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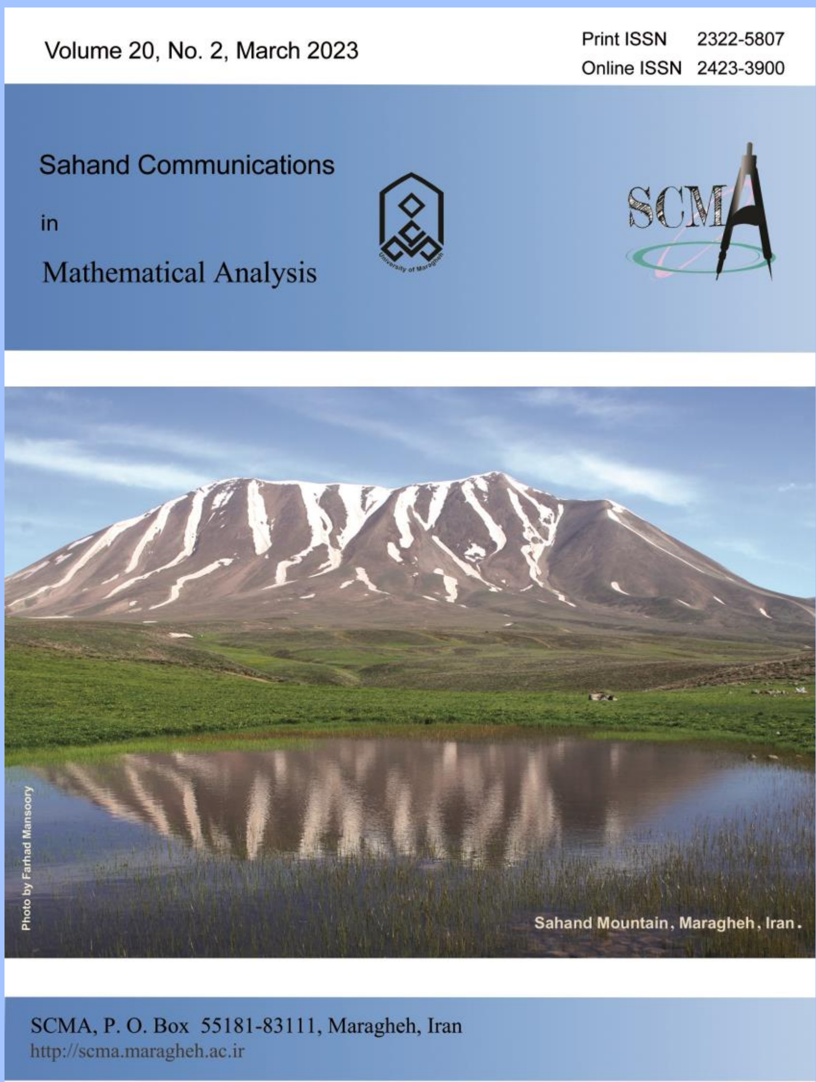

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Second Hankel Determinant for Certain Subclasses of Bi-starlike Functions Defined by Differential Operators

Halit Orhan[1], Hava Arıkan[2] and Murat Çağlar[3*]

Abstract. In this paper, we obtain upper bounds of the initial Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and $|a_4|$ and of the Fekete-Szegö functional $\left|a_3-\eta a_2^2\right|$ for certain subclasses of analytic and bi-starlike functions $\mathcal{S}_\sigma^*(\beta,\theta,n,m)$ in the open unit disk. We have also obtained an upper bound of the functional $\left|a_2a_4-a_3^2\right|$ for the functions in the class $\mathcal{S}_\sigma^*(\beta,\theta,n,m)$. Moreover, several interesting applications of the results presented here are also discussed.

1. Introduction and Definitions

Let $\mathcal{A}$ denotes the family of functions f analytic in the open unit disk

$$\mathfrak{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\},$$

of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k. \tag{1.1}$$

Let $\mathcal{S}$ denotes the class of all functions in $\mathcal{A}$ that are univalent in $\mathfrak{U}$. The Koebe one-quarter theorem (see, for example, [10]) ensures that the image of $\mathfrak{U}$ under every $f\in\mathcal{S}$ contains a disk of radius 1/4. Clearly, every $f\in\mathcal{S}$ has an inverse function f^{-1} satisfying $f^{-1}(f(z))=z$ $(z\in\mathfrak{U})$ and $f\left(f^{-1}(w)\right)=w$ $(|w|<r_0(f);\ r_0(f)\geqq 1/4)$, where

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathfrak{U}$. Let σ denotes the class of bi-univalent functions in $\mathfrak{U}$ given by (1.1).

In 1967, Lewin [21] showed that, for every function $f \in \sigma$ of the form (1.1), the second coefficient of f satisfies the estimate $|a_2| < 1.51$. In 1967, Brannan and Clunie [3] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \sigma$. Later, Netanyahu [22] proved that $\max\limits_{f\in\sigma}|a_2| = \frac{4}{3}$. In 1985, Kedzierawski [17] proved the Brannan–Clunie conjecture for bi-starlike functions. In 1985, Tan [35] obtained a bound for a_2, namely that $|a_2| < 1.485$, which is the best known estimate for functions in the class σ. Brannan and Taha [4] obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in the classes of bi-starlike functions of order β and bi-convex functions of order β $(0 \leqq \beta < 1)$.

The study of bi-univalent functions was revived in recent years by Srivastava et al. [33] and a considerably large number of sequels to the work of Srivastava et al. [33] have appeared in the literature since then. In particular, several results on coefficient estimates for the initial coefficients $|a_2|$, $|a_3|$, and $|a_4|$ were proved for various subclasses of σ (see, for example, [1, 2, 8, 11, 13, 16, 25, 31, 32, 34, 36, 37]).

Recently, Deniz [9] and Kumar et al. [19] both extended and improved the results of Brannan and Taha [4] by generalizing their classes by means of the principle of subordination between analytic functions. The problem of estimating the coefficients $|a_k|$ $(k \geqq 2)$ is still open (see also [32] in this connection).

