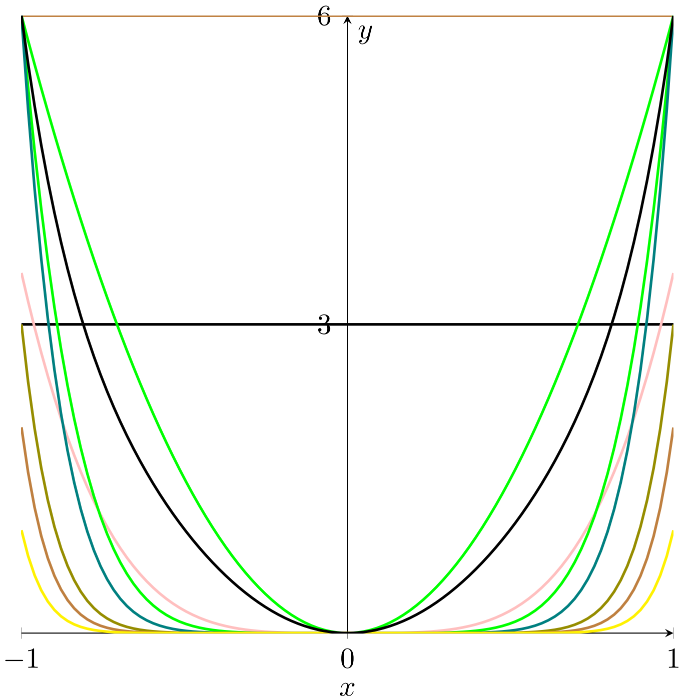
and

$$\begin{aligned}-\sqrt{|f(0)|^2-(\mathrm{Re}g(x)-\mathrm{Re}f(x))^2}&\leq\mathrm{Im}g(x)-\mathrm{Im}f(x)\\&\leq\sqrt{|f(0)|^2-(\mathrm{Re}g(x)-\mathrm{Re}f(x))^2},\end{aligned}$$

for all $x\in K$, providing 1 and 2. □

Corollary 4.10. *Let $M_0=\left\{g\in C^b(K),\quad g(0)=0\right\}$ and $f=c\in\mathbb{R}$ be a constant function in $C^b(K)$. Also let $g\in M_0$ be a real-valued function. Then $g\in P^\infty_{M_0}(f)$ if and only if $c-|c|\leq g(x)\leq c+|c|$ for all $x\in K$.*

For a special case, if $K = [-1, 1]$ and $f = 3 \in C^b([-1, 1])$ is a constant function, then by Corollary 4.10 some of the best approximations for f from $M_0 = \{g \in C^b([-1, 1]) \mid g(0) = 0\}$ are presented in the following figure.



Elements of the following set are some of the best approximations of $f = 3$ from M_0 which are given in the above figure.

$$\left\{6x^2, 6x^8, 3x^{10}, 2x^{12}, 2x^8 + 4x^2, \dots\right\} \subseteq P^{\infty}_{M_0}(f).$$

Proposition 4.11. *Let* $M_0 = \{g \in C^b(K), \quad g(0) = 0\}$. *Then* M_0 *is not quasi Chebyshev.*

Proof. Let $f(x) = \dfrac{1}{2}$ for all $x \in K$. Define $g_n : K \longrightarrow \mathbb{C}$ by

$$g_n(x) = \begin{cases} n\,\|x\|; & \|x\| \le \dfrac{1}{n}, \\ 1; & o.w. \end{cases}$$

for all $n \in \mathbb{N}$ and $x \in K$. One can easily check that $g_n \in M_0$. By Corollary 4.10, $g_n \in P^{\infty}_{M_0}\left(\dfrac{1}{2}\right)$ for all $n \in \mathbb{N}$. We shall show that $g_n \xrightarrow{p.w.} g$

(pointwise on K), where

$$g(x)=\begin{cases}0; & x=0,\\ 1; & o.w.\end{cases}$$

It is obvious that $\lim\limits_{n\to\infty} g_n(0)=0$. If $x\in K\setminus\{0\}$, choose $N\in\mathbb{N}$ such that $\|x\|>\dfrac{1}{N}$. So for all $n\geq N$ we have $\|x\|>\dfrac{1}{n}$. Hence $g_n(x)=1$ for all $n\geq N$. This shows that $\lim\limits_{n\to\infty} g_n(x)=1$. Therefore $g_n\overset{p.w.}{\longrightarrow} g$ on K as $n\longrightarrow\infty$. Since g is not continuous on K, there is no a convergent subsequence of the sequence $\{g_n\}_n$ with respect to $\|\cdot\|_\infty$. This shows that M_0 is not quasi Chebyshev. $\square$

Theorem 4.12. *Let $X=C(T)$ where T is a compact Hausdorff space. Then every proximinal hyperplane in X is E-proximinal [3].*

Corollary 4.13. *If $\dim(A)<\infty$, then M_0 is E-proximinal.*

Proof. Since $\dim(A)<\infty$, $K=\overline{B_1^{(0)}}$ is compact. Define $\phi: C^b(K)\longrightarrow \mathbb{C}$ by $\phi(f)=f(0)$. Since $\phi\in C^b(K)^*$ and $M_0=\ker(\phi)$, M_0 is a hyperplane in $C^b(K)$. Hence by Theorem 4.12, M_0 is E-proximinal. $\square$

In the sequel, we present some results concerning strongly proximinality and E-proximinality of M and $M\varphi$.

