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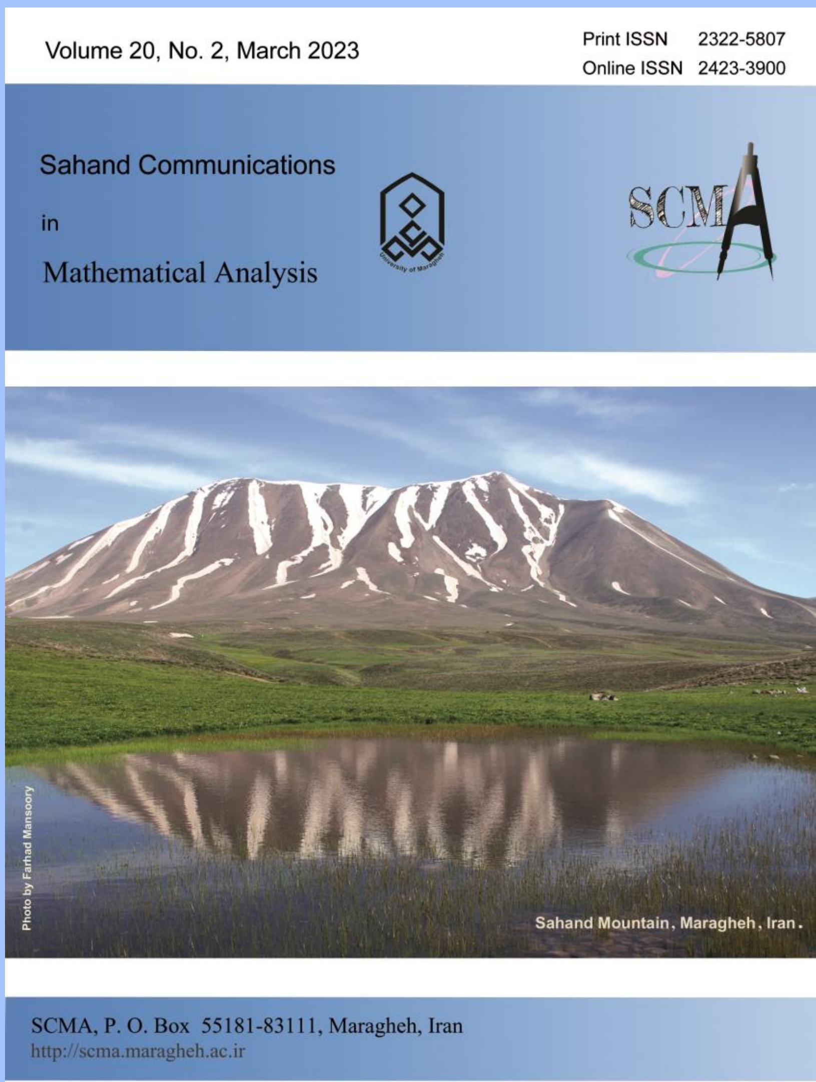

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On Fixed Points of a General Class of Hybrid Contractions with Ulam-Type Stability

Jamilu Abubakar Jiddah[1], Mohammed Shehu Shagari[2*] and Abdussamad Tanko Imam[3]

Abstract. In this paper, a new general class of contraction, namely admissible hybrid (G-α-ϕ)-contraction is introduced and some fixed point theorems that cannot be deduced from their corresponding ones in metric spaces are proved. The distinction of this family of contractions is that its contractive inequality can be specialized in several ways, depending on multiple parameters. Consequently, several corollaries, including some recently announced results in the literature are highlighted and analyzed. Nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. We further examine Ulam-type stability and well-posedness for the new contraction proposed herein. In addition, one of our obtained corollaries is applied to set up novel existence conditions for the solution of a class of integral equations. There is an open problem concerning the discretized population balance model, whose solution may be analyzed using the methods established here.

1. Introduction

The well-known Banach contraction in metric spaces has paved the way for a new dawn in metric fixed point theory, which is proven to have many applications in inequalities, approximation theory, optimization and so on. Researchers in this area have introduced several new concepts in metric spaces and obtained a wealth of fixed point results for linear and nonlinear contractions (e.g., see [2, 9, 17] and the references therein). Recently, Karapınar and Fulga [7] introduced a new

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notion of hybrid contraction which combined and unified some existing linear and nonlinear contractions in metric spaces. For some related results on generalized contractions, see [8, 13] and the references therein.

On the other hand, Mustafa [10] introduced an extension of metric space, called generalized metric space (or more specifically, G-metric space) and proved some fixed point results for Banach-type contraction mappings. The idea was brought to the limelight by Mustafa and Sims [11]. Subsequently, Mustafa et al. [12] obtained some fixed point results for Lipschitzian-type mappings on G-metric space, which in turn attracted the attention of many researchers in fixed point theory. However, Jleli and Samet [5] and Samet et al. [15] published observations that most of the fixed point results in G-metric spaces are direct consequences of existence results in metric spaces. In fact, Jleli and Samet [5] noted that if a G-metric can be reduced to a quasi-metric, then the related fixed point results become the known fixed point results in the context of a quasi-metric space. Motivated by the latter observation, many investigators (see, e.g. [4, 6]) have developed techniques for establishing fixed point results in G-metric spaces that cannot be followed from their corresponding ones in ordinary or quasi-metric spaces.

Following the existing literature, we realize that hybrid fixed point results in G-metric spaces are not sufficiently investigated. Hence, motivated by the ideas in [4, 6, 7], we introduce a new concept of admissible hybrid (G-α-ϕ)-contraction in G-metric space and prove some related fixed point theorems. An example is given to demonstrate the validity of our result and to show that the main ideas obtained herein do not reduce to any existing result in metric spaces. Some corollaries are presented to show that the concept proposed in this paper is an extension and generalization of well-known fixed point theorems in G-metric space. Finally, Ulam-type stability and well-posedness of this type of hybrid contraction in G-metric space are demonstrated. Additionally, one of our obtained corollaries is applied to establish novel existence conditions for the solution of a class of integral equations. As an open problem for future research and application of our results, we highlight the discretized population balance model, whose solution can be analyzed using our established techniques.

2. Preliminaries

In this section, we will present some fundamental notations and results that will be applied subsequently.

Throughout, every set X is considered non-empty, $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ represents the set of real numbers and $\mathbb{R}_+$ is the set of non-negative real numbers.

Definition 2.1 ([11])**.** Let X be a non-empty set and let $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;
(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G_3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(G_4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = ...$ (symmetry in all three variables);
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangular inequality).

The function G is called a generalized metric, or more generally, a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.2. [12]] Let (X, d) be a usual metric space, then (X, G_s) and (X, G_m) are G-metric spaces, where

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad \forall x, y, z \in X. \tag{2.1}$$

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \quad \forall x, y, z \in X. \tag{2.2}$$

Definition 2.3 ([12])**.** Let (X, G) be a G-metric space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. We say that $\{x_n\}_{n\in\mathbb{N}}$ is G-convergent to x if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, $\forall n, m \geq n_0$. We refer to x as the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

Proposition 2.4 ([12])**.** *Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) $\{x_n\}_{n\in\mathbb{N}}$ *is G-convergent to x;*
(ii) $G(x, x_n, x_m) \longrightarrow 0$*, as* $n, m \to \infty$*;*
(iii) $G(x_n, x, x) \longrightarrow 0$*, as* $n \to \infty$*;*
(iv) $G(x_n, x_n, x) \longrightarrow 0$*, as* $n \to \infty$*.*

Definition 2.5 ([12])**.** Let (X, G) be a G-metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called G-Cauchy if given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, $\forall n, m, l \geq n_0$. That is, $G(x_n, x_m, x_l) \longrightarrow 0$, as $n, m, l \to \infty$.

