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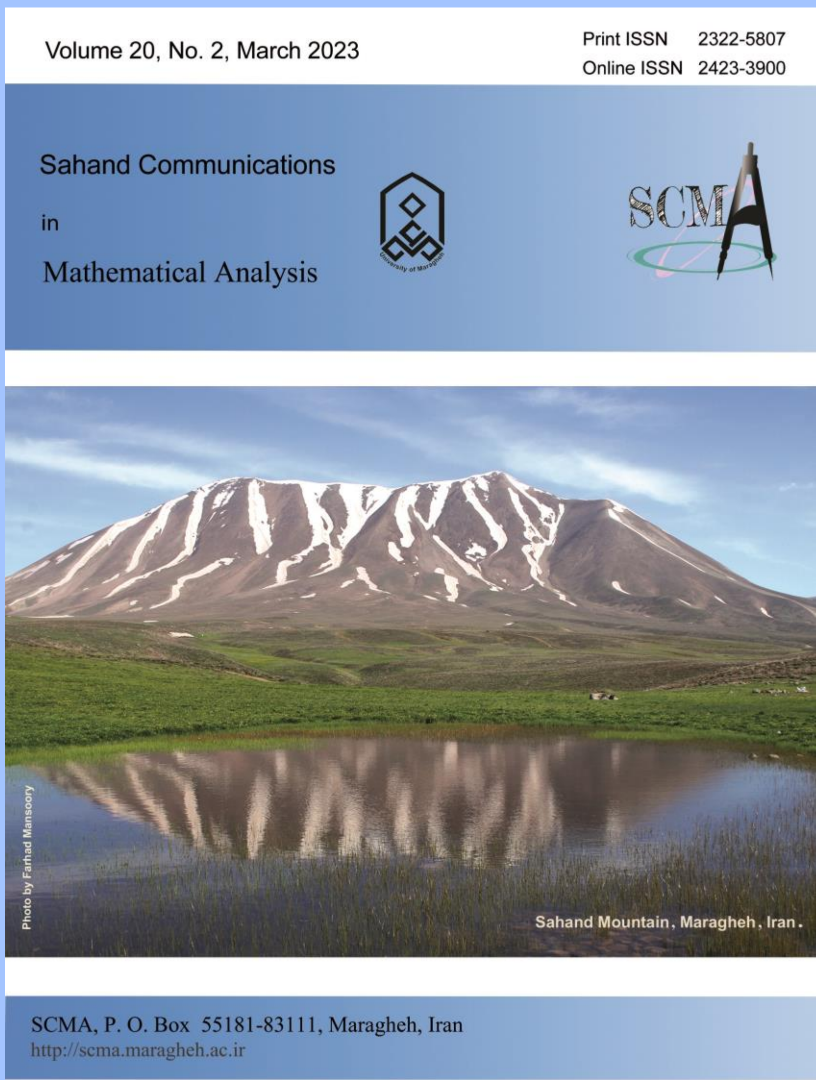
Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 2
Pages: 109-121

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.549903.1069



Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 2 (2023), 109-121
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.549903.1069

Fixed Point Results for Multivalued Mapping in R-Metric Space

Astha Malhotra[1] and Deepak Kumar[2*]

ABSTRACT. This paper explores certain fixed point results for multivalued mapping in a metric space endowed with an arbitrary binary relation R, briefly written as R-metric space. The fixed point results proved are subjected to contraction conditions corresponding to the multivalued counterpart of F-contraction and F-weak contraction in R-metric space. The main results unify, extend and generalize the results on multivalued and single-valued mapping in the literature. To support the conclusion, several examples have been provided.

1. Introduction

Fixed point theory gained much of its popularity with the discovery of the famous Banach Contraction Principle [16], named after the Polish mathematician S. Banach. The result deals with the existence and uniqueness of the fixed point for a single-valued self-map satisfying a contraction condition on a complete metric space. However, the study of fixed point theory is not just restricted to a single-valued self-map but can be extended to a multivalued mapping as well.

In 1969, S. B. Nadler [20] introduced the idea of multivalued Lipschitz mapping along with fixed point results for multivalued contraction mapping in a complete metric space. The concept of multivalued mapping has been improved many times since then in the literature (see [4, 5, 9, 11, 15] and references cited therein). Wardowski [6] originally proposed F-contraction in 2012, but Altun et al. generalized this idea

2020 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. R-metric space, Multivalued mapping, Fixed point.
Received: 03 March 2022, Accepted: 23 October 2022.
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to a multivalued mapping and presented the fixed point result in a complete metric space in [10]. In 2015, A. Alam and M. Imdad [1] coined the notion of a relation theoretic metric space wherein a given metric space is combined with an arbitrary binary relation. The relation theoretic metric has since been used to study fixed point results for contractive mapping (see [8, 12]), expansive mapping (see [13]) and non-expansive mapping (see [2, 3]).

Based on the work done in some of the recent literature, in this paper, we prove fixed point results in relational theoretic metric space for a multivalued mapping equipped with generalized contractions and authenticate them further with the help of examples.

