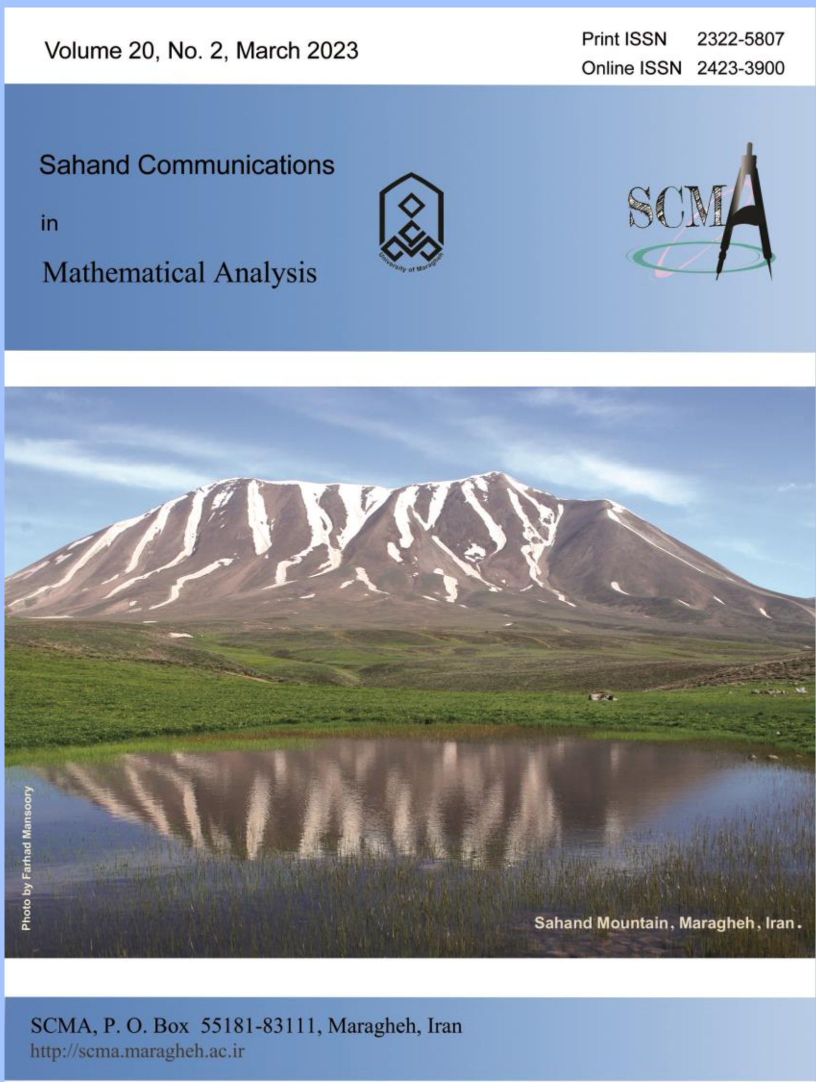
Hermite-Hadamard, Trapezoid and Midpoint Type Inequalities Involving Generalized Fractional Integrals for Convex Functions

Hasan Kara, Samet Erden and Huseyin Budak

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 2
Pages: 85-107

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.552775.1102



Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 2 (2023), 85-107
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.552775.1102

Hermite-Hadamard, Trapezoid and Midpoint Type Inequalities Involving Generalized Fractional Integrals for Convex Functions

Hasan Kara[1*], Samet Erden[2] and Hüseyin Budak[3]

ABSTRACT. We first construct new Hermite-Hadamard type inequalities which include generalized fractional integrals for convex functions by using an operator which generates some significant fractional integrals such as Riemann-Liouville fractional and the Hadamard fractional integrals. Afterwards, Trapezoid and Midpoint type results involving generalized fractional integrals for functions whose the derivatives in modulus and their certain powers are convex are established. We also recapture the previous results in the particular situations of the inequalities which are given in the earlier works.

1. INTRODUCTION

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g.,[14, 21], [37, p.137]). These inequalities state that if $F : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)d\varkappa \le \frac{F(\kappa_1)+F(\kappa_2)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if F is concave.

Over the last few decades, many papers have focused on generalization of the inequality (1.1) and obtaining trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side

2020 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.
Key words and phrases. Hermite-Hadamard inequalities, Generalized fractional integrals, Convex functions.
Received: 27 April 2022, Accepted: 10 October 2022.
* Corresponding author.

of (1.1), respectively. For instance, in [15] and [29] authors first obtained trapezoid and midpoint inequalities for convex functions, respectively. Sarikaya et al. extended the inequalities in (1.1) for Riemann-Liouville fractional integrals and the authors also proved some corresponding trapezoid type inequalities in [41]. On the other hand in [22], Iqbal et al. obtained some midpoint type inequalities for convex functions via Riemann-Liouville fractional integrals. Moreover, Jleli and Samet obtained Hermite-Hadamard type inequalities and some corresponding trapezoid type inequalities for generalized fractional integrals in [25]. Fractional calculus, on which many studies have been made in recent years, can be said to be a generalization of classical calculus. Generalized fractional integrals are one of the cornerstones of fractional calculus. Generalized fractional operators generalize many fractional types. For the other similar inequalities, please refer to [2, 9, 12, 27, 30, 31].