Among important tools in the theory of univalent functions are Hankel determinants, which are used, for example, in showing that a function of bounded characteristic in $\mathfrak{U}$, that is, a function that is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [5]. The Hankel determinants $H_q(k)$ $(k = 1, 2, 3, \ldots,\ q = 1, 2, 3, \ldots)$ of the function f are defined by (see [23])

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q} \\ \vdots & \vdots & & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

This determinant was discussed by several authors with $q = 2$ (see [6, 7, 15, 20, 26, 29, 30, 38]). For example, we know that the functional $H_2(1) = a_3 - a_2^2$ is known as the Fekete-Szegö functional and one usually considers the further generalized functional $a_3 - \mu a_2^2$ where μ is some real number (see [12]). Estimating for the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem. In 1969, Keogh and

Merkes [18] solved the Fekete–Szegö problem for the classes of starlike and convex functions. One can see the Fekete-Szegö problem for the classes of starlike functions of order β and convex functions of order β in special cases in the paper of Orhan et al. [24]. On the other hand, quite recently, Zaprawa (see [38, 39]) studied the Fekete-Szegö problem for some classes of bi-univalent functions. In special cases, he gave the Fekete-Szegö problem for the classes of bi-starlike functions of order β and bi-convex functions of order β.

The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$. The bounds for the second Hankel determinant $H_2(2)$ were obtained for the classes of starlike and convex functions in [15]. Lee et al. [20] established a sharp bound for $|H_2(2)|$ by generalizing their classes by means of the principle of subordination between analytic functions. In their paper [20], one can find the sharp bound for $|H_2(2)|$ for the functions in the classes of starlike functions of order β and convex functions of order β. Recently, Deniz et al. [6], Soh and Mohamad [30] and Orhan et al. [26] found some upper bounds for the functional $H_2(2) = a_2a_4 - a_3^2$ for the subclasses of bi-univalent functions.

Let $f \in \mathcal{A}$. In [28], Salagean introduced the following differential operator:

$$\begin{aligned}
&\mathcal{D}^0 f(z) = f(z),\\
&\mathcal{D}^1 f(z) = \mathcal{D}f(z) = zf'(z),\\
&\quad\vdots\\
&\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1}f(z)), \quad (n \in \mathbb{N} = 1, 2, 3, ...).
\end{aligned}$$

Note that

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Definition 1.1 ([29])**.** A function $f(z)$ given by (1.1) is said to be in the class $f \in S_\sigma^*(\beta, \theta, n, m)$, if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad \operatorname{Re}\left\{e^{i\theta}\left[\frac{\mathcal{D}^n f(z)}{\mathcal{D}^m f(z)}\right]\right\} > \beta,$$

$$(z \in \mathfrak{U};\ n > m,\ 0 \leqq \beta < 1,\ |\theta| < \pi \text{ and } \cos\theta > \beta),$$

and

$$\operatorname{Re}\left\{e^{i\theta}\left[\frac{\mathcal{D}^n g(w)}{\mathcal{D}^m g(w)}\right]\right\} > \beta,$$

$$(w \in \mathfrak{U};\ n > m,\ 0 \leqq \beta < 1,\ |\theta| < \pi \text{ and } \cos\theta > \beta),$$

where the function g is given by

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots.$$

Upon allocating the parameters n, m and θ, one can obtain several new subclasses of σ, as illustrated in the following three examples.

Example 1.2. A function $f(z)$ given by (1.1) is said to be in the class $f \in S_\sigma^*(\beta,\theta,1,0) = S_\sigma^*(\beta,\theta)$, if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad \operatorname{Re}\left\{e^{i\theta}\frac{zf'(z)}{f(z)}\right\} > \beta,$$

$$\left(z \in \mathfrak{U};\ 0 \leqq \beta < 1,\ \ |\theta| < \pi \text{ and } \cos\theta > \beta\right),$$

and

$$\operatorname{Re}\left\{e^{i\theta}\frac{wg'(w)}{g(w)}\right\} > \beta,$$

$$\left(w \in \mathfrak{U};\ 0 \leqq \beta < 1,\ \ |\theta| < \pi \text{ and } \cos\theta > \beta\right),$$

where the function g is given by

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Example 1.3. A function $f(z)$ given by (1.1) is said to be in the class $f \in S_\sigma^*(\beta,0,1,0) = S_\sigma^*(\beta)$, if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad \left(z \in \mathfrak{U};\ 0 \leqq \beta < 1\right),$$

and

$$\operatorname{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > \beta,\ \ \left(w \in \mathfrak{U};\quad 0 \leqq \beta < 1\right),$$

where the function g is given by

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Example 1.4. A function $f(z)$ given by (1.1) is said to be in the class $f \in S_\sigma^*(0,0,1,0) = S_\sigma^*$, if the following conditions are satisfied:

$$f \in \sigma \ \text{ and } \ \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad \left(z \in \mathfrak{U}\right),$$

and

$$\operatorname{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > 0, \quad \left(w \in \mathfrak{U}\right),$$

where the function g is given by

$$g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $\mathcal{P} : \mathfrak{U} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\operatorname{Re}\left(p(z)\right) > 0$.

To establish our main results, we shall require the following lemmas.

Lemma 1.5 ([27]). *If the function $p \in \mathcal{P}$ is given by the following series:*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \tag{1.2}$$

then the sharp estimate given by

$$|c_k| \leqq 2, \quad (k = 1, 2, 3, \ldots),$$

holds.

Lemma 1.6 ([14]). *If the function $p \in \mathcal{P}$ is given by the series (1.2), then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right),$$
$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z,$$

for some x and z with $|x| \leqq 1$ and $|z| \leqq 1$.

In this paper, we determine the coefficients of a_2, a_3, a_4 and obtain the second Hankel determinant, which satisfy the condition and is to seek an upper bound for the functional $\left|a_2 a_4 - a_3^2\right|$ for $f \in \mathcal{S}_\sigma^*(\beta, \theta, n, m)$.