2. Preliminaries

This section of the paper deals with some of the basic notations, definitions and lemma that shall be required to prove the main results. Throughout the scope of this paper, X denotes a non-empty set together with metric d, $\mathcal{N}(X)$ represents the set of all non-empty subsets of X, $\mathcal{K}(X)$ represents the set of all compact subsets of X, $\mathcal{CB}(X)$ represents the set of all closed and bounded subsets of X. In addition, the symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Q}^c$ denote the set of natural numbers, real numbers, rational numbers and irrational numbers respectively.

Also, we define functional $D : \mathcal{N}(X) \times \mathcal{N}(X) \to [0, +\infty)$ such that $D(M, N) = \inf\{d(\sigma, \upsilon) \text{ such that } \sigma \in M, \upsilon \in N\}$ and Hausdorff-Pompeiu functional $\mathcal{H} : \mathcal{N}(X) \times \mathcal{N}(X) \to [0, +\infty) \cup \{+\infty\}$ such that $\mathcal{H}(M, N) = \max\{\sup_{m\in M} d(m, N), \sup_{n\in N} d(n, M)\}$.

Definition 2.1 ([19])**.** For a non-empty set X, a binary relation R on X is a subset of $X \times X$.

Definition 2.2 ([18])**.** For a binary relation R defined on a non-empty set X, a sequence $\{\sigma_n\}_{n\in\mathbb{N}} \in X$ is said to be an R-sequence if $(\sigma_n, \sigma_{n+1}) \in$ R for all $n \in \mathbb{N}$.

Definition 2.3 ([18])**.** A metric space (X, d) together with a binary relation R is said to be an R-metric space. It is usually written as (X, d, R).

Definition 2.4 ([18])**.** For an R-metric space (X, d, R), a self-map $g : X \to X$ is said to be R-continuous at $\sigma \in X$ if for any R-sequence $\{\sigma_n\}_{n\in\mathbb{N}} \in X$ with $\sigma_n \to \sigma$ as $n \to +\infty$ then, $g\sigma_n \to g\sigma$ as $n \to +\infty$. Also, g is said to be R-continuous on X if it is R-continuous at each point of X.

Definition 2.5 ([18])**.** An R-metric space (X, d, R) is said to be an R-complete metric space if every R-Cauchy sequence in X is convergent.

Definition 2.6 ([6])**.** Let $F : (0, +\infty) \to (-\infty, +\infty)$ such that:

(F_1) for $\sigma, \upsilon \in (0, +\infty)$ if $\sigma < \upsilon$ implies $F(\sigma) < F(\upsilon)$;

(F_2) for each sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\lim_{n\to+\infty} \sigma_n = 0 \text{ if and only if } \lim_{n\to+\infty} F(\sigma_n) = -\infty;$$

(F_3) If there exists some $k \in (0, 1)$ then $\lim_{\eta\to 0^+} \eta^k F(\eta) = 0$.

Denote $\mathfrak{F}$, the family of all mappings $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying $(F_1) - (F_3)$.

If in addition, F satisfies (F_4) where (F_4): $F(\inf M) = \inf F(M)$ for all $M \subset (0, +\infty)$ with $\inf M > 0$, then the family of all mappings $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying $(F_1) - (F_4)$ is denoted by $\mathfrak{F}'$.

It is easy to prove the following property for a metric space (X, d) based on the proof for a b-metric space found in [17].

Lemma 2.7. *For a metric space (X, d), $D(m, N) \le d(m, n) + D(n, N)$ for all $m, n \in X$ and $N \subset X$.*

3. Main Results

We now prove fixed point results for a multivalued mapping in an R-metric space subject to generalized contractions. But before proceeding to the results, we first define the relation between two non-empty subsets of an R-metric space and R-continuity for a multivalued mapping.

Definition 3.1. Let (X, d, R) be an R-metric space. Then for two non-empty subsets M, N of X we say $(M, N) \in \mathrm{R}$ if $(m, n) \in \mathrm{R}$ for all $m \in M$ and $n \in N$.

Example 3.2. Let $X = \mathbb{R}$ in the usual metric space and define $\mathrm{R} = \{(\sigma, \upsilon) \in X^2 \text{ such that } \sigma.\upsilon < 0\}$. Then for subsets $M = (-\infty, 0)$ and $N = (0, +\infty)$ of X, we have $(M, N) \in \mathrm{R}$.

Definition 3.3. Let (X, d, R) be an R-metric space then a multivalued map $g : X \to \mathcal{K}(X)$ is said to be $\mathrm{R}_{\mathcal{H}}$-continuous at $\sigma \in X$ if for any R-sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ in X with $d(\sigma_n, \sigma) \to 0$ as $n \to +\infty$, we have $\mathcal{H}(g\sigma_n, g\sigma) \to 0$ as $n \to +\infty$. Also, g is said to be $\mathrm{R}_{\mathcal{H}}$-continuous on X if it is $\mathrm{R}_{\mathcal{H}}$-continuous at each point of X.

Readers should note that the above definition holds true when considering multivalued map $g : X \to \mathcal{CB}(X)$ where (X, d, R) is an R-metric space.