Proposition 2.6 ([12])**.** *In a G-metric space (X, G), the following are equivalent:*

(i) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is G-Cauchy.*
(ii) *For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, $\forall n, m \geq n_0$.*

Definition 2.7 ([12])**.** Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \longrightarrow (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if given $\epsilon > 0$, there exists $\delta > 0$

such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 2.8 ([12]). *Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : (X, G) \longrightarrow (X', G')$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x. That is, whenever $\{x_n\}_{n\in\mathbb{N}}$ is G-convergent to x, $\{fx_n\}$ is G-convergent to fx.*

Definition 2.9 ([12]). A G-metric space (X, G) is called a symmetric G-metric space if

$$G(x, x, y) = G(y, x, x), \quad \forall x, y \in X.$$

Proposition 2.10 ([12]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 2.11 ([12]). *Every G-metric space (X, G) defines a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{2.3}$$

Note that if (X, G) is a symmetric G-metric space, then

$$(X, d_G) = 2G(x, y, y), \quad \forall x, y \in X. \tag{2.4}$$

However, if (X, G) is not symmetric, then it holds by the G-metric properties that

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X, \tag{2.5}$$

and that in general, these inequalities are sharp.

Definition 2.12 ([12]). A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 2.13 ([12]). *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Mustafa [10] proved the following result in the framework of G-metric space.

Theorem 2.14 ([10]). *Let (X, G) be a complete G-metric space, and let $T : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{2.6}$$

for all $x, y, z \in X$ where $0 \leq k < 1$, then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G-continuous at u.

Consistent with [16], let Φ be the set of all functions ϕ such that $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-decreasing function with $\lim\limits_{n\to+\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-map.

Let $\phi \in \Phi$ be a Φ-map such that there exist $n_0 \in \mathbb{N}$, $k \in (0,1)$ and a convergent series of non-negative terms $\sum_{n=1}^{\infty} v_n$ satisfying

$$\phi^{n+1}(t) \leq k\phi^n(t) + v_n,$$

for $n \geq n_0$ and any $t > 0$. Then ϕ is called a (c)-comparison function [1].

Lemma 2.15 ([1])**.** *If $\phi \in \Phi$, then the following hold:*

(i) $\{\phi^n(t)\}_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for* $t \geq 0$*;*
(ii) $\phi(t) < t$ *for any* $t \in \mathbb{R}_+$*;*
(iii) ϕ *is continuous at* 0*;*
(iv) *the series* $\sum_{i=1}^{\infty} \phi^i(t)$ *is convergent for* $t \geq 0$*.*

Popescu [14] gave the following definitions in the setting of metric spaces.

Definition 2.16. ([14]). Let $\alpha : X \times X \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $T : X \longrightarrow X$ is called α-orbital admissible if for all $x \in X$, $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

Definition 2.17 ([14])**.** Let $\alpha : X \times X \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $T : X \longrightarrow X$ is called triangular α-orbital admissible if for all $x \in X$, T is α-orbital admissible and $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ implies $\alpha(x, Ty) \geq 1$.

We modify the above definitions in the framework of G-metric space as follows:

Definition 2.18. Let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $T : X \longrightarrow X$ is called $(G\text{-}\alpha)$-orbital admissible if for all $x \in X$, $\alpha(x, Tx, T^2x) \geq 1$ implies $\alpha(Tx, T^2x, T^3x) \geq 1$.

Definition 2.19. Let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function. A self-mapping $T : X \longrightarrow X$ is called triangular $(G\text{-}\alpha)$-orbital admissible if for all $x \in X$, T is $(G\text{-}\alpha)$-orbital admissible and $\alpha(x, y, Ty) \geq 1$ and $\alpha(y, Ty, T^2y) \geq 1$ implies $\alpha(x, Ty, T^2y) \geq 1$.

Lemma 2.20. *Let $T : X \longrightarrow X$ be a triangular $(G\text{-}\alpha)$-orbital admissible mapping. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$, then*

$$\alpha(x_n, x_m, x_l) \geq 1, \quad \forall n, m, l \in \mathbb{N}, \tag{2.7}$$

where the sequence $\{x_n\}_{n\in\mathbb{N}}$ is defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$.

Proof. Since T is (G-α)-orbital admissible mapping and $\alpha(x_0, Tx_0, T^2x_0) \geq 1$, then we deduce that $\alpha(x_1, x_2, x_3) = \alpha(Tx_0, Tx_1, Tx_2) \geq 1$. Continuing in this manner, we obtain $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$ for all $n \geq 1$. Assume that $\alpha(x_n, x_m, x_{m+1}) \geq 1$, where $m > n$. Since T is triangular (G-α)-orbital admissible mapping and $\alpha(x_m, x_{m+1}, x_{m+2}) \geq 1$, then clearly, $\alpha(x_n, x_{m+1}, x_{m+2}) \geq 1$ for all $m, n \in \mathbb{N}$. This validates our assumption that $\alpha(x_n, x_m, x_{m+1}) \geq 1$. Letting $l = m + 1$ completes the proof. □

Definition 2.21. ([3]). Let $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be a mapping. The set X is called regular with respect to α if and only if for a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, we have $\alpha(x_n, x, x) \geq 1$ for all n.

Karapınar and Fulga [7] gave the following definition of admissible hybrid contraction in metric space.

Definition 2.22. ([7]). Let (X, d) be a metric space. A self-mapping $T : X \longrightarrow X$ is called an admissible hybrid contraction, if there exist $\phi \in \Phi$ and $\alpha : X \times X \longrightarrow \mathbb{R}_+$ such that:

$$\alpha(x, y)d(Tx, Ty) \leq \phi(M(x, y)), \tag{2.8}$$

where

$$M(x,y) = \begin{cases} \Big[\lambda_1 d(x,y)^q + \lambda_2 d(x,Tx)^q + \lambda_3 d(y,Ty)^q \\ \quad + \lambda_4 \left(\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right)^q + \lambda_5 \left(\frac{d(y,Tx)(1+d(x,Ty))}{1+d(x,y)}\right)^q\Big]^{\frac{1}{q}}, \\ \qquad\qquad \text{for } q > 0,\ x, y \in X, \\ \\ [d(x,y)]^{\lambda_1} \cdot [d(x,Tx)]^{\lambda_2} \cdot [d(y,Ty)]^{\lambda_3} \cdot \left[\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}\right]^{\lambda_4} \\ \quad \cdot \left[\frac{d(x,Ty)+d(y,Tx)}{2}\right]^{\lambda_5}, \quad \text{for } q = 0,\ x, y \in X \backslash Fix(T), \end{cases} \tag{2.9}$$

$q \geq 0$, $\lambda_i \geq 0$; $i = 1, 2, \ldots, 5$ such that $\sum\limits_{i=1}^{5} \lambda_i = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

3. Main Results

We begin this section by defining the notion of admissible hybrid (G-α-ϕ)-contraction in G-metric space.

Definition 3.1. Let (X, G) be a G-metric space. A self-mapping $T : X \longrightarrow X$ is called an admissible hybrid (G-α-ϕ)-contraction, if there exist $\phi \in \Phi$ and $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ such that:

$$\alpha(x, y, Ty)G\left(Tx, Ty, T^2y\right) \leq \phi(M(x, y, Ty)), \tag{3.1}$$

where
(3.2)

$$M(x,y,Ty) = \begin{cases} \Big[\lambda_1 G(x,y,Ty)^q + \lambda_2 G(x,Tx,T^2x)^q + \lambda_3 G(y,Ty,T^2y)^q \\ \quad +\lambda_4 \left(\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q \\ \quad +\lambda_5 \left(\frac{G(x,y,Ty)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right)^q\Big]^{\frac{1}{q}}, \quad \text{for } q>0,\ x,y \in X, \\ \\ G(x,y,Ty)^{\lambda_1} \cdot G(x,Tx,T^2x)^{\lambda_2} \cdot G(y,Ty,T^2y)^{\lambda_3} \\ \quad \cdot \left[\frac{G(y,Ty,T^2y)(1+G(x,Tx,T^2x))}{1+G(x,y,Ty)}\right]^{\lambda_4} \\ \quad \cdot \left[\frac{G(x,y,Ty)+G(y,Ty,T^2y)}{2}\right]^{\lambda_5}, \quad \text{for } q=0,\ x,y \in X \backslash Fix(T), \end{cases}$$

$q \geq 0,\ \lambda_i \geq 0;\ i = 1, 2, \ldots, 5$ such that $\sum_{i=1}^{5} \lambda_i = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

Our main result is the following.