The overall structure of the study takes the form of five sections with introduction. The remainder of this work is organized as follows: we first give definitions of some kinds of fractional integrals and we also mention some works which focus on fractional version of Hermite-Hadamard inequality. In Section 2, the new version of Hermite-Hadamard type inequalities for generalized fractional integrals is proved. By utilizing generalized fractional integrals, we present some midpoint and trapezoid type inequalities for functions whose first derivatives in absolute value are convex in Section 3 and Section 4, respectively. Finally, some conclusions and further directions of research are discussed in Section 5.

Definitions of Riemann-Liouville and Hadamard fractional integrals are given as follows.

Definition 1.1 (see, [28]). Suppose that $\digamma$ is the element of $L_1[\kappa_1, \kappa_2]$. The left-sided Riemann-Liouville fractional integral $J^{\alpha}_{\kappa_1+}\digamma$ and right-sided Riemann-Liouville fractional integral $J^{\alpha}_{\kappa_2-}\digamma$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$J^{\alpha}_{\kappa_1+}\digamma(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\varkappa} (\varkappa - \xi)^{\alpha-1} \digamma(\xi) d\xi, \quad \varkappa > \kappa_1,$$

and

$$J^{\alpha}_{\kappa_2-}\digamma(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\kappa_2} (\xi - \varkappa)^{\alpha-1} \digamma(\xi) d\xi, \quad \varkappa < \kappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and

$$J^{0}_{\kappa_1+}\digamma(\varkappa) = J^{0}_{\kappa_2-}\digamma(\varkappa) = \digamma(\varkappa).$$

Definition 1.2 (see, [20, 28]). Suppose that $\digamma$ is an element of $L_1[\kappa_1, \kappa_2]$. The left-sided Hadamard fractional integral $\mathbf{J}^{\alpha}_{\kappa_1+}\digamma$ and right-sided.

Hadamard fractional integral $\mathbf{J}^{\alpha}_{\kappa_2-}F$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$\mathbf{J}^{\alpha}_{\kappa_1+}F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\varkappa} \left(\ln \frac{\varkappa}{\xi}\right)^{\alpha-1} F(\xi)\frac{d\xi}{\xi}, \quad \varkappa > \kappa_1,$$

and

$$\mathbf{J}^{\alpha}_{\kappa_2-}F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\kappa_2} \left(\ln \frac{\xi}{\varkappa}\right)^{\alpha-1} F(\xi)\frac{d\xi}{\xi}, \quad \varkappa < \kappa_2,$$

respectively.

We now give generalized fractional integrals which will be used in our main results.

Definition 1.3 (see, [28]). Assume that $\rho : [\kappa_1, \kappa_2] \to \mathbb{R}$ is positive and monotone increasing function on $(\kappa_1, \kappa_2]$ such that the derivative $\rho'(\varkappa)$ is a continuous on (κ_1, κ_2) and let $\alpha > 0$. The left-sided ($I^{\alpha}_{\kappa_1^+;\rho}F(\varkappa)$) and right-sided ($I^{\alpha}_{\kappa_2^-;\rho}F(\varkappa)$) generalized fractional integrals of F with respect to the function g on $[\kappa_1, \kappa_2]$ of order α are defined by

$$I^{\alpha}_{\kappa_1+;\rho}F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\varkappa} \frac{\rho'(\xi)F(\xi)}{[\rho(\varkappa) - \rho(\xi)]^{1-\alpha}} d\xi, \quad \varkappa > \kappa_1,$$

and

$$I^{\alpha}_{\kappa_2-;\rho}F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\kappa_2} \frac{\rho'(\xi)F(\xi)}{[\rho(\xi) - \rho(\varkappa)]^{1-\alpha}} d\xi, \quad \varkappa < \kappa_2,$$

respectively.

The Hermite-Hadamard type inequalities involving left and right-sided Riemann-Liouville fractional integrals are first proved by Sarikaya et al. in [41].