Theorem 3.4. *Let (X, d, R) be an R-complete metric space and let $g : X \to \mathcal{K}(X)$ be a multivalued map satisfying the following:*

(i) *there exists $\sigma_0 \in X$ such that $(\sigma_0, \upsilon) \in \mathrm{R}$ for all $\upsilon \in g\sigma_0$;*

(ii) *for all* $(\sigma, \upsilon) \in \mathrm{R}$*, we have* $(g\sigma, g\upsilon) \in \mathrm{R}$*;*
(iii) *either* g *is* $\mathrm{R}_{\mathcal{H}}$*-continuous or for any* R*-sequence* $\{\sigma_n\}_{n\in\mathbb{N}}$ *such that* $\sigma_n \to \sigma$ *as* $n \to +\infty$*, we have* $(\sigma_n, \sigma) \in \mathrm{R}$ *for all* $n \in \mathbb{N}$*;*
(iv) *if for some* $F \in \mathfrak{F}$*, there exists* $\tau > 0$ *such that for all* $(\sigma, \upsilon) \in \mathrm{R}$ *with* $\mathcal{H}(g\sigma, g\upsilon) > 0$ *we have*

$$\tau + F(\mathcal{H}(g\sigma, g\upsilon)) \leq F(d(\sigma, \upsilon)).$$

Then, g *has a fixed point.*

Proof. Define a sequence $\{\sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$ in X such that $\sigma_{n+1} \in g\sigma_n$ for all $n \in \mathbb{N} \cup \{0\}$. By condition (i), we obtain $(\sigma_0, \sigma_1) \in \mathrm{R}$.
On using condition (ii), we have

$$(g\sigma_0, g\sigma_1) = (\sigma_1, \sigma_2) \in \mathrm{R}. \tag{3.1}$$

On repeated use of condition (ii) in (3.1), we get

$$(\sigma_n, \sigma_{n+1}) \in \mathrm{R},$$

for all $n \in \mathbb{N}\cup\{0\}$. Thus $\{\sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$ is an R-sequence in X. If $\sigma_1 \in g\sigma_1$ then we are done. Suppose $\sigma_1 \notin g\sigma_1$ and since $g\sigma_1$ is compact subset of X then $d(\sigma_1, g\sigma_1) > 0$. Now, $d(\sigma_1, g\sigma_1) \leq \mathcal{H}(g\sigma_0, g\sigma_1)$ thus by (F_1), we have

$$\begin{aligned} F(d(\sigma_1, g\sigma_1)) &\leq F(\mathcal{H}(g\sigma_0, g\sigma_1)) \\ &\leq F(d(\sigma_0, \sigma_1)) - \tau. \end{aligned}$$

If $\sigma_k \in g\sigma_k$ for some $k \in \mathbb{N} \cup \{0\}$ then we are done. Suppose $\sigma_n \notin g\sigma_n$ for all $n \in \mathbb{N} \cup \{0\}$ and since $g\sigma_n$ is a compact subset of X then, $d(\sigma_n, g\sigma_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. As

$$\begin{aligned} d(\sigma_n, g\sigma_n) &\leq d(\sigma_n, \sigma_{n+1}) \\ &\leq \mathcal{H}(g\sigma_{n-1}, g\sigma_n) \end{aligned}$$

that is,

$$\begin{aligned} F(d(\sigma_n, g\sigma_n)) &\leq F(d(\sigma_n, \sigma_{n+1})) \\ &\leq F(\mathcal{H}(g\sigma_{n-1}, g\sigma_n)) \\ &\leq F(d(\sigma_{n-1}, \sigma_n)) - \tau \\ &< F(d(\sigma_{n-1}, \sigma_n)). \end{aligned} \tag{3.2}$$

Thus $\{\zeta_n = d(\sigma_n, \sigma_{n+1})\}_{n\in\mathbb{N}\cup\{0\}}$ is decreasing sequence of non-negative real number. Let $\lim\limits_{n\to+\infty} \zeta_n = \zeta \geq 0$.

Next, by (3.2) we have

$$F(\zeta_n) \leq F(\zeta_{n-1}) - \tau \leq F(\zeta_{n-2}) - 2\tau \leq \cdots \leq F(\zeta_0) - n\tau. \tag{3.3}$$

By letting $n \to +\infty$ in (3.3) we get $\lim_{n\to+\infty} \zeta_n = 0$. By (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to+\infty} \zeta_n^k F(\zeta_n) = 0.$$

Using (3.3), we have

$$\zeta_n^k F(\zeta_n) - \zeta_n^k F(\zeta_0) \leq -n\zeta_n^k \tau. \tag{3.4}$$

When we take limit as $n \to +\infty$ in (3.4), we get $\lim_{n\to+\infty} n\zeta_n^k = 0$. Thus there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\zeta_n \leq \frac{1}{n^{1/k}}.$$