Theorem 3.2. *Let (X, G) be a complete G-metric space and let $T : X \longrightarrow X$ be an admissible hybrid (G-α-ϕ)-contraction. Assume further that:*

(i) *T is triangular (G-α)-orbital admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$;*
(iii) *either T is continuous or;*
(iv) *T^3 is continuous and $\alpha(x, Tx, T^2x) \geq 1$ for any $x \in Fix(T^3)$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Assume there exists some $m \in \mathbb{N}$ such that $Tx_m = x_{m+1} = x_m$. Then clearly, x_m is a fixed point of T. Assume now that $x_n \neq x_{n-1}$ for any $n \in \mathbb{N}$. Since T is an admissible hybrid (G-α-ϕ)-contraction, then we have from (3.1) that

$$\alpha(x_{n-1}, x_n, Tx_n)G(Tx_{n-1}, Tx_n, T^2x_n) \leq \phi(M(x_{n-1}, x_n, Tx_n)). \tag{3.3}$$

Owing to the fact that T is triangular (G-α)-orbital admissible together with (2.7) and (3.1), we have

$$G(x_n, x_{n+1}, Tx_{n+1}) = G(x_n, x_{n+1}, x_{n+2}) \tag{3.4}$$

$$\leq \alpha(x_{n-1}, x_n, x_{n+1})G(Tx_{n-1}, Tx_n, Tx_{n+1}) < \phi(M(x_{n-1}, x_n, x_{n+1})).$$

We now consider the following cases:
Case 1: For $q > 0$, we have

(3.5)

$$\begin{aligned}
M(x_{n-1},&x_n, Tx_n)\\
=&\Big[\lambda_1 G(x_{n-1}, x_n, Tx_n)^q + \lambda_2 G\left(x_{n-1}, Tx_{n-1}, T^2x_{n-1}\right)^q\\
&+ \lambda_3 G\left(x_n, Tx_n, T^2x_n\right)^q\\
&+ \lambda_4\left(\frac{G\left(x_n, Tx_n, T^2x_n\right)\left(1 + G\left(x_{n-1}, Tx_{n-1}, T^2x_{n-1}\right)\right)}{1 + G(x_{n-1}, x_n, Tx_n)}\right)^q\\
&+ \lambda_5\left(\frac{G(x_{n-1}, x_n, Tx_n)(1 + G\left(x_{n-1}, Tx_{n-1}, T^2x_{n-1}\right))}{1 + G(x_{n-1}, x_n, Tx_n)}\right)^q\Bigg]^{\frac{1}{q}}\\
=&\Big[\lambda_1 G(x_{n-1}, x_n, x_{n+1})^q + \lambda_2 G(x_{n-1}, x_n, x_{n+1})^q\\
&+ \lambda_3 G(x_n, x_{n+1}, x_{n+2})^q\\
&+ \lambda_4\left(\frac{G(x_n, x_{n+1}, x_{n+2})(1 + G(x_{n-1}, x_n, x_{n+1}))}{1 + G(x_{n-1}, x_n, x_{n+1})}\right)^q\\
&+ \lambda_5\left(\frac{G(x_{n-1}, x_n, x_{n+1})(1 + G(x_{n-1}, x_n, x_{n+1}))}{1 + G(x_{n-1}, x_n, x_{n+1})}\right)^q\Bigg]^{\frac{1}{q}}\\
=&\big[\lambda_1 G(x_{n-1}, x_n, x_{n+1})^q + \lambda_2 G(x_{n-1}, x_n, x_{n+1})^q\\
&+ \lambda_3 G(x_n, x_{n+1}, x_{n+2})^q\\
&+ \lambda_4 G(x_n, x_{n+1}, x_{n+2})^q + \lambda_5 G(x_{n-1}, x_n, x_{n+1})^q\big]^{\frac{1}{q}}\\
=&\big[(\lambda_1 + \lambda_2 + \lambda_5)G(x_{n-1}, x_n, x_{n+1})^q\\
&+ (\lambda_3 + \lambda_4)G(x_n, x_{n+1}, x_{n+2})^q\big]^{\frac{1}{q}}.
\end{aligned}$$

Since ϕ is non-decreasing, if we assume that

$$G(x_{n-1}, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}),$$

then (3.4) becomes

(3.6)

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &\leq \alpha(x_{n-1}, x_n, x_{n+1})G(Tx_{n-1}, Tx_n, Tx_{n+1})\\
&\leq \phi\big(\big[(\lambda_1 + \lambda_2 + \lambda_5)G(x_{n-1}, x_n, x_{n+1})^q
\end{aligned}$$

$$
\begin{aligned}
&+ (\lambda_3+\lambda_4)G(x_n,x_{n+1},x_{n+2})^q\big]^{\frac{1}{q}}\big)\\
&\le \phi\big(\,[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)G(x_n,x_{n+1},x_{n+2})^q]^{\frac{1}{q}}\,\big)\\
&= \phi((\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}}G(x_n,x_{n+1},x_{n+2}))\\
&< (\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}}G(x_n,x_{n+1},x_{n+2})\\
&\le G(x_n,x_{n+1},x_{n+2}),
\end{aligned}
$$

which is a contradiction. Therefore, for every $n\in\mathbb{N}$, we have

$$G(x_n,x_{n+1},x_{n+2}) < G(x_{n-1},x_n,x_{n+1}),$$

so that (3.4) becomes

(3.7)

$$
\begin{aligned}
G(x_n,x_{n+1},x_{n+2}) &\le \phi\big([(\lambda_1+\lambda_2+\lambda_5)G(x_{n-1},x_n,x_{n+1})^q\\
&\quad + (\lambda_3+\lambda_4)G(x_n,x_{n+1},x_{n+2})^q]^{\frac{1}{q}}\big)\\
&\le \phi\big(\,[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)G(x_{n-1},x_n,x_{n+1})^q]^{\frac{1}{q}}\,\big)\\
&= \phi((\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}}G(x_{n-1},x_n,x_{n+1}))\\
&\le \phi(G(x_{n-1},x_n,x_{n+1}))\\
&< \phi^2(G(x_{n-1},x_n,x_{n+1})).
\end{aligned}
$$

Continuing inductively, we have

$$G(x_n,x_{n+1},x_{n+2}) < \phi^n(G(x_0,x_1,x_2)). \tag{3.8}$$

Now, since

$$G(x_n,x_n,x_{n+1}) \le G(x_n,x_{n+1},x_{n+2}) \le \phi^n(G(x_0,x_1,x_2)),$$

for all n in $\mathbb{N}$ with $x_{n+1}\neq x_{n+2}$, then for any $n,m\in\mathbb{N}$ with $n<m$ and by rectangular inequality, we have

$$
\begin{aligned}
G(x_n,x_n,x_m) &\le G(x_n,x_n,x_{n+1}) + G(x_{n+1},x_{n+1},x_{n+2}) + \cdots\\
&\quad + G(x_{m-1},x_{m-1},x_m)\\
&\le (\phi^n+\phi^{n+1}+\phi^{n+2}+\cdots+\phi^{m-1})G(x_0,x_1,x_2)\\
&= \sum_{i=n}^{m-1}\phi^i(G(x_0,x_1,x_2))\\
&\le \sum_{i=n}^{\infty}\phi^i(G(x_0,x_1,x_2)).
\end{aligned} \tag{3.9}
$$