2. Main Results

Our first main result for the class $f \in \mathcal{S}_\sigma^*(\beta, \theta, n, m)$ is stated as follows:

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_\sigma^*(\beta, \theta, n, m)$ for $n > m + 1$, $0 \leqq \beta < 1$, $|\theta| < \pi$ and $\cos\theta > \beta$. Then*

$$|a_2| \leq \frac{2(\cos\theta - \beta)}{2^n - 2^m}, \tag{2.1}$$

$$|a_3| \leq \frac{4(\cos\theta - \beta)^2}{(2^n - 2^m)^2} + \frac{2(\cos\theta - \beta)}{3^n - 3^m}, \tag{2.2}$$

$$|a_4| \leq \frac{8(\cos\theta - \beta)^3\left[\begin{array}{l}(2^n - 2^m)\left(3^m - 2^{m-1} - 2^{2m-1}\right)\\ +2^m(3^n - 3^m)\end{array}\right]}{(2^n + 2^m)(2^n - 2^m)^4} + \frac{10(\cos\theta - \beta)^2}{(2^n - 2^m)(3^n - 3^m)} + \frac{2(\cos\theta - \beta)}{4^n - 4^m}, \tag{2.3}$$

and for $\eta \in \mathbb{C}$

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{\cos\theta - \beta}{3^n - 3^m}, & 0 \leq |h(\eta, n, m))| \leq \frac{1}{2(3^n - 3^m)}, \\ 2\,|h(\eta, n, m)|\,[\cos\theta - \beta], & |h(\eta, n, m)| \geq \frac{1}{2(3^n - 3^m)}, \end{cases}$$

where

$$h(\eta, n, m) = \frac{1 - \eta}{2\left[(3^n - 3^m) - (2^{m+n} - 2^{2m})\right]}.$$

Proof. Let $f \in \mathcal{S}^*_\sigma(\beta, \theta, n, m)$ and $g = f^{-1}$. Then

$$\frac{\mathcal{D}^n f(z)}{\mathcal{D}^m f(z)} = 1 + \sum_{k=1}^{\infty} h_k z^k,$$

and

$$\frac{\mathcal{D}^n g(w)}{\mathcal{D}^m g(w)} = 1 + \sum_{k=1}^{\infty} h_k w^k.$$

Hence

$$e^{i\theta}\left[\frac{\mathcal{D}^n f(z)}{\mathcal{D}^m f(z)}\right] - \beta = e^{i\theta}\left(1 + \sum_{k=1}^{\infty} h_k z^k\right) - \beta,$$

and

$$e^{i\theta}\left[\frac{\mathcal{D}^n g(w)}{\mathcal{D}^m g(w)}\right] - \beta = e^{i\theta}\left(1 + \sum_{k=1}^{\infty} h_k w^k\right) - \beta.$$

Next, by simplifying the equation, we obtain

$$e^{i\theta}\left[\frac{\mathcal{D}^n f(z)}{\mathcal{D}^m f(z)}\right] - \beta - i\sin\theta = \cos\theta - \beta + e^{i\theta}\left(\sum_{k=1}^{\infty} h_k z^k\right),$$

and

$$e^{i\theta}\left[\frac{\mathcal{D}^n g(w)}{\mathcal{D}^m g(w)}\right] - \beta - i\sin\theta = \cos\theta - \beta + e^{i\theta}\left(\sum_{k=1}^{\infty} h_k w^k\right),$$

which result

$$\begin{aligned} \frac{e^{i\theta}\left[\frac{\mathcal{D}^n f(z)}{\mathcal{D}^m f(z)}\right] - \beta - i\sin\theta}{\cos\theta - \beta} &= 1 + \frac{e^{i\theta}\left(\sum_{k=1}^{\infty} h_k z^k\right)}{\cos\theta - \beta}, \\ \frac{e^{i\theta}\left[\frac{\mathcal{D}^n g(w)}{\mathcal{D}^m g(w)}\right] - \beta - i\sin\theta}{\cos\theta - \beta} &= 1 + \frac{e^{i\theta}\left(\sum_{k=1}^{\infty} h_k w^k\right)}{\cos\theta - \beta}. \end{aligned} \tag{2.4}$$

Therefore, from the left hand side of equation (2.4) and Lemma 1.5, we get

$$\begin{aligned} \frac{e^{i\theta}\left[\frac{\mathcal{D}^n f(z)}{\mathcal{D}^m f(z)}\right] - \beta - i\sin\theta}{\cos\theta - \beta} &= p(z), \\ \frac{e^{i\theta}\left[\frac{\mathcal{D}^n g(w)}{\mathcal{D}^m g(w)}\right] - \beta - i\sin\theta}{\cos\theta - \beta} &= q(w). \end{aligned} \tag{2.5}$$

Thus,

$$\frac{e^{i\theta}\left\{\begin{array}{c} 1+\left(2^n-2^m\right)a_2 z+\left[\left(3^n-3^m\right)a_3-\left(2^{m+n}-2^{2m}\right)a_2^2\right]z^2 \\ +\left[\left(4^n-4^m\right)a_4+\left(2^{2m+n}-2^{3m}\right)a_2^3\right. \\ \left.-\left(3^m\left(2^n-2^m\right)+2^m\left(3^n-3^m\right)\right)a_2a_3\right]z^3 \end{array}\right\} -(\beta+i\sin\theta)}{\cos\theta-\beta}$$

$$=p(z),$$

$$\frac{e^{i\theta}\left\{\begin{array}{l} 1-\left(2^n-2^m\right)a_2 w \\ +\left[\left(3^n-3^m\right)\left(2a_2^2-a_3\right)-\left(2^{m+n}-2^{2m}\right)a_2^2\right]w^2 \\ -\left[\left(4^n-4^m\right)\left(5a_2^3-5a_2a_3+a_4\right)\right. \\ +\left[3^m\left(2^n-2^m\right)\left(a_3-2a_2^2\right)a_2+2^{2m}\left(2^n-2^m\right)a_2^3\right] \\ \left.+2^m\left(3^n-3^m\right)\left(a_3-2a_2^2\right)a_2\right]w^3 \end{array}\right\} -(\beta+i\sin\theta)}{\cos\theta-\beta}$$

$$=q(w),$$

where the functions $p(z)$ and $q(w)$ given by

$$p(z)=1+c_1z+c_2z^2+\cdots,$$

and

$$q(w)=1+d_1w+d_2w^2+\cdots,$$

are in class $\mathcal{P}$.