Consider $m, n \in \mathbb{N}$ where $m > n > n_0$ such that

$$\begin{aligned} d(\sigma_n, \sigma_m) &\leq d(\sigma_n, \sigma_{n+1}) + d(\sigma_{n+1}, \sigma_{n+2}) + \cdots + d(\sigma_{m-1}, \sigma_m) \\ &= \zeta_n + \zeta_{n+1} + \cdots + \zeta_{m-1} \\ &= \sum_{j=n}^{m-1} \zeta_j \\ &\leq \sum_{j=n}^{+\infty} \zeta_j \\ &\leq \sum_{j=n}^{+\infty} \frac{1}{j^{1/k}}. \end{aligned} \tag{3.5}$$

Using the convergence of the series $\sum_{j=n}^{+\infty} \frac{1}{j^{1/k}}$ and on letting $n \to +\infty$ in (3.5), we obtain that $\{\sigma_n\}_{n\in\mathbb{N}}$ is an R-Cauchy sequence and since X is an R-complete space, so there exists $\sigma \in X$ such that

$$\lim_{n\to+\infty} \sigma_n = \sigma.$$

We now claim that $\sigma \in g\sigma$.

Case 1: Let g be an $\mathrm{R}_{\mathcal{H}}$-continuous mapping. Since $\sigma_{n+1} \in g\sigma_n$, we have

$$d(\sigma_{n+1}, g\sigma) \leq \mathcal{H}(g\sigma_n, g\sigma). \tag{3.6}$$

Taking the limit as $n \to +\infty$ in (3.6) and using the R-continuity of g, we get

$$\begin{aligned} d(\sigma, g\sigma) &= \lim_{n\to+\infty} d(\sigma_{n+1}, g\sigma) \\ &\leq \lim_{n\to+\infty} \mathcal{H}(g\sigma_n, g\sigma) \\ &= 0. \end{aligned}$$

As $g\sigma \subset K(X)$ so, we conclude that $\sigma \in g\sigma$.

Case 2: We have an R-sequence $\{\sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $\sigma_{n+1} \in g\sigma_n$ for all $n \in \mathbb{N}\cup\{0\}$ and $\sigma_n \to \sigma$ as $n \to +\infty$ then, $(\sigma_n, \sigma) \in \mathrm{R}$ for all $n \in \mathbb{N}$. On the contrary, suppose that $\sigma \notin g\sigma$ then there exists $n' \in \mathbb{N}$ such that $\sigma \notin \{\sigma_n\}$ for all $n > n'$ which further gives $\mathcal{H}(g\sigma, g\sigma_n) > 0$ and also by the given condition, we have $(\sigma_n, \sigma) \in \mathrm{R}$ for all $n \in \mathbb{N}\cup\{0\}$.

$$\begin{aligned} F(d(\sigma_{n+1}, g\sigma)) &\le \tau + F(\mathcal{H}(g\sigma_n, g\sigma)) \\ &< F(d(\sigma_n, \sigma)). \end{aligned} \tag{3.7}$$

By letting $n \to +\infty$ in (3.7), we get $d(\sigma, g\sigma) = 0$ which is a contradiction. Hence, $\sigma \in g\sigma$. □

Theorem 3.5. *Let (X, d, R) be an* R*-complete metric space and let $g : X \to \mathcal{CB}(X)$ be a multivalued map satisfying the following:*

(i) *there exists $\sigma_0 \in X$ such that $(\sigma_0, \upsilon) \in \mathrm{R}$ for all $\upsilon \in g\sigma_0$;*
(ii) *for all $(\sigma, \upsilon) \in \mathrm{R}$, we have $(g\sigma, g\upsilon) \in \mathrm{R}$;*
(iii) *either g is $\mathrm{R}_{\mathcal{H}}$-continuous or for any* R*-sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n \to \sigma$ as $n \to +\infty$, we have $(\sigma_n, \sigma) \in \mathrm{R}$ for all $n \in \mathbb{N}$;*
(iv) *if for some $F \in \mathfrak{F}'$, there exists $\tau > 0$ such that for all $(\sigma, \upsilon) \in \mathrm{R}$ with $\mathcal{H}(g\sigma, g\upsilon) > 0$ we have*

$$\tau + F(\mathcal{H}(g\sigma, g\upsilon)) \le F(d(\sigma, \upsilon)).$$

Then, g has a fixed point.

Proof. Proceeding on the lines of Theorem 3.4, we obtain an R-sequence $\{\sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$, where $\sigma_{n+1} \in g\sigma_n$ for all $n \in \mathbb{N}\cup\{0\}$. If $\sigma_1 \in g\sigma_1$ then, we are done. Suppose $\sigma_1 \notin g\sigma_1$ and since $g\sigma_1$ is a closed subset of X then, $d(\sigma_1, g\sigma_1) > 0$. Now, $d(\sigma_1, g\sigma_1) \le \mathcal{H}(g\sigma_0, g\sigma_1)$ thus by (F_1), we have

$$\begin{aligned} F(d(\sigma_1, g\sigma_1)) &\le F(\mathcal{H}(g\sigma_0, g\sigma_1)) \\ &\le F(d(\sigma_0, \sigma_1)) - \tau. \end{aligned} \tag{3.8}$$