Since ϕ is a (c)-comparison function, then the series $\sum_{i=0}^{\infty} \phi^i(G(x_0, x_1, x_2))$ is convergent. Hence, denoting by $S_p = \sum_{i=0}^{\infty} \phi^i(G(x_0, x_1, x_2))$, we have

$$G(x_n, x_n, x_m) \leq S_{m-1} - S_{n-1}.$$

Therefore, as $n, m \to \infty$, we see that

$$G(x_n, x_n, x_m) \longrightarrow 0.$$

Thus $\{x_n\}_{n\in\mathbb{N}}$ is a G-Cauchy sequence in (X, G) and so by the completeness of (X, G), there exists $z \in X$ such that $\{x_n\}_{n\in\mathbb{N}}$ is G-convergent to z, that is,

$$\lim_{n\to\infty} G(x_n, x_n, z) = 0.$$

We will now show that z is a fixed point of T. By the assumption that T is continuous, we have

$$\begin{aligned}\lim_{n\to\infty} G(z, z, Tz) &= \lim_{n\to\infty} G(x_{n+1}, x_{n+1}, Tz) = \lim_{n\to\infty} G(Tx_n, Tx_n, Tz) \\ &= \lim_{n\to\infty} G(Tx_n, Tx_n, Tx_n) = 0,\end{aligned}$$

so we get $Tz = z$, that is, z is a fixed point of T.
Alternatively, on the assumption that (iv) holds, we have

$$T^3 z = \lim T^3 x_n = z.$$

In order to demonstrate that $Tz = z$, assume $Tz \neq z$ on the contrary. Then by (3.1) and Proposition 2.4, we obtain

$$\begin{aligned}G(z, Tz, T^2z) &\leq \alpha(z, Tz, T^2z) G(Tz, T^2z, T^3z) \\ &= \alpha(z, Tz, T^2z) G(Tz, T^2z, z) \\ &\leq \phi(M(z, Tz, T^2z)) < M(z, Tz, T^2z),\end{aligned}$$

where
(3.10)

$$\begin{aligned}M(z, Tz, &T^2z) \\ &= \Bigg[\lambda_1 G(z, Tz, T^2z)^q + \lambda_2 G(z, Tz, T^2z)^q + \lambda_3 G(Tz, T^2z, T^3z)^q \\ &\quad + \lambda_4 \left(\frac{G(Tz, T^2z, T^3z)(1 + G(z, Tz, T^2z))}{1 + G(z, Tz, T^2z)}\right)^q \\ &\quad + \lambda_5 \left(\frac{G(z, Tz, T^2z)(1 + G(z, Tz, T^2z))}{1 + G(z, Tz, T^2z)}\right)^q\Bigg]^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&= \big[\lambda_1 G(z,Tz,T^2z)^q + \lambda_2 G(z,Tz,T^2z)^q + \lambda_3 G(z,Tz,T^2z)^q \\
&\quad + \lambda_4 G(z,Tz,T^2z)^q + \lambda_5 G(z,Tz,T^2z)^q\big]^{\frac{1}{q}} \\
&= \big[(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5) G(z,Tz,T^2z)^q\big]^{\frac{1}{q}} \\
&= (\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)^{\frac{1}{q}} G(z,Tz,T^2z) \\
&\leq G(z,Tz,T^2z),
\end{aligned}
$$

which is a contradiction. Hence, $Tz = z$.
Case 2: For $q = 0$, we have

$$
\begin{aligned}
&M(x_{n-1}, x_n, Tx_n) \\
&\quad= G(x_{n-1},x_n,Tx_n)^{\lambda_1} \cdot G\left(x_{n-1},Tx_{n-1},T^2x_{n-1}\right)^{\lambda_2} \\
&\qquad \cdot G\left(x_n,Tx_n,T^2x_n\right)^{\lambda_3} \\
&\qquad \cdot \left[\frac{G\left(x_n,Tx_n,T^2x_n\right)\left(1+G\left(x_{n-1},Tx_{n-1},T^2x_{n-1}\right)\right)}{1+G(x_{n-1},x_n,Tx_n)}\right]^{\lambda_4} \\
&\qquad \cdot \left[\frac{G(x_{n-1},x_n,Tx_n)+G\left(x_n,Tx_n,T^2x_n\right)}{2}\right]^{\lambda_5} \\
&\quad= G(x_{n-1},x_n,x_{n+1})^{\lambda_1} \cdot G(x_{n-1},Tx_{n-1},x_{n+1})^{\lambda_2} \\
&\qquad \cdot G(x_n,x_{n+1},x_{n+2})^{\lambda_3} \\
&\qquad \cdot \left[\frac{G(x_n,x_{n+1},x_{n+2})(1+G(x_{n-1},x_n,x_{n+1}))}{1+G(x_{n-1},x_n,x_{n+1})}\right]^{\lambda_4} \\
&\qquad \cdot \left[\frac{G(x_{n-1},x_n,x_{n+1})+G(x_n,x_{n+1},x_{n+2})}{2}\right]^{\lambda_5} \\
&\quad= G(x_{n-1},x_n,x_{n+1})^{(\lambda_1+\lambda_2)} \cdot G(x_n,x_{n+1},x_{n+2})^{(\lambda_3+\lambda_4)} \\
&\qquad \cdot \left[\frac{G(x_{n-1},x_n,x_{n+1})+G(x_n,x_{n+1},x_{n+2})}{2}\right]^{\lambda_5}.
\end{aligned}
$$

Now, if $G(x_{n-1},x_n,x_{n+1}) \leq G(x_n,x_{n+1},x_{n+2})$, then

$$
\begin{aligned}
M(x_{n-1},x_n,Tx_n) &= G(x_n,x_{n+1},x_{n+2})^{(\lambda_1+\lambda_2)} \cdot G(x_n,x_{n+1},x_{n+2})^{(\lambda_3+\lambda_4)} \\
&\quad \cdot G(x_n,x_{n+1},x_{n+2})^{\lambda_5} \\
&= G(x_n,x_{n+1},x_{n+2})^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \\
&\leq G(x_n,x_{n+1},x_{n+2}),
\end{aligned}
$$

and so (3.4) becomes

$$G(x_n,x_{n+1},x_{n+2}) < G(x_n,x_{n+1},x_{n+2}),$$

which is a contradiction. Therefore,

$$G(x_n, x_{n+1}, x_{n+2}) < G(x_{n-1}, x_n, x_{n+1}).$$

Hence, by (3.4) we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+2}) &< \phi(G(x_{n-1}, x_n, x_{n+1})) \\ &< \phi^2(G(x_{n-1}, x_n, x_{n+1})) \\ &\;\;\vdots \\ &< \phi^n G(x_0, x_1, x_2). \end{aligned}$$

By a similar argument in the case of $q > 0$, we can show that there exists a G-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, G), hence a z in X such that $\lim_{n\to\infty} x_n = z$. To see that z is a fixed point of T, under the assumption that T is continuous, we have

$$\begin{aligned} \lim_{n\to\infty} G(z, z, Tz) &= \lim_{n\to\infty} G(x_{n+1}, x_{n+1}, Tz) \\ &= \lim_{n\to\infty} G(Tx_n, Tx_n, Tz) \\ &= \lim_{n\to\infty} G(Tx_n, Tx_n, Tx_n) \\ &= 0, \end{aligned}$$

and by the uniqueness of limit, $Tz = z$. Similarly, if T^3 is continuous, as in case 1, we have that $T^3z = z$. Suppose the contrary that $Tz \neq z$. Then

$$\begin{aligned} G(z, Tz, T^2z) &\leq \alpha\left(z, Tz, T^2z\right) G\left(Tz, T^2z, T^3z\right) \\ &= \alpha\left(z, Tz, T^2z\right) G\left(Tz, T^2z, z\right) \\ &\leq \phi\left(M(z, Tz, T^2z)\right) \\ &< M\left(z, Tz, T^2z\right) \\ &= G\left(z, Tz, T^2z\right), \end{aligned}$$

a contradiction. Hence, $Tz = z$. □

Theorem 3.3. *If in Theorem 3.2, in the case of $q > 0$, we suppose supplementary that (X, G) is regular with respect to α and $\alpha(x, y, Ty) \geq 1$ for any $x, y \in Fix(T)$, then the fixed point of T is unique.*