Theorem 1.4. *Assume that $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a positive mapping for $0 \leq \kappa_1 < \kappa_2$, and let F be an element of $L_1[\kappa_1, \kappa_2]$. If F is a convex function on $[\kappa_1, \kappa_2]$, then one has the inequalities*

$$\begin{aligned} F\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{\Gamma(\alpha + 1)}{2(\kappa_2 - \kappa_1)^{\alpha}} \left[J^{\alpha}_{\kappa_1+}F(\kappa_2) + J^{\alpha}_{\kappa_2-}F(\kappa_1)\right] \\ &\leq \frac{F(\kappa_1) + F(\kappa_2)}{2}, \quad \text{for } \alpha > 0. \end{aligned} \tag{1.2}$$

Also, a new version of the inequalities (1.2) is provided by Sarıkaya and Yildirim in [40].

Theorem 1.5. *Assume that $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a positive mapping for $\kappa_1 < \kappa_2$, and let F be an element of $L_1[\kappa_1, \kappa_2]$. If F is a convex function* *on* $[\kappa_1, \kappa_2]$, *then we have the inequalities*

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}F(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}F(\kappa_1)\right] \leq \frac{F(\kappa_1)+F(\kappa_2)}{2}. \tag{1.3}$$

Whereupon the Hermite-Hadamard type inequalities (1.2) which involve left and right sided Riemann-Liouville fractional integrals are given by Sarikaya et al., a great many mathematicians have studied to establish new Hermite-Hadamard type inequalities including various fractional integrals such as k-fractional, Hadamard, Katugampola, Conformable and local fractional integrals. For some of them, please see [1, 3–6, 8, 10, 11, 13, 17–19, 22–24, 26, 33–36, 39, 42–53] and the references included there. Additionally, the interested reader is also referred to [7, 20, 28, 32, 38] for more details about fractional calculus.

In [25], Jleli and Samet proved the following Hermite-Hadamard type inequality:

Theorem 1.6. *Suppose that* $\rho : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is a positive and monotone increasing function on* $(\kappa_1, \kappa_2]$ *such that the derivative* $\rho'(\varkappa)$ *a continuous on* (κ_1, κ_2) *and let* $\alpha > 0$. *If* F *is convex on* $[\kappa_1, \kappa_2]$, *then, for* $\varkappa \in [\kappa_1, \kappa_2]$, *one has*

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4\left[\rho(\kappa_2)-\rho(\kappa_1)\right]^{\alpha}}\left(I^{\alpha}_{\kappa_1+;\rho}\Psi(\kappa_2)+I^{\alpha}_{\kappa_2-;\rho}\Psi(\kappa_1)\right) \leq \frac{F(\kappa_1)+F(\kappa_2)}{2}, \tag{1.4}$$

where

$$\Psi(\varkappa) = F(\varkappa) + F(\kappa_1+\kappa_2-\varkappa). \tag{1.5}$$

The main aim of this work is to establish new Hermite-Hadamard type integral inequalities including generalized fractional integrals for convex mappings. Moreover, some trapezoid and midpoint type inequalities which involve generalized fractional integrals are obtained. We should also note that the results developed in this paper produce the inequalities involving Riemann-Liouville and Hadamard fractional integrals.

2. Generalized Hermite-Hadamard Type Inequalities

In this section, we first define some notations. Supposing that $\rho : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a positive and monotone increasing function on $(\kappa_1, \kappa_2]$ such that the derivative $\rho'(\varkappa)$ is continuous on (κ_1, κ_2) and let $\alpha > 0$.

We define the following positive mapping on $[0,1]$,

$$\Lambda_{\rho}^{\alpha}(\xi):=\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]^{\alpha}+\left[\rho\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}. \tag{2.1}$$

Specifically, if we choose $\xi=1$ in (2.1), we possess the notation

$$\Lambda_{\rho}^{\alpha}(1):=\left[\rho(\kappa_2)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}+\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho(\kappa_1)\right]^{\alpha}.$$

We also consider the identity mapping ℓ instead of ρ (i.e. $\rho(\xi)=\ell(\xi)=\xi$), then we have

$$\Lambda_{\ell}^{\alpha}(1)=\frac{(\kappa_2-\kappa_1)^{\alpha}}{2^{\alpha-1}}. \tag{2.2}$$

Moreover, for $\rho(\xi)=\ln\xi$, one has

$$\Lambda_{\ln}^{\alpha}(\xi):=\left[\ln\left(\frac{\kappa_1+\kappa_2}{(1+\xi)\,\kappa_1+(1-\xi)\kappa_2}\right)\right]^{\alpha}+\left[\ln\left(\frac{(1-\xi)\kappa_1+(1+\xi)\,\kappa_2}{\kappa_1+\kappa_2}\right)\right]^{\alpha}, \tag{2.3}$$

and if we take $\xi=1$ in (2.3), then we possess

$$\Lambda_{\ln}^{\alpha}(1):=\left[\ln\frac{\kappa_1+\kappa_2}{2\kappa_1}\right]^{\alpha}+\left[\ln\frac{2\kappa_2}{\kappa_1+\kappa_2}\right]^{\alpha}.$$

In this section, we examine how Hermite-Hadamard type inequalities come out for convex functions and generalized fractional integrals.