Theorem 4.14. *Let $M\subseteq C^b(K)$. Then $M\subseteq C^{b\varphi}(K)$ is strongly proximinal if and only if $M\varphi\subseteq C^b(K)\varphi$ is strongly proximinal.*

Proof. Let $M\subseteq C^{b\varphi}(K)$ be strongly proximinal. Also let $f\varphi\in C^b(K)\varphi$ and $\{g_n\varphi\}_n\subseteq M\varphi$ be a minimizing sequence for $f\varphi$. Since

$$\lim_{n\to\infty}\|f\varphi-g_n\varphi\|_\infty=d^\infty(f\varphi,M\varphi),$$

$\lim\limits_{n\to\infty}\|f-g_n\|_\varphi=d^\varphi(f,M)$. This shows that $\{g_n\}_n\subseteq M\subseteq C^{b\varphi}(K)$ is a minimizing sequence for f. As $M\subseteq C^{b\varphi}(K)$ is strongly proximinal, there exists a subsequence $\{g_{n_k}\}_k$ and a sequence $\{z_k\}_k\subseteq P_M^\varphi(f)$ such that $\lim\limits_{k\to\infty}\|g_{n_k}-z_k\|_\varphi=0$. So $\lim\limits_{k\to\infty}\|g_{n_k}\varphi-z_k\varphi\|_\infty=0$. By the equality $P_{M\varphi}^\infty(f\varphi)=P_M^\varphi(f)\varphi$ of Theorem 4.5, it is obvious that $\{z_k\varphi\}_k\subseteq P_{M\varphi}^\infty(f\varphi)$ and also $\{g_{n_k}\varphi\}_k$ is a subsequence of $\{g_n\varphi\}_n$. Hence $M\varphi\subseteq C^b(K)\varphi$ is strongly proximinal.

Conversely, let $M\varphi\subseteq C^b(K)\varphi$ be strongly proximinal. Also let $f\in C^{b\varphi}(K)$ and $\{g_n\}_n\subseteq M$ be a minimizing sequence for f. So

$$\lim_{n\to\infty}\|f-g_n\|_\varphi=d^\varphi(f,M).$$

It follows that $\lim_{n\to\infty} \|f\varphi - g_n\varphi\|_\infty = d^\infty (f\varphi, M\varphi)$. Hence $\{g_n\varphi\}_n \subseteq M\varphi$ is a minimizing sequence for $f\varphi$. Strongly proximinality of $M\varphi \subseteq C^b(K)\varphi$ implies that there exists a subsequence $\{g_{n_k}\varphi\}_k \subseteq \{g_n\varphi\}_n$ and a sequence $\{z_k\varphi\}_k \subseteq P^\infty_{M\varphi}(f\varphi) = P^\varphi_M(f)\varphi$ such that

$$\lim_{k\to\infty} \|g_{n_k}\varphi - z_k\varphi\|_\infty = 0.$$

Clearly $\{g_{n_k}\}_k \subseteq \{g_n\}_n$, $\{z_k\}_k \subseteq P^\varphi_M(f)$ and $\lim_{k\to\infty} \|g_{n_k} - z_k\|_\varphi = 0$. This shows that $M \subseteq C^{b\varphi}(K)$ is strongly proximinal. $\square$

By Theorem 4.5, for $M \subseteq C^b(K)$ and $f \in C^b(K)$ we have $P^\varphi_M(f)\varphi = P^\infty_{M\varphi}(f\varphi)$. Hence if $M \subseteq C^{b\varphi}(K)$ is proximinal, then so is $M\varphi \subseteq C^b(K)\varphi$. For the next result we define the following notations.

$$\alpha^\varphi(f) = \inf\Big\{r \geq 0 \;\Big|\; \exists \{g_n\}_n \subseteq M \text{ s.t. } \|g_n\|_\varphi \leq r, \lim_{n\to\infty} \|f - g_n\|_\varphi = d^\varphi(f, M)\Big\},$$

and

$$\alpha^\infty(f\varphi) = \inf\Big\{r \geq 0 \;\Big|\; \exists \{g_n\varphi\}_n \subseteq M\varphi \text{ s.t. } \|g_n\varphi\|_\infty \leq r, \lim_{n\to\infty} \|f\varphi - g_n\varphi\|_\infty = d^\infty(f\varphi, M\varphi)\Big\}.$$

One can easily check that $\alpha^\varphi(f) = \alpha^\infty(f\varphi)$ for all $f \in C^b(K)$.

Theorem 4.15. *$M \subseteq C^{b\varphi}(K)$ is E-proximinal if and only if $M\varphi \subseteq C^b(K)\varphi$ is E-proximinal.*

Proof. Let $M \subseteq C^{b\varphi}(K)$ be E-proximinal, $f\varphi \in C^b(K)\varphi$ and $\epsilon > 0$ be given. Since M is E-proximinal at f, there exists $g \in P^\varphi_M(f)$ such that $\|g\|_\varphi < \alpha^\varphi(f) + \epsilon$. Hence $\|g\varphi\|_\infty < \alpha^\infty(f\varphi) + \epsilon$. Since $g\varphi \in P^\infty_{M\varphi}(f\varphi)$, $M\varphi \subseteq C^b(K)\varphi$ is E-proximinal at $f\varphi$. Hence $M\varphi$ is E-proximinal in $C^b(K)\varphi$.

Conversely, let $M\varphi \subseteq C^b(K)\varphi$ be E-proximinal, $f \in C^{b\varphi}(K)$ and $\epsilon > 0$ be given. Since $M\varphi$ is E-proximinal at $f\varphi \in C^b(K)\varphi$, there exists $g\varphi \in P^\infty_{M\varphi}(f\varphi)$ such that $\|g\varphi\|_\infty < \alpha^\infty(f\varphi) + \epsilon$. Hence $\|g\|_\varphi < \alpha^\varphi(f) + \epsilon$. Since $g \in P^\varphi_M(f)$, M is E-proximinal at $f \in C^{b\varphi}(K)$. Hence M is E-proximinal in $C^{b\varphi}(K)$. $\square$

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