Comparing the coefficients in (2.5), we have

$$a_2=\frac{c_1\left[\cos\theta-\beta\right]}{e^{i\theta}\left(2^n-2^m\right)}, \tag{2.6}$$

$$\left(3^n-3^m\right)a_3-\left(2^{m+n}-2^{2m}\right)a_2^2=\frac{c_2\left[\cos\theta-\beta\right]}{e^{i\theta}}, \tag{2.7}$$

$$\left[\begin{array}{l} \left(4^n-4^m\right)a_4+\left(2^{2m+n}-2^{3m}\right)a_2^3 \\ -\left[3^m\left(2^n-2^m\right)+2^m\left(3^n-3^m\right)\right]a_2a_3 \end{array}\right]=\frac{c_3\left[\cos\theta-\beta\right]}{e^{i\theta}}, \tag{2.8}$$

$$-a_2=\frac{d_1\left[\cos\theta-\beta\right]}{e^{i\theta}\left(2^n-2^m\right)}, \tag{2.9}$$

$$\left[\begin{array}{l} 2\left(3^n-3^m\right)a_2^2-\left(2^{m+n}-2^{2m}\right)a_2^2 \\ -\left(3^n-3^m\right)a_3 \end{array}\right]=\frac{d_2\left[\cos\theta-\beta\right]}{e^{i\theta}}, \tag{2.10}$$

$$\left[\begin{array}{l} -\left(4^n-4^m\right)\left[5a_2^3-5a_2a_3+a_4\right] \\ -3^m\left(2^n-2^m\right)\left(a_3-2a_2^2\right)a_2 \\ -2^{2m}\left(2^n-2^m\right)a_2^3 \\ -2^m\left(3^n-3^m\right)\left(a_3-2a_2^2\right)a_2 \end{array}\right]=\frac{d_3\left[\cos\theta-\beta\right]}{e^{i\theta}}. \tag{2.11}$$

From (2.6) and (2.9), we find that

$$c_1 = -d_1, \tag{2.12}$$

and

$$a_2 = \frac{c_1 [\cos\theta - \beta]}{e^{i\theta}(2^n - 2^m)}. \tag{2.13}$$

Now, from (2.7), (2.10) and (2.13), we get

$$a_3 = \frac{c_1^2 [\cos\theta - \beta]^2}{e^{i2\theta}(2^n - 2^m)^2} + \frac{1}{2}\frac{(c_2 - d_2)[\cos\theta - \beta]}{e^{i\theta}(3^n - 3^m)}. \tag{2.14}$$

Also, from (2.8) and (2.11), (2.13) and (2.14), we find that

$$\begin{aligned} a_4 = {} & \frac{[\cos\theta - \beta]^3 \begin{bmatrix} (2^n - 2^m)\left(3^m - 2^{m-1} - 2^{2m-1}\right) \\ +2^m (3^n - 3^m) \end{bmatrix} c_1^3}{e^{i3\theta}(2^n - 2^m)^3 (4^n - 4^m)} \\ & + \frac{5}{4(3^n - 3^m)} \frac{c_1 (c_2 - d_2)[\cos\theta - \beta]^2}{e^{i2\theta}(2^n - 2^m)} \\ & + \frac{1}{2(4^n - 4^m)} \frac{(c_3 - d_3)[\cos\theta - \beta]}{e^{i\theta}}. \end{aligned} \tag{2.15}$$

If we apply Lemma 1.5 to (2.13), (2.14) and (2.15), we obtain (2.1), (2.2) and (2.3).

Adding (2.7) to (2.10), we get

$$2\left[(3^n - 3^m) - \left(2^{m+n} - 2^{2m}\right)\right] a_2^2 = \frac{(c_2 + d_2)[\cos\theta - \beta]}{e^{i\theta}}. \tag{2.16}$$

Also by subtracting (2.10) from (2.7), we get

$$a_3 = \frac{1}{2}\frac{(c_2 - d_2)[\cos\theta - \beta]}{e^{i\theta}(3^n - 3^m)} + a_2^2. \tag{2.17}$$

From equations (2.16) and (2.17), we get

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{(c_2 - d_2)[\cos\theta - \beta]}{2(3^n - 3^m)e^{i\theta}} + (1-\eta) a_2^2 \\ &= \frac{(c_2 - d_2)[\cos\theta - \beta]}{2(3^n - 3^m)e^{i\theta}} + (1-\eta)\frac{(c_2 + d_2)[\cos\theta - \beta]}{2[(3^n - 3^m) - (2^{m+n} - 2^{2m})]e^{i\theta}} \\ &= \frac{[\cos\theta - \beta]}{e^{i\theta}} \left\{ \left(\frac{(1-\eta)}{2[(3^n - 3^m) - (2^{m+n} - 2^{2m})]} + \frac{1}{2(3^n - 3^m)} \right) c_2 \right. \\ &\quad \left. + \left(\frac{(1-\eta)}{2[(3^n - 3^m) - (2^{m+n} - 2^{2m})]} - \frac{1}{2(3^n - 3^m)} \right) d_2 \right\} \\ &= \frac{[\cos\theta - \beta]}{e^{i\theta}} \left\{ \left(h(\eta, n, m) + \frac{1}{2(3^n - 3^m)} \right) c_2 \right. \end{aligned}$$

$$+\left(h(\eta,n,m)-\frac{1}{2\left(3^n-3^m\right)}\right)d_2\Big\},$$

where

$$h(\eta,n,m)=\frac{1-\eta}{2\left[\left(3^n-3^m\right)-\left(2^{m+n}-2^{2m}\right)\right]}.$$

The proof of Theorem 2.1 is completed. □

Our second main result for the class $\mathcal{S}_\sigma^*(\beta,\theta,n,m)$ is given by Theorem 2.2 below.