Using the (F_4) property of F, we get

$$\begin{aligned} F(d(\sigma_1, g\sigma_1)) &= F(\inf_{\upsilon\in g\sigma_1} d(\sigma_1, \upsilon)) \\ &= \inf_{\upsilon\in g\sigma_1} F(d(\sigma_1, \upsilon)), \end{aligned} \tag{3.9}$$

using (3.8) in (3.9) and the fact that $\sigma_2 \in g\sigma_1$, we observe that

$$\begin{aligned} \inf_{\upsilon\in g\sigma_1} F(d(\sigma_1, \upsilon)) &\le F(d(\sigma_1, \sigma_2)) \\ &\le F(\mathcal{H}(g\sigma_0, g\sigma_1)) \\ &\le F(d(\sigma_0, \sigma_1)) - \tau. \end{aligned}$$

So from above equation, we have $F(d(\sigma_1,\sigma_2)) \leq F(d(\sigma_0,\sigma_1)) - \tau$. If $\sigma_2 \in g\sigma_2$ then we are done and for $\sigma_2 \notin g\sigma_2$ we have $\sigma_3 \in g\sigma_2$ such that

$$F(d(\sigma_2,\sigma_3)) \leq F(d(\sigma_1,\sigma_2)) - \tau.$$

In a similar manner, we obtain

$$F(d(\sigma_n,\sigma_{n+1})) \leq F(d(\sigma_{n-1},\sigma_n)) - \tau,$$

and thus $d(\sigma_n,\sigma_{n+1}) < d(\sigma_{n-1},\sigma_n)$ that is $\{d(\sigma_n,\sigma_{n+1})\}_{n\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence of non-negative real numbers.

Now, by the working of Theorem 3.4, we conclude that g possesses a fixed point. □

Theorem 3.6. *Let (X,d,R) be an R-complete metric space and let $g:\ X \to \mathcal{K}(X)$ be a multivalued map such that the following holds:*

(i) *there exists $\sigma_0 \in X$ such that $(\sigma_0,\upsilon) \in \mathrm{R}$ for all $\upsilon \in g\sigma_0$;*
(ii) *for all $(\sigma,\upsilon) \in \mathrm{R}$, we have $(g\sigma,g\upsilon) \in \mathrm{R}$;*
(iii) *either g is $\mathrm{R}_{\mathcal{H}}$-continuous or for any R-sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n \to \sigma$ as $n \to +\infty$, we have $(\sigma_n,\sigma) \in \mathrm{R}$ for all $n \in \mathbb{N}$;*
(iv) *if for some $F \in \mathfrak{F}$, there exists $\tau > 0$ such that for all $(\sigma,\upsilon) \in \mathrm{R}$ with $\mathcal{H}(g\sigma,g\upsilon) > 0$, we have*

$$\tau + F(\mathcal{H}(g\sigma,g\upsilon)) \leq F\left(\max\left\{d(\sigma,\upsilon), D(\sigma,g\sigma), D(\upsilon,g\upsilon), \frac{D(\sigma,g\upsilon)+D(\upsilon,g\sigma)}{2}\right\}\right).$$

Then, g has a fixed point.

Proof. Proceeding along the lines of Theorem 3.4, we get an R-sequence $\{\sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $\sigma_{n+1} \in g\sigma_n$ for all $n \in \mathbb{N}\cup\{0\}$. Suppose $\sigma_1 \notin g\sigma_1$, then we have

$$0 < d(\sigma_1,g\sigma_1) \leq \mathcal{H}(g\sigma_0,g\sigma_1).$$

Further, using condition (iv), we have

(3.10)

$$\begin{aligned}\tau &+ F(\mathcal{H}(g\sigma_0,g\sigma_1)) \\ &\leq F\left(\max\left\{d(\sigma_0,\sigma_1), D(\sigma_0,g\sigma_0), D(\sigma_1,g\sigma_1), \frac{D(\sigma_0,g\sigma_1)+D(\sigma_1,g\sigma_0)}{2}\right\}\right).\end{aligned}$$

Next, the following observations can be easily made for a multivalued map and in addition, using Lemma 2.7, we have

$$\begin{aligned}&D(\sigma_0,g\sigma_0) \leq d(\sigma_0,\sigma_1)\\ &D(\sigma_1,g\sigma_1) \leq \mathcal{H}(g\sigma_0,g\sigma_1)\\ &D(\sigma_1,g\sigma_0) = 0\\ &D(\sigma_0,g\sigma_1) \leq d(\sigma_0,\sigma_1) + \mathcal{H}(g\sigma_0,g\sigma_1).\end{aligned}$$

Using the above in (3.10), we get

$$\tau + F(\mathcal{H}(g\sigma_0, g\sigma_1)) \le F\left(\max\left\{d(\sigma_0,\sigma_1), \mathcal{H}(g\sigma_0, g\sigma_1), \frac{D(\sigma_0, g\sigma_1)}{2}\right\}\right) \le F\left(\max\left\{d(\sigma_0,\sigma_1), \mathcal{H}(g\sigma_0, g\sigma_1), \frac{d(\sigma_0,\sigma_1) + \mathcal{H}(g\sigma_0, g\sigma_1)}{2}\right\}\right). \tag{3.11}$$