Theorem 2.1. *Assume that $\rho:[\kappa_1,\kappa_2]\to\mathbb{R}$ is a positive and monotone increasing function on $(\kappa_1,\kappa_2]$ such that the derivative $\rho'(\varkappa)$ is continuous on (κ_1,κ_2), and let $\alpha>0$. If F is a convex function on $[\kappa_1,\kappa_2]$, then we have Hermite-Hadamard type inequalities including generalized fractional integrals*

$$\begin{aligned}F\left(\frac{\kappa_1+\kappa_2}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{2\Lambda_{\rho}^{\alpha}(1)}\left[I_{\kappa_1+;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_2-;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&\leq\frac{F(\kappa_1)+F(\kappa_2)}{2},\end{aligned} \tag{2.4}$$

where the mappings Ψ and Λ_{ρ}^{α} are defined as in (1.5) *and* (2.1)*, respectively.*

Proof. Due to the fact that F is a convex mapping on $[\kappa_1,\kappa_2]$, we can write

$$F\left(\frac{\varkappa+y}{2}\right)\leq\frac{F(\varkappa)+F(y)}{2}, \tag{2.5}$$

for $\varkappa, y \in [\kappa_1, \kappa_2]$. Using convexity of $\digamma$ after taking $\varkappa = \frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2$ and $y = \frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2$ for $\xi \in [0,1]$, we find that

(2.6)
$$\begin{aligned}\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) &\leq \frac{1}{2}\digamma\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right) + \frac{1}{2}\digamma\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right) \\ &\leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2}.\end{aligned}$$

After integrating the resulting inequality with respect to ξ over $(0,1)$ after multiplying both sides of (2.6) by

$$\frac{\kappa_2-\kappa_1}{2\Gamma(\alpha)} \frac{\rho'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^{1-\alpha}},$$

then one has

$$\begin{aligned}&\frac{\kappa_2-\kappa_1}{2\Gamma(\alpha)}\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_0^1 \frac{\rho'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^{1-\alpha}}d\xi \\ &\leq \frac{\kappa_2-\kappa_1}{4\Gamma(\alpha)}\int_0^1 \frac{\rho'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^{1-\alpha}} \\ &\quad \times \left[\digamma\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right) + \digamma\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right)\right]d\xi \\ &\leq \frac{\kappa_2-\kappa_1}{2\Gamma(\alpha)}\left[\frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2}\right]\int_0^1 \frac{\rho'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^{1-\alpha}}d\xi.\end{aligned}$$

By change of the variable $u = \frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2$ with $du = -\frac{\kappa_2-\kappa_1}{2}d\xi$, it is observed that

$$\begin{aligned}&\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right)\frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\rho'(u)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(u)\right]^{1-\alpha}}du \\ &\leq \frac{1}{2\Gamma(\alpha)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\rho'(u)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(u)\right]^{1-\alpha}}\left[\digamma(u) + \digamma(\kappa_1+\kappa_2-u)\right]d\xi \\ &\leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2\Gamma(\alpha)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\rho'(u)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(u)\right]^{1-\alpha}}du.\end{aligned}$$

If we use the Definition 1.3 and the integral equality

$$\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\rho'(u)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho(u)\right]^{1-\alpha}}du=\frac{1}{\alpha}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho(\kappa_1)\right]^{\alpha},$$

we establish the

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}&\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho(\kappa_1)\right]^{\alpha} F\left(\frac{\kappa_1+\kappa_2}{2}\right)\\ &\leq \frac{1}{2} I_{\kappa_1+;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\\ &\leq \frac{F(\kappa_1)+F(\kappa_2)}{2\Gamma(\alpha+1)}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho(\kappa_1)\right]^{\alpha}.\end{aligned}\tag{2.7}$$

In a similar way, integrating the resulting inequality with respect to ξ over $(0,1)$ after multiplying both sides of (2.6) by

$$\frac{\kappa_2-\kappa_1}{2\Gamma(\alpha)}\frac{\rho'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{1-\alpha}},$$

we conclude that

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}&\left[\rho(\kappa_2)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha} F\left(\frac{\kappa_1+\kappa_2}{2}\right)\\ &\leq \frac{1}{2} I_{\kappa_2-;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\\ &\leq \frac{F(\kappa_1)+F(\kappa_2)}{2\Gamma(\alpha+1)}\left[\rho(\kappa_2)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}.\end{aligned}\tag{2.8}$$

If we sum the inequalities in (2.7) and (2.8), then we have

$$\begin{aligned}\frac{\Lambda_\rho^\alpha(1)}{\Gamma(\alpha+1)}F\left(\frac{\kappa_1+\kappa_2}{2}\right)&\leq\frac{1}{2}\left[I_{\kappa_1+;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_2-;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\ &\leq\frac{\Lambda_\rho^\alpha(1)}{\Gamma(\alpha+1)}\left[\frac{F(\kappa_1)+F(\kappa_2)}{2}\right],\end{aligned}$$

which completes the proof of the theorem. $\square$

Remark 2.2. If we choose $\rho(\xi)=\xi$ in (2.4), then we possess

$$\begin{aligned}F\left(\frac{\kappa_1+\kappa_2}{2}\right)&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J_{\kappa_1+}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+J_{\kappa_2-}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\ &\leq\frac{F(\kappa_1)+F(\kappa_2)}{2},\end{aligned}$$

which is given by Dragomir in [16].