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_\sigma^*(\beta,\theta,n,m)$ and if $A=\cos\theta-\beta$ for $n>m+1$,* $\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\ +2^m(3^n-3^m)-(4^n-4^m)\end{array}\right]\neq 0$, *$|\theta|<\pi$ and $\cos\theta>\beta$. Then*

$$\left|a_2a_4-a_3^2\right|$$
$$\leqq\begin{cases}\dfrac{4A^2}{4^n-4^m}\left\{\begin{array}{l}\dfrac{4A^2\left[\begin{array}{l}\left(2^n-2^m\right)\left(\begin{array}{l}3^m-2^{m-1}\\-2^{2m-1}\end{array}\right)\\+2^m\left(3^n-3^m\right)\\-\left(4^n-4^m\right)\end{array}\right]}{\left(2^n-2^m\right)^4}\\+\dfrac{\left(3^n-3^m\right)^2-\left(2^n-2^m\right)\left(4^n-4^m\right)}{\left(2^n-2^m\right)\left(3^n-3^m\right)^2}\\+\dfrac{\left(4^n-4^m\right)}{\left(3^n-3^m\right)^2}\end{array}\right\}, & A\in\left[0,\Phi_{(n,m)}\right],\\[2ex] \dfrac{A^4}{16\left(4^n-4^m\right)}\left\{\begin{array}{l}\dfrac{64\left(4^n-4^m\right)}{\left(3^n-3^m\right)^2}\\+\dfrac{\left[\mathfrak{B}+\frac{2A\left(2^n+2^m\right)}{\left(2^n-2^m\right)\left(3^n-3^m\right)}\right]^2\left(4\mathfrak{P}^2+5\mathfrak{P}\mathcal{E}+\mathcal{E}^2\right)}{\left(\mathfrak{P}+\mathcal{E}\right)^3+\left[\mathfrak{B}+\frac{2A\left(2^n+2^m\right)}{\left(2^n-2^m\right)\left(3^n-3^m\right)}\right]^2}\end{array}\right\}, & A\in\left[\Phi_{(n,m)},1\right],\end{cases}$$

where

$$\Phi_{(n,m)}=\frac{\left(4^n-4^m\right)\left(2^n-2^m\right)^2}{8\left(3^n-3^m\right)\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\+2^m(3^n-3^m)-(4^n-4^m)\end{array}\right]}$$
$$\times\left(1+\sqrt{1-\frac{16\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\+2^m(3^n-3^m)-(4^n-4^m)\end{array}\right]\times\left\{(4^n-4^m)(2^n-2^m)-2(3^n-3^m)^2\right\}}{(4^n-4^m)^2(2^n-2^m)}}\right),$$

$$\mathfrak{B}=\frac{6(3^n-3^m)^2-4(4^n-4^m)(2^n-2^m)}{(2^n-2^m)(3^n-3^m)^2},$$

$$\mathfrak{P}=\frac{-2(3^n-3^m)^2+(2^n-2^m)(4^n-4^m)}{(2^n-2^m)(3^n-3^m)^2}-\frac{A(2^n+2^m)}{(2^n-2^m)(3^n-3^m)},$$

$$\mathcal{E}=\frac{4A^2\left[\left(3^m-2^{m-1}-2^{2m-1}\right)+2^m\left(3^n-3^m\right)\left(2^n+2^m\right)\right]}{\left(2^n-2^m\right)^3}.$$

Proof. From (2.13), (2.14) and (2.15) and letting $A=\cos\theta-\beta$, we have

$$a_2=\frac{c_1A}{e^{i\theta}\left(2^n-2^m\right)},$$
$$a_3=\frac{c_1^2A^2}{e^{i2\theta}\left(2^n-2^m\right)^2}+\frac{1}{2}\frac{\left[c_2-d_2\right]A}{e^{i\theta}\left(3^n-3^m\right)},$$

and

$$a_4=\frac{\left[\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)+2^m\left(3^n-3^m\right)\right]A^3c_1^3}{e^{i3\theta}\left(2^n-2^m\right)^3\left(4^n-4^m\right)}+\frac{5}{4\left(3^n-3^m\right)}\frac{c_1\left(c_2-d_2\right)A^2}{e^{i2\theta}\left(2^n-2^m\right)}+\frac{1}{2\left(4^n-4^m\right)}\frac{\left(c_3-d_3\right)A}{e^{i\theta}}.$$

Hence, the functional $a_2a_4-a_3^2$ will become

(2.18)
$$a_2a_4-a_3^2=\frac{\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\+2^m\left(3^n-3^m\right)-\left(4^n-4^m\right)\end{array}\right]A^4c_1^4}{e^{i4\theta}\left(2^n-2^m\right)^4\left(4^n-4^m\right)}+\frac{1}{4\left(3^n-3^m\right)}\frac{c_1^2\left(c_2-d_2\right)A^3}{e^{i3\theta}\left(2^n-2^m\right)^2}+\frac{1}{2\left(4^n-4^m\right)}\frac{c_1\left(c_3-d_3\right)A^2}{e^{i2\theta}\left(2^n-2^m\right)}-\frac{1}{4\left(3^n-3^m\right)^2}\frac{\left(c_2-d_2\right)^2A^2}{e^{i2\theta}}.$$

According to Lemma 1.6 and (2.12), we write

$$\left.\begin{array}{l}2c_2=c_1^2+x\left(4-c_1^2\right)\\2d_2=d_1^2+y(4-d_1^2)\end{array}\right\}\quad\Rightarrow\quad c_2-d_2=\frac{4-c_1^2}{2}(x-y),\tag{2.19}$$

and

(2.20)
$$4c_3=c_1^3+2\left(4-c_1^2\right)c_1x-c_1\left(4-c_1^2\right)x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z,$$
$$4d_3=d_1^3+2(4-d_1^2)d_1y-d_1(4-d_1^2)y^2+2(4-d_1^2)\left(1-|y|^2\right)w.$$

Moreover, we have

$$c_3-d_3=\frac{c_1^3}{2}+\frac{c_1\left(4-c_1^2\right)}{2}(x+y)-\frac{c_1\left(4-c_1^2\right)}{4}(x^2+y^2)+\frac{\left(4-c_1^2\right)}{2}\left(\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right),\tag{2.21}$$

$$c_2 + d_2 = c_1^2 + \frac{\left(4 - c_1^2\right)}{2}(x + y), \tag{2.22}$$

for some x, y and z, w with $|x| \leqq 1$, $|y| \leqq 1$, $|z| \leqq 1$, $|w| \leqq 1$ and $\left|e^{i\theta}\right| = 1$. Using (2.19) and (2.21) in (2.18), and applying the triangle inequality, we have