If $d(\sigma_0,\sigma_1) < \mathcal{H}(g\sigma_0, g\sigma_1)$, then by (3.11) we obtain

$$\tau + F(\mathcal{H}(g\sigma_0, g\sigma_1)) \le F(\mathcal{H}(g\sigma_0, g\sigma_1)),$$

which is a contradiction. Thus, we have $\mathcal{H}(g\sigma_0, g\sigma_1) < d(\sigma_0,\sigma_1)$ and by (3.11), we get

$$\tau + F(\mathcal{H}(g\sigma_0, g\sigma_1)) \le F(d(\sigma_0,\sigma_1)).$$

Further, suppose $\sigma_n \notin g\sigma_n$ for all $n \in \mathbb{N} \cup \{0\}$,

$$\tau + F(\mathcal{H}(g\sigma_n, g\sigma_{n+1})) \le F\left(\max\left\{d(\sigma_n,\sigma_{n+1}), D(\sigma_n, g\sigma_n), D(\sigma_{n+1}, g\sigma_{n+1}), \frac{D(\sigma_n, g\sigma_{n+1}) + D(\sigma_{n+1}, g\sigma_n)}{2}\right\}\right). \tag{3.12}$$

Again, we have the following observations:

$$\begin{aligned} &D(\sigma_n, g\sigma_n) \le d(\sigma_n,\sigma_{n+1}),\\ &D(\sigma_{n+1}, g\sigma_{n+1}) \le \mathcal{H}(g\sigma_n, g\sigma_{n+1}),\\ &D(\sigma_{n+1}, g\sigma_n) = 0,\\ &D(\sigma_n, g\sigma_{n+1}) \le d(\sigma_n,\sigma_{n+1}) + \mathcal{H}(g\sigma_n, g\sigma_{n+1}). \end{aligned}$$

Using the above in (3.12), we get

$$\tau + F(\mathcal{H}(g\sigma_n, g\sigma_{n+1})) \le F\left(\max\left\{d(\sigma_n,\sigma_{n+1}), \mathcal{H}(g\sigma_n, g\sigma_{n+1}), \frac{d(\sigma_n,\sigma_{n+1}) + \mathcal{H}(g\sigma_n, g\sigma_{n+1})}{2}\right\}\right). \tag{3.13}$$

If $d(\sigma_n,\sigma_{n+1}) \le \mathcal{H}(g\sigma_n, g\sigma_{n+1})$, then by (3.13) we have

$$\tau + F(\mathcal{H}(g\sigma_n, g\sigma_{n+1})) \le F(\mathcal{H}(g\sigma_n, g\sigma_{n+1})),$$

which is a contradiction. Thus, we have $\mathcal{H}(g\sigma_n, g\sigma_{n+1}) < d(\sigma_n, \sigma_{n+1})$ and by (3.13), we get

$$\tau + F(\mathcal{H}(g\sigma_n, g\sigma_{n+1})) \leq F(d(\sigma_n, \sigma_{n+1})).$$

Now, by the working of Theorem 3.4, we conclude that g possesses a fixed point. □

Theorem 3.7. *Let (X, d, R) be an* R*-complete metric space and let $g: X \to \mathcal{CB}(X)$ be a multivalued map such that the following holds:*

(i) *there exists $\sigma_0 \in X$ such that $(\sigma_0, \upsilon) \in \mathrm{R}$ for all $\upsilon \in g\sigma_0$;*
(ii) *for all $(\sigma, \upsilon) \in \mathrm{R}$, we have $(g\sigma, g\upsilon) \in \mathrm{R}$;*
(iii) *either g is $\mathrm{R}_{\mathcal{H}}$-continuous or for any* R*-sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ such that $\sigma_n \to \sigma$ as $n \to +\infty$, we have $(\sigma_n, \sigma) \in \mathrm{R}$ for all $n \in \mathbb{N}$;*
(iv) *if for some $F \in \mathfrak{F}'$, there exists $\tau > 0$ such that for all $(\sigma, \upsilon) \in \mathrm{R}$ with $\mathcal{H}(g\sigma, g\upsilon) > 0$, we have*

$$\tau + F(\mathcal{H}(g\sigma, g\upsilon)) \leq F\left(\max\left\{d(\sigma, \upsilon), D(\sigma, g\sigma), D(\upsilon, g\upsilon), \frac{D(\sigma, g\upsilon) + D(\upsilon, g\sigma)}{2}\right\}\right).$$

Then, g has a fixed point.