Proof. Let $v, z \in Fix(T)$ be such that $v \neq z$. By replacing this in (3.1) and noting the additional hypothesis, we have

$$\begin{aligned} G(z, v, Tv) &\leq \alpha(z, v, Tv) G(Tz, Tv, T^2v) \\ &\leq \phi(M(z, v, Tv)) \\ &< M(z, v, Tv) \end{aligned}$$

$$
\begin{aligned}
&= \Bigg[\lambda_1 G(z,v,Tv)^q + \lambda_2 G(z,Tz,T^2z)^q + \lambda_3 G(v,Tv,T^2v)^q \\
&\quad + \lambda_4 \left(\frac{G(v,Tv,T^2v)(1+G(z,Tz,T^2z))}{1+G(z,v,Tv)}\right)^q \\
&\quad + \lambda_5 \left(\frac{G(z,v,Tv)(1+G(z,Tz,T^2z))}{1+G(z,v,Tv)}\right)^q\Bigg]^{\frac{1}{q}} \\
&= \left[\lambda_1 G(z,v,Tv)^q + \lambda_5 \left(\frac{G(z,v,Tv)}{1+G(z,v,Tv)}\right)^q\right]^{\frac{1}{q}} \\
&\le \left[\lambda_1 G(z,v,Tv)^q + \lambda_5 G(z,v,Tv)^q\right]^{\frac{1}{q}} \\
&= \left[(\lambda_1+\lambda_5) G(z,v,Tv)^q\right]^{\frac{1}{q}} \\
&= (\lambda_1+\lambda_5)^{\frac{1}{q}} G(z,v,Tv) \\
&\le G(z,v,Tv),
\end{aligned}
$$

which is a contradiction. Hence, $v = z$, and so the fixed point of T is unique. □

Example 3.4. Let $X = [-1,1]$ and $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be defined by

$$G(x,y,Ty) = |x-y| + |x-Ty| + |y-Ty|, \quad \forall x,y \in X.$$

Then (X,G) is a complete G-metric space. Define $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $\phi(t) = \frac{t}{2}$ for all $t \ge 0$, $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{x}{5}, & \text{if} \quad x \in \{-1,1\}, \\ \frac{1}{5}, & \text{if} \quad x \in (-1,1), \end{cases}$$

for all $x \in X$ and $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ by

$$\alpha(x,y,Ty) = \begin{cases} 1, & \text{if} \quad x,y \in \{-1\} \cup [0,1], \\ 0, & \text{otherwise}. \end{cases} \tag{3.11}$$

Then obviously, $\phi \in \Phi$, T is triangular (G-α)-orbital admissible, T is continuous for all $x \in X$ and T^3 is continuous for any $x \in Fix(T^3)$. Also, there exists $x_0 = \frac{1}{2} \in X$ such that $\alpha\left(\frac{1}{2}, T(\frac{1}{2}), T^2(\frac{1}{2})\right) = \alpha(\frac{1}{2}, \frac{1}{5}, \frac{1}{5}) \ge 1$. Hence, conditions (i)-(iv) of Theorem 3.2 are satisfied.

To see that T is an admissible hybrid (G-α-ϕ)-contraction, notice that $\alpha(x,y,Ty) = 0$ for all $x,y \in (-1,0)$ and $G\left(Tx,Ty,T^2y\right) = 0$ for all $x,y \in (-1,1)$. Hence, inequality (3.1) holds for all $x,y \in (-1,1)$.

Now for $x,y \in \{-1,1\}$, if $x = y = 1$, then $G\left(Tx,Ty,T^2y\right) = 0$ for all $q \ge 0$. If $x = y = -1$, letting $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = \frac{1}{5}$, $\lambda_5 = \frac{3}{10}$ and

$q = 2$, we obtain

$$\begin{aligned}\alpha(x,y,Ty)G\left(Tx,Ty,T^2y\right) &= \alpha\left(-1,-1,\frac{-1}{5}\right)G\left(\frac{-1}{5},\frac{-1}{5},\frac{1}{5}\right)\\ &= \frac{4}{5} < 1 = \frac{1}{2}(2)\\ &= \frac{1}{2}\left(M\left(-1,-1,\frac{-1}{5}\right)\right)\\ &= \phi(M(x,y,Ty)).\end{aligned}$$

Also, for $q = 0$, we have

$$\begin{aligned}\alpha(x,y,Ty)G\left(Tx,Ty,T^2y\right) &= \frac{4}{5}\\ &< \frac{1}{2}(2)\\ &= \phi(M(x,y,Ty)).\end{aligned}$$

If $x \neq y$, then letting $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = \frac{2}{5}$, $\lambda_5 = \frac{1}{10}$ and $q = 2$, we obtain

$$\begin{aligned}\alpha(x,y,Ty)G\left(Tx,Ty,T^2y\right) &= \alpha\left(-1,1,\frac{1}{5}\right)G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right)\\ &= \alpha\left(1,-1,\frac{-1}{5}\right)G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right)\\ &= \frac{4}{5}\\ &< \frac{3}{2}\\ &= \frac{1}{2}(3)\\ &= \frac{1}{2}\left(M\left(-1,1,\frac{1}{5}\right)\right)\\ &= \frac{1}{2}\left(M\left(1,-1,\frac{-1}{5}\right)\right)\\ &= \phi(M(x,y,Ty)).\end{aligned}$$

Also, for $q = 0$, we take $\lambda_1 = \lambda_4 = \frac{1}{2}$ and $\lambda_2 = \lambda_3 = \lambda_5 = 0$. Then

$$\begin{aligned}\alpha(x,y,Ty)G\left(Tx,Ty,T^2y\right) &= \alpha\left(-1,1,\frac{1}{5}\right)G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right)\\ &= \alpha\left(1,-1,\frac{-1}{5}\right)G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right)\\ &= \frac{4}{5}\end{aligned}$$

$$
\begin{aligned}
&< 1 \\
&= \frac{1}{2}(2) \\
&= \frac{1}{2}\left(M\left(-1,1,\frac{1}{5}\right)\right) \\
&= \frac{1}{2}\left(M\left(1,-1,\frac{-1}{5}\right)\right) \\
&= \phi(M(x,y,Ty)).
\end{aligned}
$$

Hence, inequality (3.1) is satisfied for all $x, y \in X$. Therefore, T is an admissible hybrid (G-α-ϕ)-contraction which satisfies all the assumptions of Theorem 3.2 and $x = \frac{1}{5}$ is the fixed point of T.

We now demonstrate that our result is independent of the result of Karapınar and Fulga [7]. Let $\alpha : X \times X \longrightarrow \mathbb{R}_+$ be as given by Definition 2.22, $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ and $d : X \times X \longrightarrow \mathbb{R}_+$ be defined by

$$d(x,y) = |x-y|, \quad \forall x, y \in X.$$

Consider $x, y \in \{-1, 1\}$ and take for case 1, $x \neq y$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_5 = \frac{1}{5}$, $\lambda_4 = \frac{3}{5}$ and $q = 2$. Then inequality (3.1) becomes

$$
\begin{aligned}
\alpha(x,y,Ty)G\left(Tx,Ty,T^2y\right) &= \alpha\left(-1,1,\frac{1}{5}\right)G\left(\frac{-1}{5},\frac{1}{5},\frac{1}{5}\right) \\
&= \alpha\left(1,-1,\frac{-1}{5}\right)G\left(\frac{1}{5},\frac{-1}{5},\frac{1}{5}\right) \\
&= \frac{4}{5} \\
&< \frac{41}{50} \\
&= \frac{1}{2}\left(\frac{82}{50}\right) \\
&= \frac{1}{2}\left(M\left(-1,1,\frac{1}{5}\right)\right) \\
&= \frac{1}{2}\left(M\left(1,-1,\frac{-1}{5}\right)\right) \\
&= \phi(M(x,y,Ty)),
\end{aligned}
$$

while inequality (2.8) due to Karapınar and Fulga [7] yields

$$
\begin{aligned}
\alpha(x,y)d(Tx,Ty) &= \alpha(-1,1)d\left(\frac{-1}{5},\frac{1}{5}\right) \\
&= \alpha(1,-1)d\left(\frac{1}{5},\frac{-1}{5}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{5} > \frac{37}{100} \\
&= \frac{1}{2}\left(\frac{74}{100}\right) \\
&= \frac{1}{2}(M(-1,1)) \\
&= \frac{1}{2}(M(1,-1)) \\
&= \phi(M(x,y)).
\end{aligned}
$$