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| = &\left| \frac{\left[\begin{array}{c} \left(2^n - 2^m\right)\left(3^m - 2^{m-1} - 2^{2m-1}\right) \\ +2^m\left(3^n - 3^m\right) - \left(4^n - 4^m\right) \end{array}\right] A^4 c_1^4}{e^{i4\theta}\left(2^n - 2^m\right)^4\left(4^n - 4^m\right)} \right. \\
&+ \frac{1}{8}\frac{c_1^2\left(4 - c_1^2\right)A^3}{\left(2^n - 2^m\right)^2\left(3^n - 3^m\right)}(x - y) \\
&+ \frac{c_1 A^2}{2\left(2^n - 2^m\right)\left(4^n - 4^m\right)}\left[\frac{c_1^3}{2} + \frac{c_1\left(4 - c_1^2\right)}{2}(x + y)\right. \\
&- \frac{c_1\left(4 - c_1^2\right)}{4}\left(x^2 + y^2\right) \\
&\left. + \frac{\left(4 - c_1^2\right)}{2}\left((1 - |x|^2)z - (1 - |y|^2)w\right)\right] \\
&\left. - \frac{\left(4 - c_1^2\right)^2 A^2}{16\left(3^n - 3^m\right)^2}(x - y)^2 \right| \\
\leqq &\frac{\left[\begin{array}{c} \left(2^n - 2^m\right)\left(3^m - 2^{m-1} - 2^{2m-1}\right) \\ +2^m\left(3^n - 3^m\right) - \left(4^n - 4^m\right) \end{array}\right] A^4 c_1^4}{\left(2^n - 2^m\right)^4\left(4^n - 4^m\right)} \\
&+ \frac{c_1^4 A^2}{4\left(2^n - 2^m\right)\left(4^n - 4^m\right)} \\
&+ \frac{c_1\left(4 - c_1^2\right)A^2}{2\left(2^n - 2^m\right)\left(4^n - 4^m\right)} \\
&+ \left[\begin{array}{c} \frac{c_1^2\left(4 - c_1^2\right)A^3}{8\left(2^n - 2^m\right)^2\left(3^n - 3^m\right)} \\ + \frac{c_1^2\left(4 - c_1^2\right)A^2}{4\left(2^n - 2^m\right)\left(4^n - 4^m\right)} \end{array}\right](|x| + |y|) \\
&+ \left[\begin{array}{c} \frac{c_1^2\left(4 - c_1^2\right)A^2}{8\left(2^n - 2^m\right)\left(4^n - 4^m\right)} \\ - \frac{c_1\left(4 - c_1^2\right)A^2}{4\left(2^n - 2^m\right)\left(4^n - 4^m\right)} \end{array}\right]\left(|x|^2 + |y|^2\right) \\
&+ \frac{\left(4 - c_1^2\right)^2 A^2}{16\left(3^n - 3^m\right)^2}(|x| + |y|)^2.
\end{aligned}$$

Since $p \in \mathcal{P}$, the function $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ is also in the class $\mathcal{P}$ and therefore we can assume without loss of generality that $c_1 = c \in [0,2]$. Thus, for $\lambda = |x| \leqq 1$ and $\mu = |y| \leqq 1$, we obtain

$$\left|a_2a_4 - a_3^2\right| \leqq M_1 + M_2(\lambda+\mu) + M_3(\lambda^2+\mu^2) + M_4(\lambda+\mu)^2 = F(\lambda,\mu),$$

where

$$M_1 = M_1(c) = \frac{A^2}{4\left(4^n-4^m\right)}\left[\begin{array}{c}\left(\frac{1}{(2^n-2^m)}+\frac{4A^2}{(2^n-2^m)^4}\right)c^4 \\ +\frac{8c-2c^3}{(2^n-2^m)}\end{array}\right] \geqq 0,$$

$$M_2 = M_2(c)$$

$$= \frac{A^2}{24\left(3^n-3^m\right)}\left[c^2(4-c^2)\frac{\left\{\begin{array}{c}\left(4^n-4^m\right)A \\ +6\left(3^n-3^m\right)\left(2^n-2^m\right)\end{array}\right\}}{\left(2^n-2^m\right)^2\left(4^n-4^m\right)}\right] \geqq 0,$$

$$M_3 = M_3(c) = \frac{A^2}{8\left(4^n-4^m\right)\left(2^n-2^m\right)}c(4-c^2)(c-2) \leqq 0,$$

$$M_4 = M_4(c) = \frac{A^2}{16\left(3^n-3^m\right)^2}(4-c^2)^2 \geqq 0.$$

Now we need to maximize $F(\lambda,\mu)$ in the closed square

$$\mathbb{S} = \left\{(\lambda,\mu) : 0 \leqq \lambda \leqq 1, 0 \leqq \mu \leqq 1\right\},$$

for $c \in [0,2]$. By differentiating the function $F(\lambda,\mu)$ partially, we have

$$\frac{\partial F}{\partial \lambda} = M_2 + 2M_3\lambda + 2M_4\left(\lambda+\mu\right) = 0, \tag{2.23}$$

and

$$\frac{\partial F}{\partial \mu} = M_2 + 2M_3\mu + 2M_4\left(\lambda+\mu\right) = 0. \tag{2.24}$$

By equating (2.23) and (2.24), we obtain

$$\lambda = \mu, \qquad \lambda = \left(-M_2\right) \diagup 2\left(M_3+2M_4\right).$$

Since, the function $F(\lambda,\mu)$ cannot have a local maximum, we investigate the maximum of $F(\lambda,\mu)$ on the boundary. For $\lambda = 0$ and $0 \leqq \lambda \leqq 1$ (similar to $\mu = 0$ and $0 \leqq \mu \leqq 1$), we obtain $F(0,\mu) = M_1 + M_2\mu + \left(M_3+M_4\right)\mu^2 = G(\mu)$. We attained the interior point of $0 \leqq c \leqq 2$ for $0 \leqq \mu \leqq 1$ when $M_3 + M_4 \geqq 0$. The function $G'(\mu) > 0$ for $\lambda > 0$ indicates that F is an increasing function. Therefore, the upper bound for functional $\left|a_2a_4 - a_3^2\right|$ corresponds to $\mu = 1$ and $c = 0$, which can be simplified into $G'(\mu) = 2\left(M_3+M_4\right)\mu + M_2 \geqq 0$. Hence, the maximum of $G(\mu)$ occurs at $\mu = 1$ and