Corollary 2.3. *Under assumptions of Theorem 2.1 with $\rho(\xi) = \ln \xi$, one has the recent result including Hadamard fractional integrals*

$$\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\Lambda_{\ln}^{\alpha}(1)}\left[\mathbf{J}_{\kappa_1+}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathbf{J}_{\kappa_2-}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2},$$

where the mapping $\Lambda_{\ln}^{\alpha}$ is defined as in (2.3)*.*

3. Generalized Trapezoid Type Inequalities

We present some trapezoid type inequalities involving generalized fractional integrals and their particular results in this section. For this, we first give an equality involving generalized fractional integrals in the following Lemma.

Lemma 3.1. *Assume that the mapping ρ is defined as in Theorem 2.1 and let $\alpha > 0$. If $\digamma : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$, then we possess the identity including generalized fractional integrals as*

$$\begin{aligned} &\frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2}-\frac{\Gamma(\alpha+1)}{2\Lambda_{\rho}^{\alpha}(1)}\times\left[I_{\kappa_2-;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_1+;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \\ &= \frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}\int_0^1 \Lambda_{\rho}^{\alpha}(\xi) \\ &\quad\times\left[\digamma'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-\digamma'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]d\xi, \end{aligned} \tag{3.1}$$

where the mappings Ψ and Λ_{ρ}^{α} are defined as in (1.5) *and* (2.1)*, respectively.*

Proof. By integration by parts, it is easy to see that

$$\begin{aligned} I_1 &= \int_0^1 \left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]^{\alpha} \\ &\quad\times\Psi'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)d\xi \\ &= -\frac{2}{\kappa_2-\kappa_1}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]^{\alpha} \\ &\quad\times\Psi\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\Bigg|_0^1 \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&+\alpha \int_0^1 \frac{\rho'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]^{1-\alpha}} \\
&\times \Psi\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right) d\xi \\
=& -\frac{2}{\kappa_2-\kappa_1}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\kappa_1\right)\right]^{\alpha} \Psi\left(\kappa_1\right) \\
&+\frac{2\alpha}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\rho'(u)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho(u)\right]^{1-\alpha}} \Psi(u)\, d\xi \\
=& \frac{2\Gamma(\alpha+1)}{\kappa_2-\kappa_1} I^{\alpha}_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&-\frac{2}{\kappa_2-\kappa_1}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\kappa_1\right)\right]^{\alpha}\left[F\left(\kappa_1\right)+F\left(\kappa_2\right)\right].
\end{aligned}
$$

Similarly, by integration by parts, it is observed that

$$
\begin{aligned}
I_2 =& \int_0^1 \left[\rho\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha} \\
&\times \Psi'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right) d\xi \\
=& \frac{2}{\kappa_2-\kappa_1}\left[\rho\left(\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}\left[F\left(\kappa_1\right)+F\left(\kappa_2\right)\right] \\
&-\frac{2\Gamma(\alpha+1)}{\kappa_2-\kappa_1} I^{\alpha}_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned} \tag{3.3}
$$

From (3.2) and (3.3), it follows that

$$
\begin{aligned}
\frac{\kappa_2-\kappa_1}{4\Lambda^{\alpha}_{\rho}(1)}(I_2-I_1) =& \frac{F\left(\kappa_1\right)+F\left(\kappa_2\right)}{2}-\frac{\Gamma(\alpha+1)}{2\Lambda^{\alpha}_{\rho}(1)} \\
&\times\left[I^{\alpha}_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I^{\alpha}_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right].
\end{aligned} \tag{3.4}
$$

Also, owing to the fact that $\Psi'(\varkappa)=F'(\varkappa)-F'(\kappa_1+\kappa_2-\varkappa)$, it is found that

$$
\begin{aligned}
I_1 =& \int_0^1 \left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]^{\alpha} \\
&\times\left[F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)-F'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)\right] d\xi
\end{aligned}
$$

and

$$I_2 = \int_0^1 \left[\rho\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right) - \rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^\alpha \times \left[F'\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right) - F'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right] d\xi.$$