Take $x = y$ for case 2 and choose $\lambda_1 = 0$. Then $M(x, y)$ is undefined for any choice of λ_i with $i = 2, 3, 4, 5$ and $\sum\limits_{2}^{5} \lambda_i = 1$, since

$$
\begin{aligned}
[d(x,y)]^{\lambda_1} &= |x-y|^{\lambda_1} \\
&= |-1+1|^0 \\
&= |1-1|^0 \\
&= 0^0.
\end{aligned}
$$

Therefore, admissible hybrid (G-α-ϕ)-contraction is not admissible hybrid contraction defined by Karapınar and Fulga [7], and so Theorem 2.2 due to Karapınar and Fulga [7] is not applicable to this example.

Corollary 3.5 (see Theorem 2.14)**.** *Let (X, G) be a complete G-metric space, and let $T : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{3.12}$$

for all $x, y, z \in X$ where $0 \leq k < 1$, then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Consider Definition 3.1 and let $\alpha(x, y, Ty) = 1$ for all $x, y \in X$, $Ty = z$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\phi(t) = kt$ for all $t \geq 0$ and $k \in [0, 1)$. Clearly, $\phi \in \Phi$ and T is an admissible hybrid (G-α-ϕ)-contraction. Hence, (3.1) coincides with (2.6) of Theorem 2.14 due to Mustafa [10]. Therefore, it is easy to see that we can find a unique point u in X such that $Tu = u$ and T is G-continuous at u. □

Corollary 3.6 (see [16, Theorem 3.1])**.** *Let (X, G) be a complete G-metric space. Suppose the map $T : X \longrightarrow X$ satisfies*

$$G(Tx, Ty, Tz) \leq \phi(G(x, y, z)),$$

for all $x, y, z \in X$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. Consider Definition 3.1 and let $\alpha(x, y, Ty) = 1$ for all $x, y \in X$, $Ty = z$, $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Then,

$$M(x, y, z) = G(x, y, z),$$

for all $x, y, z \in X$ and $q \geq 0$. Hence, inequality (3.1) becomes

$$G(Tx, Ty, Tz) \leq \phi(G(x, y, z)),$$

for all $x, y, z \in X$ and $\phi \in \Phi$. This coincides with Theorem 3.1 due to Shatanawi [16] and so the proof follows in a similar manner. □

Definition 3.7. ([1]). Let $T : X \longrightarrow X$ and $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be two mappings. Then T is said to be α-admissible if for all $x, y, z \in X$, $\alpha(x, y, z) \geq 1$ implies $\alpha(Tx, Ty, Tz) \geq 1$.

Definition 3.8. ([1]). Let (X, G) be a G-metric space and let $T : X \longrightarrow X$ be a given mapping. Then T is said to be a G-α-ϕ-contractive mapping of type I if there exist two functions $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ and $\phi \in \Phi$ such that for all $x, y, z \in X$,

$$\alpha(x, y, z)G(Tx, Ty, Tz) \leq \phi(G(x, y, z)).$$

Corollary 3.9 (see [1, Theorem 29]). *Let (X, G) be a complete G-metric space. Suppose that $T : X \longrightarrow X$ is a G-α-ϕ-contractive mapping of type I and satisfies the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq 1$;*
(iii) *T is G-continuous.*

Then there exists $u \in X$ such that $Tu = u$.

Proof. Consider Definition 3.1 and let $Ty = z$, $\alpha : X \times X \times X \longrightarrow \mathbb{R}_+$ be α-admissible mapping and $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Then T is a G-α-ϕ-contractive mapping of type I and so inequality (3.1) becomes

$$\alpha(x, y, z)G(Tx, Ty, Tz) \leq \phi(G(x, y, z)),$$

for all $x, y, z \in X$ and $\phi \in \Phi$. Hence, the proof follows Theorem 29 of Alghamdi and Karapınar [1]. □

By specializing the parameters λ_i $(i = 1, 2, ..., 5)$ and q, as well as letting $\alpha(x, y, Ty) = 1$ for all $x, y \in X$ and $\phi(t) = kt$ for all $t \geq 0$, $k \in (0, 1)$, the following result is also a direct consequence of Theorem 3.2.

Corollary 3.10. *Let (X, G) be a complete G-metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, the mapping $T : X \longrightarrow X$ satisfies:*

$$G\left(Tx, Ty, T^2y\right) \leq kG(x, y, Ty), \tag{3.13}$$

then T has a fixed point in X.

4. Ulam-Type Stability

Ulam stability was introduced by Ulam and is seen as type of data dependence. This notion was further developed by Hyers and other researchers (see [7]). Karapınar and Fulga [7] investigated general Ulam-type stability in the sense of a fixed point problem in metric spaces. Here, we consider general Ulam-type stability as a fixed point problem in the framework of G-metric space.

Suppose that $T : X \longrightarrow X$ is a self-mapping in a G-metric space (X, G). Then we say that the fixed point problem

$$Tx = x, \tag{4.1}$$

has the general Ulam-type stability if and only if there exists an increasing function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, continuous at 0, $\mu(0) = 0$ such that for every $\epsilon > 0$ and for each $y' \in X$ which satisfies the inequality

$$G(y', Ty', T^2y') \leq \epsilon, \tag{4.2}$$

there exists a solution $z \in X$ of (4.1) such that

$$G(z, y', Ty') \leq \mu(\epsilon). \tag{4.3}$$

For a positive number C, we take $\mu(t) = Ct$ for all $t \geq 0$. Then the fixed point of (4.1) is said to be Ulam type stable.
On a G-metric space (X, G), the fixed point problem (4.1) is said to be well-posed if the following assumptions are satisfied:

(i) T has a unique fixed point $z \in X$;
(ii) $G(x_n, z, z) = 0$ for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} G\left(x_n, Tx_n, T^2x_n\right) = 0$.

Theorem 4.1. *Let (X, G) be a complete G-metric space. If in addition to the assumptions of Theorem 3.3, we have $\lambda_1 + \lambda_5 < \frac{1}{K}$ where $K = \max\{1, 2^{q-1}\}$, then the following hold:*

(i) *the fixed point equation (4.1) is Ulam-Hyers stable if $\alpha(u, v, Tv) \geq 1$ for any u, v satisfying (4.2);*
(ii) *the fixed point equation (4.1) is well-posed if $\alpha(z, x_n, Tx_n) \geq 1$ for any $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} G\left(x_n, Tx_n, T^2x_n\right) = 0$ and $Fix(T) = \{z\}$.*

Proof. (i) In Theorem 3.3, we have shown that there exists a unique $z \in X$ such that $Tz = z$. Let $y' \in X$ such that for any $\epsilon > 0$, we have

$$G(y', Ty', T^2y') \leq \epsilon.$$

Then obviously, z satisfies (4.2) and so we have $\alpha(z, y', Ty') \geq 1$. Hence, by rectangular inequality,