Proof. Using the work done in Theorem 3.6, we obtain an R-sequence $\{\sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$ such that $\sigma_{n+1} \in g\sigma_n$ for all $n \in \mathbb{N}\cup\{0\}$. Suppose $\sigma_1 \notin g\sigma_1$, then we have

$$\begin{aligned} \tau + F(\mathcal{H}(g\sigma_0, g\sigma_1)) \leq F\Bigg(&\max\Bigg\{d(\sigma_0, \sigma_1), D(\sigma_0, g\sigma_0), D(\sigma_1, g\sigma_1), \\ &\frac{D(\sigma_0, g\sigma_1) + D(\sigma_1, g\sigma_0)}{2}\Bigg\}\Bigg), \\ &\leq d(\sigma_0, \sigma_1)). \end{aligned}$$

Also since, $g\sigma_1$ is closed and for some $\sigma_2 \in g\sigma_1$, then by working done in Theorem 3.5, we obtain

$$\begin{aligned} \inf_{\upsilon \in g\sigma_1} F(d(\sigma_1, \upsilon)) &\leq F(d(\sigma_1, \sigma_2)) \\ &\leq F(\mathcal{H}(g\sigma_0, g\sigma_1)) \\ &\leq F(d(\sigma_0, \sigma_1)) - \tau, \end{aligned}$$

and

$$F(d(\sigma_1, \sigma_2)) \leq F(d(\sigma_0, \sigma_1)) - \tau.$$

Following the lines of Theorem 3.5, we obtain that g possesses a fixed point. □

Example 3.8. Let $X = \{0,1,2,3,4\}$ and $d(\sigma,\upsilon) = |\sigma - \upsilon|$. Define $g : X \to \mathcal{CB}(X)$ as:

$$g\sigma = \begin{cases} \{0,1\}, & \text{for } \sigma = 3, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $\mathrm{R} = \{(0,0),(0,1),(0,3)\}$ and $F(\rho) = \ln\rho$. We have (X,d,R) as an R-complete metric space. Also, g is $\mathrm{R}_{\mathcal{H}}$-continuous and satisfies the condition (ii) of Theorem 3.7. Next, for $(\sigma,\upsilon) \in \mathrm{R}$ with $\mathcal{H}(g\sigma, g\upsilon)) > 0$, we have $\sigma = 0$, $\upsilon = 3$ and

$$\tau + F(\mathcal{H}(g\sigma, g\upsilon)) \\ \leq F\left(\max\left\{d(\sigma,\upsilon), D(\sigma,g\sigma), D(\upsilon,g\upsilon), \frac{D(\sigma,g\upsilon)+D(\upsilon,g\sigma)}{2}\right\}\right).$$

Thus, the given R-metric space satisfies all the conditions of Theorem 3.7 and hence, g has a fixed point which is $\sigma = 0$. Also, note that the given example in the absence of relation R does not satisfy the multivalued contraction condition given in [10] and [14]. It should also be noted that the given relation R is not orthogonal thus, the results of [15] cannot be applied.

Example 3.9. Let $X = X_1 \cup X_2$, where $X_1 = [0,1/2]$ and $X_2 = (1/2,1)$. Define the metric $d : X \times X \to \mathbb{R}^+$ as:

$$g\sigma = \begin{cases} 0.01 + \mid \sigma - \upsilon \mid, & \text{if } \sigma \neq \upsilon, \\ 0, & \text{if } \sigma = \upsilon. \end{cases}$$

Also, define $\mathrm{R} = \{(\sigma,\upsilon) \in X^2 \text{ such that } \sigma.\upsilon \in \{\sigma,\upsilon\}\}$. Then (X,d,R) is an R-metric space. Let the multivalued map $g : X \to \mathcal{CB}(X)$ be defined as:

$$g\sigma = \begin{cases} \{0\}, & \text{if } \sigma \in X_1, \\ \{0,1/3\}, & \text{if } \sigma \in X_2. \end{cases}$$

Let $F(\rho) = \rho$, then we have (X,d,R) as an R-complete metric space and in addition, the following conditions hold:

(i) for $0 \in X$, $(0, g0) \in \mathrm{R}$;
(ii) for any $(\sigma,\upsilon) \in \mathrm{R}$, $(g\sigma, g\upsilon) \in \mathrm{R}$;
(iii) for any R-sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ with $\sigma_n \to \sigma$ then $\sigma = 0$ and thus $(\sigma_n, \sigma) \in \mathrm{R}$ for all $n \in \mathbb{N}$;
(iv) let $(\sigma,\upsilon) \in \mathrm{R}$ then $\sigma = 0$ or/and $\upsilon = 0$. Consider $\sigma = 0$ (the case for $\upsilon = 0$ follows similarly), then we have the following cases:

Case (1): let $\upsilon \in X_1$ then

$$\tau + \ln(\mathcal{H}(g\sigma, g\upsilon)) = \tau + \ln(\mathcal{H}(0,0)) \\ \to -\infty,$$

and

$$\begin{aligned}\ln(d(\sigma,\upsilon)) &= \ln(d(0,\upsilon))\\ &\geq -\infty.\end{aligned}$$

Thus, the condition:

$$\tau + \ln(\mathcal{H}(g\sigma, g\upsilon)) \leq \ln(d(\sigma,\upsilon)),$$

holds for any finite $\tau > 0$.