Thus, one possesses the integral identity

$$I_2 - I_1 = \int_0^1 \Lambda_\rho^\alpha(\xi)\left[F'\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right) - F'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right] d\xi. \tag{3.5}$$

If we substitute the equality (3.5) in (3.4), then we can readily attain the required identity (3.1). □

Theorem 3.2. *Assume that the mapping ρ is defined as in Theorem 2.1 and let $\alpha > 0$. If $|F'|$is a convex mapping on $[\kappa_1, \kappa_2]$, then we have the trapezoid type inequality including generalized fractional integrals as*

$$\begin{aligned} &\left|\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)} \times \left[I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) + I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left[|F'(\kappa_1)| + |F'(\kappa_2)|\right]\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right| d\xi. \end{aligned} \tag{3.6}$$

where the mappings Ψ and Λ_ρ^α are defined as in (1.5) *and* (2.1)*, respectively.*

Proof. Taking absolute value of both sides of (3.1), we find that

$$\begin{aligned} &\left|\frac{F(\kappa_1)+F(\kappa_2)}{2} - \frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)} \times \left[I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) + I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right| \left|F'\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right)\right| d\xi \\ &\quad + \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right| \left|F'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right| d\xi. \end{aligned} \tag{3.7}$$

Because $|F'|$ is a convex function on $[\kappa_1, \kappa_2]$, it is easy to see that

$$\left|F'\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right)\right| \leq \frac{1-\xi}{2}\left|F'(\kappa_1)\right| + \frac{1+\xi}{2}\left|F'(\kappa_2)\right|,$$

and

$$\left|F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right| \leq \frac{1+\xi}{2}\left|F'(\kappa_1)\right|+\frac{1-\xi}{2}\left|F'(\kappa_2)\right|.$$

Finally, substituting the above inequalities in (3.7), by the elementary analysis operations, it is found that

$$\begin{aligned}&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right]\int_0^1\left|\Lambda_\rho^\alpha(\xi)\right|d\xi,\end{aligned}$$

which finishes the proof. □

Remark 3.3. If we write ξ instead of $\rho(\xi)$ in (3.6), we reach the result

$$\begin{aligned}&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)+J^\alpha_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4(\alpha+1)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right].\end{aligned}$$

Corollary 3.4. *Under assumptions of Theorem 3.2 with $\rho(\xi)=\ln\xi$, one has the inequality involving Hadamard fractional integrals as*

$$\begin{aligned}&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{\Gamma(\alpha+1)}{2\Lambda_{\ln}^\alpha(1)}\left[\mathbf{J}^\alpha_{\kappa_2-}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathbf{J}^\alpha_{\kappa_1+}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4\Lambda_{\ln}^\alpha(1)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right]\int_0^1\left|\Lambda_{\ln}^\alpha(\xi)\right|d\xi,\end{aligned}$$

where the mapping $\Lambda_{\ln}^\alpha$ is defined as in (2.3).

Theorem 3.5. *Suppose that the mapping ρ is defined as in Theorem 2.1 and let $\alpha>0$. If $|F'|^q$, $q>1$, is a convex function on $[\kappa_1,\kappa_2]$ with $\frac{1}{p}+\frac{1}{q}=1$, then we possess the inequality including generalized fractional integrals as*

(3.8)

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I^\alpha_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+;\rho}\Psi(\kappa_2)+I^\alpha_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-;\rho}\Psi(\kappa_1)\right]-F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\ &\quad\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left(\int_0^1\left|\Lambda_\rho^\alpha(\xi)\right|^p d\xi\right)^{\frac{1}{p}} \\ &\qquad\times\left[\left(\frac{|F'(\kappa_1)|^q+3|F'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|F'(\kappa_1)|^q+|F'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right]\end{aligned}$$

$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left(4\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right|^p d\xi\right)^{\frac{1}{p}}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right],$$

where the mappings Ψ *and* Λ_ρ^α *are defined as in* (1.5) *and* (2.1)*, respectively.*

Proof. Applying the well known Hölder's inequality after taking modulus in both sides of (3.1), we find that

(3.9)
$$\begin{aligned}
&\left|\frac{F(\kappa_1)+F(\kappa_2)}{2}-\frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I_{\kappa_2-;\rho}^\alpha\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_1+;\rho}^\alpha\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\quad\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right|\left|F'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)\right|d\xi \\
&\qquad+\frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right|\left|F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right|d\xi \\
&\quad\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left(\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)\right|^q d\xi\right)^{\frac{1}{q}} \\
&\qquad+\frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left(\int_0^1 \left|\Lambda_\rho^\alpha(\xi)\right|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right|^q d\xi\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the fact that $|F'|^q$ is a convex mapping on $[\kappa_1,\kappa_2]$, it is seen that