$G(z, y', Ty')$

$$
\begin{aligned}
&\leq G(z,Ty',T^2y') + G(T^2y',y',Ty') \\
&= G(Tz,Ty',T^2y') + G(y',Ty',T^2y') \\
&\leq \alpha(z,y',Ty')G(Tz,Ty',T^2y') + G(y',Ty',T^2y') \\
&\leq \phi(M(z,y',Ty')) + G(y',Ty',T^2y') \\
&< M(z,y',Ty') + G(y',Ty',T^2y') \\
&= \Bigg[\lambda_1 G(z,y',Ty')^q + \lambda_2 G(z,Tz,T^2z)^q + \lambda_3 G(y',Ty',T^2y')^q \\
&\quad + \lambda_4 \left(\frac{G(y',Ty',T^2y')(1+G(z,Tz,T^2z))}{1+G(z,y',Ty')}\right)^q \\
&\quad + \lambda_5 \left(\frac{G(z,y',Ty')(1+G(z,Tz,T^2z))}{1+G(z,y',Ty')}\right)^q\Bigg]^{\frac{1}{q}} + G(y',Ty',T^2y') \\
&= \Bigg[\lambda_1 G(z,y',Ty')^q + \lambda_2 G(z,z,z)^q + \lambda_3 G(y',Ty',T^2y')^q \\
&\quad + \lambda_4 \left(\frac{G(y',Ty',T^2y')(1+G(z,z,z))}{1+G(z,y',Ty')}\right)^q \\
&\quad + \lambda_5 \left(\frac{G(z,y',Ty')(1+G(z,z,z))}{1+G(z,y',Ty')}\right)^q\Bigg]^{\frac{1}{q}} + G(y',Ty',T^2y') \\
&= \Bigg[\lambda_1 G(z,y',Ty')^q + \lambda_3 G(y',Ty',T^2y')^q + \lambda_4 \left(\frac{G(y',Ty',T^2y')}{1+G(z,y',Ty')}\right)^q \\
&\quad + \lambda_5 \left(\frac{G(z,y',Ty')}{1+G(z,y',Ty')}\right)^q\Bigg]^{\frac{1}{q}} + G(y',Ty',T^2y') \\
&\leq \big[\lambda_1 G(z,y',Ty')^q + \lambda_3 G(y',Ty',T^2y')^q + \lambda_4 G(y',Ty',T^2y')^q \\
&\quad + \lambda_5 G(z,y',Ty')^q\big]^{\frac{1}{q}} + G(y',Ty',T^2y') \\
&\leq \big[\lambda_1 G(z,y',Ty')^q + \lambda_3\epsilon^q + \lambda_4\epsilon^q + \lambda_5 G(z,y',Ty')^q\big]^{\frac{1}{q}} + \epsilon \\
&= \big[(\lambda_1+\lambda_5)G(z,y',Ty')^q + (\lambda_3+\lambda_4)\epsilon^q\big]^{\frac{1}{q}} + \epsilon.
\end{aligned}
$$

Therefore, we have

$$G(z,y',Ty')^q \leq K\big[(\lambda_1+\lambda_5)G(z,y',Ty')^q + (\lambda_3+\lambda_4)\epsilon^q + \epsilon^q\big],$$

where $K = \max\{1, 2^{q-1}\}$. Hence, the above inequality reduces to

$$G(z,y',Ty')^q \leq \frac{K(\lambda_3+\lambda_4+1)}{1-K(\lambda_1+\lambda_5)}\epsilon^q,$$

which is equivalent to

$$G(z,y',Ty') \leq C\epsilon,$$

where $C = \frac{K(\lambda_3+\lambda_4+1)}{1-K(\lambda_1+\lambda_5)}$ for any $q > 0$, $\lambda_1, \lambda_5 \in [0,1)$ such that $\lambda_1 + \lambda_5 < \frac{1}{K}$.

(ii) Taking into account the supplementary condition and since $Fix(T) = \{z\}$, then we have

$$
\begin{aligned}
&G(z, x_n, Tx_n)\\
&\quad \le G\left(z, Tx_n, T^2x_n\right) + G\left(T^2x_n, x_n, Tx_n\right)\\
&\quad = G(Tz, Tx_n, T^2x_n) + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad \le \alpha(z, x_n, Tx_n)G(Tz, Tx_n, T^2x_n) + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad \le \phi(M(z, x_n, Tx_n)) + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad < M(z, x_n, Tx_n) + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad = \Bigg[\lambda_1 G(z, x_n, Tx_n)^q + \lambda_2 G(z, Tz, T^2z)^q + \lambda_3 G\left(x_n, Tx_n, T^2x_n\right)^q\\
&\qquad + \lambda_4\left(\frac{G\left(x_n, Tx_n, T^2x_n\right)\left(1 + G(z, Tz, T^2z)\right)}{1 + G(z, x_n, Tx_n)}\right)^q\\
&\qquad + \lambda_5\left(\frac{G(z, x_n, Tx_n)(1 + G(z, Tz, T^2z))}{1 + G(z, x_n, Tx_n)}\right)^q\Bigg]^{\frac{1}{q}}\\
&\qquad + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad = \Bigg[\lambda_1 G(z, x_n, Tx_n)^q + \lambda_2 G(z, z, z)^q + \lambda_3 G\left(x_n, Tx_n, T^2x_n\right)^q\\
&\qquad + \lambda_4\left(\frac{G\left(x_n, Tx_n, T^2x_n\right)\left(1 + G(z, z, z)\right)}{1 + G(z, x_n, Tx_n)}\right)^q\\
&\qquad + \lambda_5\left(\frac{G(z, x_n, Tx_n)(1 + G(z, z, z))}{1 + G(z, x_n, Tx_n)}\right)^q\Bigg]^{\frac{1}{q}} + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad = \Bigg[\lambda_1 G(z, x_n, Tx_n)^q + \lambda_3 G\left(x_n, Tx_n, T^2x_n\right)^q\\
&\qquad + \lambda_4\left(\frac{G\left(x_n, Tx_n, T^2x_n\right)}{1 + G(z, x_n, Tx_n)}\right)^q + \lambda_5\left(\frac{G(z, x_n, Tx_n)}{1 + G(z, x_n, Tx_n)}\right)^q\Bigg]^{\frac{1}{q}}\\
&\qquad + G\left(x_n, Tx_n, T^2x_n\right)\\
&\quad \le \left[\lambda_1 G(z, x_n, Tx_n)^q + \lambda_3 G\left(x_n, Tx_n, T^2x_n\right)^q\right.\\
&\qquad \left. + \lambda_4 G\left(x_n, Tx_n, T^2x_n\right)^q + \lambda_5 G(z, x_n, Tx_n)^q\right]^{\frac{1}{q}}\\
&\qquad + G\left(x_n, Tx_n, T^2x_n\right).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned} G(z,x_n,Tx_n)^q \leq K\big[&\lambda_1 G(z,x_n,Tx_n)^q + \lambda_3 G\left(x_n,Tx_n,T^2x_n\right)^q \\ &+ \lambda_4 G\left(x_n,Tx_n,T^2x_n\right)^q + \lambda_5 G(z,x_n,Tx_n)^q \\ &+ G\left(x_n,Tx_n,T^2x_n\right)^q\big], \end{aligned}$$

where $K=\max\{1,2^{q-1}\}$. Hence, the above inequality reduces to

$$G(z,x_n,Tx_n) \leq \frac{K(\lambda_3+\lambda_4+1)}{1-K(\lambda_1+\lambda_5)} G\left(x_n,Tx_n,T^2x_n\right).$$

Letting $n\to\infty$ and keeping in mind Proposition 2.4 and

$$\lim_{n\to\infty} G\left(x_n,Tx_n,T^2x_n\right)=0,$$

we obtain

$$\begin{aligned} \lim_{n\to\infty} G(x_n,z,z) &= \lim_{n\to\infty} G(z,x_n,Tx_n) \\ &\leq \lim_{n\to\infty} G\left(x_n,Tx_n,T^2x_n\right) \\ &= 0. \end{aligned}$$

That is, the fixed point equation (4.1) is well-posed. □

5. Applications to Solution of Integral Equation

In this section, Corollary 3.10 is applied to examine the existence criteria for a solution to the following integral equation:

$$u(t)=h(t)+\int_a^b \mathcal{L}(t,s)f(s,u(s))ds, \quad t\in[a,b] \tag{5.1}$$

where $h:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$, $\mathcal{L}:[a,b]\times[a,b]\longrightarrow\mathbb{R}_+$ and $f:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ are given continuous functions, and the function u is unknown.