$$\max\{G(\mu)\} = G(1) = M_1 + M_2 + M_3 + M_4.$$

For the case when $M_3+M_4<0$, we note that $M_2+2\left(M_3+M_4\right)\mu \geqq 0$ for $0 \leqq \mu \leqq 1$ and any fixed c with $0 \leqq c<2$. It is clear that $M_2+2\left(M_3+M_4\right)<2\left(M_3+M_4\right)\mu+M_2<M_2$ and so $G'(\mu)>0$. Hence, for $c=2$, we obtain

$$\begin{aligned} F(\lambda,\mu) &= \frac{16A^4\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\ +2^m\left(3^n-3^m\right)-\left(4^n-4^m\right)\end{array}\right]}{\left(4^n-4^m\right)\left(2^n-2^m\right)^4} \\ &\quad+\frac{4A^2}{\left(2^n-2^m\right)\left(4^n-4^m\right)} \\ &= \frac{4A^2}{\left(4^n-4^m\right)}\left\{\frac{4A^2\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\ +2^m\left(3^n-3^m\right)-\left(4^n-4^m\right)\end{array}\right]}{\left(2^n-2^m\right)^4}\right. \\ &\quad\left.+\frac{1}{\left(2^n-2^m\right)}\right\}. \end{aligned}$$

Next, we looking for $\lambda=1$ and $0 \leqq \lambda \leqq 1$ (similar to $\mu=1$ and $0 \leqq \mu \leqq 1$), we obtained

$$F(1,\mu)=H(\mu)=M_1+M_2+M_3+M_4+\left(M_2+2M_4\right)\mu+\left(M_3+M_4\right)\mu^2.$$

Similarly, to the above cases of M_3+M_4 where $\mu=1$, we get

$$\max\{H(\mu)\}=H(1)=M_1+2M_2+2M_3+4M_4.$$

Since $G(1) \leqq H(1)$, we attained the interior point of $c \in [0,2]$ where maximum of F occurs at $\lambda=1$ and $\mu=1$. Therefore, $F(\lambda,\mu)=F(1,1)=M_1+2M_2+2M_3+4M_4=K(c)$. By substituting the value of $M_1+M_2+M_3+M_4$ in the function K, we have

$$\begin{aligned} K(c) &= \frac{A^2}{16\left(4^n-4^m\right)}\left\{c^4\left[\frac{16A^2\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\ +2^m\left(3^n-3^m\right)-\left(4^n-4^m\right)\end{array}\right]}{\left(2^n-2^m\right)^4}\right.\right. \\ &\quad\left.-\frac{4A\left(4^n-4^m\right)}{\left(2^n-2^m\right)^2\left(3^n-3^m\right)}+\frac{4\left(2^n-2^m\right)\left(4^n-4^m\right)-8\left(3^n-3^m\right)^2}{\left(2^n-2^m\right)\left(3^n-3^m\right)^2}\right] \\ &\quad+c^2\left[\frac{16A\left(4^n-4^m\right)}{\left(2^n-2^m\right)^2\left(3^n-3^m\right)}+\frac{\begin{array}{c}48\left(3^n-3^m\right)^2\\ -32\left(2^n-2^m\right)\left(4^n-4^m\right)\end{array}}{\left(2^n-2^m\right)\left(3^n-3^m\right)^2}\right] \\ &\quad\left.+\frac{64\left(4^n-4^m\right)^2}{\left(3^n-3^m\right)^2}\right\}. \end{aligned}$$

Assume that $K(c)$ has a maximum in an interior point c of $[0,2]$. By differentiating the function $K(c)$ with respect to c, we have

$$K'(c)=\frac{A^2}{16\left(4^n-4^m\right)}\left\{4c^3\left[\frac{16\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\+2^m\left(3^n-3^m\right)-\left(4^n-4^m\right)\end{array}\right]A^2}{\left(2^n-2^m\right)^4}\right.\right.$$
$$\left.-\frac{4A\left(4^n-4^m\right)}{\left(2^n-2^m\right)^2\left(3^n-3^m\right)}+\frac{4\left(2^n-2^m\right)\left(4^n-4^m\right)-8\left(3^n-3^m\right)^2}{\left(2^n-2^m\right)\left(3^n-3^m\right)^2}\right]$$
$$\left.+2c\left[\frac{16A\left(4^n-4^m\right)}{\left(2^n-2^m\right)^2\left(3^n-3^m\right)}+\frac{\begin{array}{c}48\left(3^n-3^m\right)^2\\-32\left(2^n-2^m\right)\left(4^n-4^m\right)\end{array}}{\left(2^n-2^m\right)\left(3^n-3^m\right)^2}\right]\right\}.$$

By letting

$$\left[\begin{array}{l}\frac{16A^2\left[\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)+2^m\left(3^n-3^m\right)-\left(4^n-4^m\right)\right]}{\left(2^n-2^m\right)^4}\\-\frac{4A\left(4^n-4^m\right)}{\left(2^n-2^m\right)^2\left(3^n-3^m\right)}\\+\frac{4\left(2^n-2^m\right)\left(4^n-4^m\right)-8\left(3^n-3^m\right)^2}{\left(2^n-2^m\right)\left(3^n-3^m\right)^2}\end{array}\right]\geqq 0$$

that is

$$A\in\left[0,\Phi_{(n,m)}\right],$$

where

$$\Phi_{(n,m)}=\frac{(4^n-4^m)(2^n-2^m)^2}{8(3^n-3^m)\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\+2^m(3^n-3^m)-(4^n-4^m)\end{array}\right]}$$
$$\times\left(1+\sqrt{1-\frac{\begin{array}{c}16\left[\begin{array}{c}\left(2^n-2^m\right)\left(3^m-2^{m-1}-2^{2m-1}\right)\\+2^m(3^n-3^m)-(4^n-4^m)\end{array}\right]\\\times\left\{(4^n-4^m)(2^n-2^m)-2(3^n-3^m)^2\right\}\end{array}}{(4^n-4^m)^2(2^n-2^m)}}\right).$$