$$\begin{aligned}
&\int_0^1 \left|F'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)\right|^q d\xi \\
&\quad\leq \int_0^1 \left[\frac{1-\xi}{2}\left|F'(\kappa_1)\right|^q+\frac{1+\xi}{2}\left|F'(\kappa_2)\right|^q\right]d\xi \\
&\quad= \frac{|F'(\kappa_1)|^q+3\,|F'(\kappa_2)|^q}{4},
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\int_0^1 \left|F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right| d\xi \\
&\quad\leq \int_0^1 \left[\frac{1+\xi}{2}\left|F'(\kappa_1)\right|^q+\frac{1-\xi}{2}\left|F'(\kappa_2)\right|^q\right]d\xi \\
&\quad= \frac{3\,|F'(\kappa_1)|^q+|F'(\kappa_2)|^q}{4}.
\end{aligned} \tag{3.11}$$

If we substitute the results (3.10) and (3.11) in (3.9), then we achieve the first inequality in (3.8).

For we to proof the second inequality, assume that $\kappa_{11} = |F'(\kappa_1)|^q$, $\kappa_{21} = 3|F'(\kappa_2)|^q$, $\kappa_{12} = 3|F'(\kappa_1)|^q$ and $\kappa_{22} = |F'(\kappa_2)|^q$. Using the inequalities

$$\sum_{k=1}^{n}(\kappa_{1k}+\kappa_{2k})^s \leq \sum_{k=1}^{n}\kappa_{1k}^s + \sum_{k=1}^{n}\kappa_{2k}^s, \quad 0 \leq s < 1, \tag{3.12}$$

and $1 + 3^{\frac{1}{q}} \leq 4$, then the desired result can be readily attained. □

Remark 3.6. If we take $\rho(\xi) = \xi$ in (3.8), we acquire the inequalities involving Riemann-Liouville fractional integrals as

(3.13)

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}F(\kappa_2) + J^\alpha_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}F(\kappa_1)\right] - F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\
&\quad \leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\qquad \times \left[\left(\frac{|F'(\kappa_1)|^q + 3|F'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|F'(\kappa_1)|^q + |F'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad \leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right].
\end{aligned}$$

Corollary 3.7. *Under assumption of Theorem 3.5 with $\rho(\xi) = \ln\xi$, one possesses the inequalities involving Hadamard fractional integrals as*

(3.14)

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\Lambda^\alpha_{\ln}(1)}\left[\mathbf{J}^\alpha_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}\Psi(\kappa_2) + \mathbf{J}^\alpha_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}\Psi(\kappa_1)\right] - F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\
&\quad \leq \frac{\kappa_2-\kappa_1}{4\Lambda^\alpha_{\ln}(1)}\left(\int_0^1 (\Lambda^\alpha_{\ln}(\xi))^p\,d\xi\right)^{\frac{1}{p}} \\
&\qquad \times \left[\left(\frac{|F'(\kappa_1)|^q + 3|F'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|F'(\kappa_1)|^q + |F'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\quad \leq \frac{\kappa_2-\kappa_1}{4\Lambda^\alpha_{\ln}(1)}\left(4\int_0^1 (\Lambda^\alpha_{\ln}(\xi))^p\,d\xi\right)^{\frac{1}{p}}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right]
\end{aligned}$$

where the expression $\Lambda^\alpha_{\ln}$ is defined as in (2.3).

4. Generalized Midpoint Type Inequalities

In this section, some generalized midpoint type inequalities involving the generalized fractional integrals are established by using the identity given in the following Lemma.

Lemma 4.1. *Supposing that the mapping ρ is defined as in Theorem 2.1 and let $\alpha > 0$. If $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a differentiable mapping on (κ_1, κ_2) with $\kappa_1 < \kappa_2$, then we have the identity including generalized fractional integrals as*

(4.1)
$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) + I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]$$
$$= \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left(\Lambda_\rho^\alpha(1) - \Lambda_\rho^\alpha(\xi)\right)$$
$$\times\left[F'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right) - F'\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right)\right] d\xi,$$

where the mappings Ψ and Λ_ρ^α are defined as in (1.5) *and* (2.1)*, respectively.*

Proof. By integration by parts, it is deduced that

$$J_1 = \int_0^1 \left(\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(\kappa_1)\right]^\alpha - \left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^\alpha\right)$$
$$\times \Psi'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right) d\xi$$
$$= -\frac{2}{\kappa_2-\kappa_1}\left(\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(\kappa_1)\right]^\alpha\right.$$
$$\left. - \left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^\alpha\right)$$
$$\times \Psi\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\Big|_0^1$$
$$-\alpha\int_0^1 \frac{\rho'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right]^{1-\alpha}}\Psi\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right) d\xi$$
$$= \frac{2}{\kappa_2-\kappa_1}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(\kappa_1)\right]^\alpha \Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)$$
$$- \frac{2\alpha}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\rho'(u)}{\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(u)\right]^{1-\alpha}}\Psi(u)\, du$$
$$= \frac{4}{\kappa_2-\kappa_1}\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right) - \rho(\kappa_1)\right]^\alpha F\left(\frac{\kappa_1+\kappa_2}{2}\right)$$

$$-\frac{2\Gamma(\alpha+1)}{\kappa_2-\kappa_1} I^{\alpha}_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right).$$