Let $X=C([a,b],\mathbb{R})$ be the set of all real-valued continuous functions defined on $[a,b]$. We equip X with the mapping:

$$G(u,v,w)=\max_{a\leq t\leq b}\left(|u(t)-v(t)|+|u(t)-w(t)|+|v(t)-w(t)|\right). \tag{5.2}$$

Then, it is clear that (X,G) is a complete G-metric space. Consider the self-mapping $T:X\longrightarrow X$ defined by

$$Tu(t)=h(t)+\int_a^b \mathcal{L}(t,s)f(s,u(s))ds, \quad t\in[a,b]. \tag{5.3}$$

One can see that u^* is a fixed point of T if and only if u^* is a solution to (5.1).

Now, we study the existing conditions of the integral equation (5.1) under the following hypotheses.

Theorem 5.1. *Assume that the following conditions are satisfied:*

(C_1) $|f(s,x)-f(s,y)|+|f(s,x)-f(s,z)|+|f(s,y)-f(s,z)| \leq |x-y|+|x-z|+|y-z|$, *for all* $t \in [a,b]$, $x,y,z \in \mathbb{R}$;

(C_2) $\max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)|ds = \eta < 1.$

Then, the integral equation (5.1) has a solution in X.

Proof. Taking (5.2) into account, we obtain

$$\begin{aligned}
&G(Tu,Tv,T^2v)\\
&\quad= \max_{t\in[a,b]} \left(|Tu(t)-Tv(t)| + \left|Tu(t)-T^2v(t)\right| + \left|Tv(t)-T^2v(t)\right|\right)\\
&\quad= \max_{t\in[a,b]} \left(\left|\int_a^b \mathcal{L}(t,s)(f(s,u(s))-f(s,v(s)))ds\right|\right.\\
&\qquad+ \left|\int_a^b \mathcal{L}(t,s)(f(s,u(s))-f(s,Tv(s)))ds\right|\\
&\qquad+ \left.\left|\int_a^b \mathcal{L}(t,s)(f(s,v(s))-f(s,Tv(s)))ds\right|\right)\\
&\quad\leq \max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)| \big[\, |f(s,u(s))-f(s,v(s))|\\
&\qquad+ |f(s,u(s))-f(s,Tv(s))| + |f(s,v(s))-f(s,Tv(s))| \,\big] ds\\
&\quad\leq \max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)| \big[|u(s)-v(s)|\\
&\qquad+ |u(s)-Tv(s)| + |v(s)-Tv(s)|\big] ds\\
&\quad\leq \left(\max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)|\, ds\right)\\
&\qquad \max_{t\in[a,b]} \int_a^b \big[\, |u(s)-v(s)| + |u(s)-Tv(s)| + |v(s)-Tv(s)| \,\big] ds\\
&\quad= \eta G(u,v,Tv).
\end{aligned}$$

Hence, all the conditions of Corollary 3.10 are satisfied. It follows that T has a fixed point u^* in X, which corresponds to a solution to the integral equation (5.1). □

Example 5.2. Let $X = C([0,1],\mathbb{R})$ and consider

$$u(t) = \frac{t^2}{(5+t)} + \frac{1}{7}\int_0^1 \frac{s^2}{(5+t)} \cdot \frac{1}{(3+u(s))} ds, \quad t \in [0,1]. \tag{5.4}$$

To obtain the solution of (5.1), we prove that $u(t)$ is a fixed point of $Tu(t)$, that is, $Tu(t) = u(t)$, where

$$Tu(t) = \frac{t^2}{(5+t)} + \frac{1}{7}\int_0^1 \frac{s^2}{(5+t)} \cdot \frac{1}{(3+u(s))} ds, \quad t \in [0,1].$$

Notice that the integral equation (5.4) is a special case of (5.1), where

$$h(t) = \frac{t^2}{(5+t)}, \qquad \mathcal{L}(t,s) = \frac{s^2}{(5+t)}, \qquad f(s,t) = \frac{1}{7(3+u(s))}.$$

Obviously, the functions $h(t)$, $\mathcal{L}(t,s)$ and $f(s,t)$ are continuous. Moreover, for all $u, v \in \mathbb{R}$,

$$\begin{aligned}
&|f(s,u) - f(s,v)| + |f(s,u) - f(s,Tv)| + |f(s,v) - f(s,Tv)| \\
&\quad = \left|\frac{1}{7(3+u(s))} - \frac{1}{7(3+v(s))}\right| + \left|\frac{1}{7(3+u(s))} - \frac{1}{7(3+Tv(s))}\right| \\
&\qquad + \left|\frac{1}{7(3+v(s))} - \frac{1}{7(3+Tv(s))}\right| \\
&\quad = \left|\frac{v-u}{7(3+u(s))(3+v(s))}\right| + \left|\frac{Tv-u}{7(3+u(s))(3+Tv(s))}\right| \\
&\qquad + \left|\frac{Tv-v}{7(3+v(s))(3+Tv(s))}\right| \\
&\quad \le \frac{1}{7}\left(|u-v| + |u-Tv| + |v-Tv|\right) \\
&\quad \le |u-v| + |u-Tv| + |v-Tv|.
\end{aligned}$$

Also, notice that

$$\begin{aligned}
\max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)| ds &= \max_{t\in[0,1]} \int_0^1 \left|\frac{s^2}{(5+t)}\right| ds \\
&= \max_{t\in[0,1]} \frac{1}{3(5+t)} \\
&\le \frac{1}{15} \\
&< 1.
\end{aligned}$$

Hence, all the conditions of Theorem 5.1 are satisfied. Therefore, the integral equation (5.1) has a solution in X.

6. Open Problem

For further research, an open problem is highlighted as follows:
A discretized population balance for continuous systems at a steady

state can be modeled by the nonlinear integral equation:

$$g(t) = \frac{\mu}{2(1+2\mu)} \int_0^t g(t-x)g(x)dx + e^{-t}, \quad \mu \in \mathbb{R}. \tag{6.1}$$

So far, it is still unclear whether or not the existing conditions for the solution of (6.1) can be obtained using any of the results established in this paper.

Remark 6.1. (i) We can deduce many other corollaries by replacing Tx with y or T^2y with z, and by particularizing some of the parameters in Definition 3.1.
(ii) None of the results presented in this work can be expressed in the form $G(x, y, y)$ or $G(x, x, y)$. Hence, they cannot be obtained from their equivalent versions in metric spaces.

7. Conclusion

An extension of metric spaces was introduced by Mustafa and Sims [11], called G-metric space (or generalized metric space) and several fixed point results were studied in that space. However, Jleli and Samet [5] as well as Samet et al. [15] noted that most fixed point theorems obtained in G-metric space are direct consequences of their analogs in metric space. In this note, we introduce a new class of contractions, the admissible hybrid (G-α-ϕ)-contraction and prove some fixed-point theorems that cannot be deduced from those in metric spaces. The key advantage of this family of contractions is the fact that its contractive inequality can be specialized in several ways, depending on multiple parameters. Consequently, a handful of corollaries, including some recently announced results in the literature are highlighted and analyzed. Nontrivial comparative examples are constructed to validate the assumptions of our obtained theorems. Furthermore, we examined Ulam-type stability and well-posedness for the new contraction proposed herein. In addition, one of our obtained corollaries is applied to setup novel existence conditions for the solution of a class of integral equations. In regard to some future aspects of our results, an open issue relating to a discretized population balance model may be analysed using the methods established in this work.

Competing Interests

The authors declare that they have no competing interests.

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