Therefore, $K'(c)>0$ for $c\in[0,2]$. Since K is an increasing function in the interval $[0,2]$ so the maximum point of K is on the boundary for $c=2$. Thus,

$$\max K(c)=K(2)$$

$$
= \frac{4A^2}{(4^n - 4^m)} \left\{ \frac{4A^2 \left[\begin{array}{c} (2^n - 2^m)\left(3^m - 2^{m-1} - 2^{2m-1}\right) \\ +2^m (3^n - 3^m) - (4^n - 4^m) \end{array} \right]}{(2^n - 2^m)^4} \right.
$$

$$
\left. + \frac{(3^n - 3^m)^2 - (2^n - 2^m)(4^n - 4^m)}{(2^n - 2^m)(3^n - 3^m)^2} + \frac{(4^n - 4^m)}{(3^n - 3^m)^2} \right\}.
$$

Hence,

$$
\left|a_2 a_4 - a_3^2\right| \leqq \frac{4A^2}{(4^n - 4^m)} \left\{ \frac{4 \left[\begin{array}{c} (2^n - 2^m)\left(3^m - 2^{m-1} - 2^{2m-1}\right) \\ +2^m (3^n - 3^m) - (4^n - 4^m) \end{array} \right] A^2}{(2^n - 2^m)^4} \right.
$$

$$
\left. + \frac{(3^n - 3^m)^2 - (2^n - 2^m)(4^n - 4^m)}{(2^n - 2^m)(3^n - 3^m)^2} + \frac{(4^n - 4^m)}{(3^n - 3^m)^2} \right\}.
$$

Also, by letting

$$
\left[\begin{array}{l} \frac{16A^2\left[(2^n-2^m)\left(3^m-2^{m-1}-2^{2m-1}\right)+2^m(3^n-3^m)-(4^n-4^m)\right]}{(2^n-2^m)^4} \\ -\frac{4A(4^n-4^m)}{(2^n-2^m)^2(3^n-3^m)} \\ +\frac{4(2^n-2^m)(4^n-4^m)-8(3^n-3^m)^2}{(2^n-2^m)(3^n-3^m)^2} \end{array} \right] \geqq 0
$$

that is

$$
A \in \left[\Phi_{(n,m)}, 1\right].
$$

We observe that $c_0 < 2$, that is c_0 is in the interval $[0, 2]$. Since $K'(c_0) \leqq 0$, the maximum of $K(c)$ occurs at $c = c_0$. Therefore,

$$
\max\{K(c_0)\} = K\left(\sqrt{\frac{\frac{2A(4^n-4^m)}{(2^n-2^m)^2(3^n-3^m)} + \frac{6(3^n-3^m)^2-4(4^n-4^m)(2^n-2^m)}{(2^n-2^m)(3^n-3^m)^2}}{\frac{4A^2\left[(2^n-2^m)\left(3^m-2^{m-1}-2^{2m-1}\right)+2^m(3^n-3^m)(4^n-4^m)\right]}{(2^n-2^m)^4} - \frac{A(4^n-4^m)}{(2^n-2^m)^2(3^n-3^m)} + \frac{(2^n-2^m)(4^n-4^m)-2(3^n-3^m)^2}{(2^n-2^m)(3^n-3^m)^2}}} \right)
$$

$$
= \frac{A^4}{16(4^n - 4^m)} \left\{ \frac{64(4^n - 4^m)}{(3^n - 3^m)^2} \right.
$$

$$
\left. + \frac{\left[\mathfrak{B} + \frac{2A(2^n+2^m)}{(2^n-2^m)(3^n-3^m)}\right]^2 \left(4\mathfrak{P}^2 + 5\mathfrak{P}\mathcal{E} + \mathcal{E}^2\right)}{(\mathfrak{P} + \mathcal{E})^3 + \left[\mathfrak{B} + \frac{2A(2^n+2^m)}{(2^n-2^m)(3^n-3^m)}\right]^2} \right\}.
$$

Hence,

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{A^4}{16(4^n-4^m)}\left\{\frac{64(4^n-4^m)}{(3^n-3^m)^2}\right. \\ \left. + \frac{\left[\mathfrak{B} + \frac{2A(2^n+2^m)}{(2^n-2^m)(3^n-3^m)}\right]^2\left(4\mathfrak{P}^2 + 5\mathfrak{P}\mathcal{E} + \mathcal{E}^2\right)}{(\mathfrak{P}+\mathcal{E})^3 + \left[\mathfrak{B} + \frac{2A(2^n+2^m)}{(2^n-2^m)(3^n-3^m)}\right]^2}\right\},$$

where

$$\mathfrak{B} = \frac{6(3^n-3^m)^2 - 4(4^n-4^m)(2^n-2^m)}{(2^n-2^m)(3^n-3^m)^2},$$
$$\mathfrak{P} = \frac{-2(3^n-3^m)^2 + (2^n-2^m)(4^n-4^m)}{(2^n-2^m)(3^n-3^m)^2} - \frac{A(2^n+2^m)}{(2^n-2^m)(3^n-3^m)},$$
$$\mathcal{E} = \frac{4A^2\left[\left(3^m - 2^{m-1} - 2^{2m-1}\right) + 2^m\left(3^n-3^m\right)\left(2^n+2^m\right)\right]}{(2^n-2^m)^3}.$$

The proof of Theorem 2.2 is completed. □

For $n = 1$, and $m = 0$ in Theorem 2.2, we obtained the result of Soh *et al.* (2021) as given in Corollary 2.3.

Corollary 2.3. *Let $f(z)$ given in (1.1) be in the class $\mathcal{S}_\sigma^*(\beta,\theta,1,0)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{4}{3}A^2\left(4A^2+1\right).$$

For $\beta = 0$, $\theta = 0$ and $n = 1$, $m = 0$ in Theorem 2.2, we obtained the result of Deniz *et al.* (2015) as given in Corollary 2.4

Corollary 2.4. *Let $f(z)$ given in (1.1) be in the class $\mathcal{S}_\sigma^*(0,0,1,0)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{20}{3}.$$

3. Conclusion

In the present paper, we found an upper bound of the initial Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and $|a_4|$ and also of the Fekete-Szegö functional for functions in the class $\mathcal{S}_\sigma^*(\beta,\theta,n,m)$, which we introduced here. We also obtained a significantly-improved upper bound of the functional $\left|a_2a_4 - a_3^2\right|$ for the functions in the class $\mathcal{S}_\sigma^*(\beta,\theta,n,m)$.

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