Case (2):Let $\upsilon \in X_2$ then,

$$\begin{aligned}\tau + \ln(\mathcal{H}(g\sigma, g\upsilon)) &= \tau + \ln(\mathcal{H}(\{0\},\{0,1/3\}))\\ &= \tau + \ln(0.3433),\end{aligned}$$

and

$$\begin{aligned}\ln(d(\sigma,\upsilon)) &= \ln(d(0,\upsilon))\\ &= \ln(0.01+\upsilon)\\ &> \ln(0.51).\end{aligned}$$

Thus, the contraction condition:

$$\tau + \ln(\mathcal{H}(g\sigma, g\upsilon)) \leq \ln(d(\sigma,\upsilon)),$$

holds for $0 < \tau < 0.396$.

Hence from Example 3.9, we conclude that multivalued map g satisfies all conditions of Theorem 3.5 and thus possesses a fixed point which is $\sigma = 0$. However, it should be noted here that since the space (X, d) is an incomplete metric space the results of [6, 7, 10, 14] are not applicable.

4. Conclusion

The purpose of the present paper is to unify, extend and generalize the results of multivalued and single-valued mapping in the literature. The main results of [10] are obtained if we consider the binary relation R in Theorem 3.4 and Theorem 3.5 as a universal relation on X. Under the condition given in Theorem 3.7, a result equivalent to the one given in [14] is obtained.

Next, if the binary relation R in Theorem 3.4 and Theorem 3.5 is considered to be an orthogonal relation, that is, there exists some $\sigma_0 \in X$ such that $(\sigma_0, \upsilon) \in$ R for all $\upsilon \in X$ or $(\upsilon, \sigma_0) \in$ R for all $\upsilon \in X$, then the main results of [15] are deduced.

Finally, on considering R as a universal relation and g as a single-valued self-map on X in Theorem 3.4-3.7, then the corresponding results of [6] and [7] are obtained.

We claim that from the results proved in [6, 7, 10, 14, 15], the result of this paper cannot be deduced which is further validated with the help of Example 3.8-3.9.

References

1. A. Alam and M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl., 17 (2015), pp. 693-702.
2. A. Alam, R. George, M. Imdad and M. Hasanuzzaman, *Fixed Point Theorems for Nonexpansive Mappings under Binary Relations*, Mathematics, 9 (2021), Article No. 2059.
3. A. Alam, M. Imdad, M. Asim and S. Sessa, *A Relation-Theoretic Formulation of Browder–Göhde Fixed Point Theorem*, Axioms, 10 (2021), Article No. 285.
4. B.S. Choudhury, N. Metiya and C. Bandyopadhyay, *Fixed points of multivalued α-admissible mappings and stability of fixed point sets in metric spaces*, Rend. Circ. Mat. Palermo, 64 (2015), pp. 43-55.
5. D. Klim and D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., 334 (2007), pp. 132-139.
6. D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., 2012 (1) (2012), Artice No. 94.
7. D. Wardowski and N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces*, Demonstr. Math., 47 (2014), pp. 146-155.
8. G. Prasad, R.C. Dimri and A. Bartwal, *Fixed points of Suzuki contractive mappings in relational metric spaces*, Rendiconti del Circolo Mat. di Palermo, 2 (2019), pp. 1347–1358.
9. I. Altun, G. Durmaz, G. Mınak and S. Romaguera, *Multivalued Almost F-contractions on Complete Metric Spaces*, Filomat, 30 (2016), pp. 441-448.
10. I. Altun, G. Minak and H. Dag, *Multivalued F-contractions on complete metric spaces*, J. Nonlinear Convex Anal., 28 (2015), pp. 659-666.
11. L. Ćirić, *Fixed point theorems for multi-valued contractions in complete metric spaces*, J. Math. Anal. Appl., 348 (2008), pp. 499-507.
12. M. Imdad, Q.H. Khan, W.M. Alfaqih and R. Gubran, *A relation theoretic (F, R)-contraction principle with applications to matrix equation*, Bull. Math. Anal. Appl., 10 (2018), pp. 1-12.
13. M. Imdad and W.M. Alfaqih, *A relation-theoretic expansion principle*, Acta Univ. Apulensis., 54 (2018), pp. 55-69.

14. Ö. Acar, G. Durmaz and G. Minak, *Generalized multivalued F-contractions on complete metric spaces*, Bull. Iranian Math. Soc., 40 (2014), pp. 1469-1478.
15. R. K. Sharma and S. Chandok, *Multivalued problems, orthogonal mappings, and fractional integro-differential equation*, J. Math., (2020), Article ID 6615478.
16. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundam. Math., 3 (1922), pp. 133-181.
17. S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena., 46 (1998), pp. 263-276.
18. S. Khalehoghli, H. Rahimi and M.E. Gordji, *Fixed point theorems in* R*-metric spaces with applications*, AIMS Math., 5 (2020), pp. 3125-3137.
19. S. Lipschutz, *Schaum's outline of theory and problems of set theory and related topics*, McGraw-Hill, New York, 1964.
20. S.B. Nadler Jr, *Multi-valued contraction mappings*, Pacific J. Math., 30 (1969), pp. 475-488.

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