In a similar way, it is easy to see that

$$\begin{aligned}
J_2 &= \int_0^1 \left(\left[\rho\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}-\left[\rho\left(\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}\right)\\
&\quad\times \Psi'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)d\xi\\
&= \frac{4}{\kappa_2-\kappa_1}\left[\rho\left(\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha} F\left(\frac{\kappa_1+\kappa_2}{2}\right)\\
&\quad -\frac{2\Gamma(\alpha+1)}{\kappa_2-\kappa_1} I^{\alpha}_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned}$$

Then, if we sum the integrals J_1 and J_2 and later we multiply the resulting equality by $\frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}$, then we possess

$$\begin{aligned}
&\frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}\left(J_1+J_2\right)=F\left(\frac{\kappa_1+\kappa_2}{2}\right)\\
&\qquad -\frac{\Gamma(\alpha+1)}{2\Lambda_{\rho}^{\alpha}(1)}\left[I^{\alpha}_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I^{\alpha}_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right].
\end{aligned}\tag{4.2}$$

Also, using the fact that $\Psi'(\varkappa)=F'(\varkappa)-F'(\kappa_1+\kappa_2-\varkappa)$, it is observed that

$$\begin{aligned}
J_1 &= \int_0^1 \left(\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\kappa_1\right)\right]^{\alpha}\right.\\
&\quad \left.-\left[\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)-\rho\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]^{\alpha}\right)\\
&\quad\times\left[F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)-F'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)\right]d\xi,
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
J_2 &= \int_0^1 \left(\left[\rho\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}\right.\\
&\quad \left.-\left[\rho\left(\kappa_2\right)-\rho\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]^{\alpha}\right)\\
&\quad\times\left[F'\left(\frac{1-\xi}{2}\kappa_1+\frac{1+\xi}{2}\kappa_2\right)-F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right]d\xi.
\end{aligned}\tag{4.4}$$

Substituting the identities (4.3) and (4.4) in (4.2), the required equality (4.1) can be easily obtained. □

Theorem 4.2. *Assuming that the mapping ρ is defined as in Theorem 2.1 and let $\alpha > 0$. If $|F'|$ is a convex function on $[\kappa_1, \kappa_2]$, then the following midpoint type inequality involving generalized fractional integrals holds:*

(4.5)
$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) + I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right]\int_0^1 \left|\Lambda_\rho^\alpha(1) - \Lambda_\rho^\alpha(\xi)\right| d\xi,$$

where the mappings Ψ and Λ_ρ^α are defined as in (1.5) and (2.1), respectively.

Proof. If we take modulus in both sides of (4.1), from the absolute value inequality for integrals, then we get

$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) + I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left|\Lambda_\rho^\alpha(1) - \Lambda_\rho^\alpha(\xi)\right| \left|F'\left(\frac{1+\xi}{2}\kappa_1 + \frac{1-\xi}{2}\kappa_2\right)\right| d\xi$$
$$+ \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\int_0^1 \left|\Lambda_\rho^\alpha(1) - \Lambda_\rho^\alpha(\xi)\right| \left|F'\left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2\right)\right| d\xi.$$

Owing to the fact that $|F'|$ is a convex mapping on $[\kappa_1, \kappa_2]$, it follows that

$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2\Lambda_\rho^\alpha(1)}\left[I^\alpha_{\kappa_1+;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right) + I^\alpha_{\kappa_2-;\rho}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)}\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right]\int_0^1 \left|\Lambda_\rho^\alpha(1) - \Lambda_\rho^\alpha(\xi)\right| d\xi,$$

which finishes the proof of the theorem. □

Remark 4.3. Taking $\rho(\xi) = \xi$ in (4.5), we have the inequality

(4.6)
$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left[J^\alpha_{\kappa_1+}F\left(\frac{\kappa_1+\kappa_2}{2}\right) + J^\alpha_{\kappa_2-}F\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{\alpha}{\alpha+1}\right)\left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right|\right].$$

Corollary 4.4. *Under the same assumptions of Theorem 4.2 with $\rho(\xi) = \ln \xi$, the following inequality involving Hadamard fractional integrals* *holds:*

(4.7)

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2\Lambda_{\ln}^{\alpha}(1)}\left[\mathbf{J}_{\kappa_1+}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathbf{J}_{\kappa_2-}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda_{\ln}^{\alpha}(1)}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right]\int_0^1\left|\Lambda_{\ln}^{\alpha}(1)-\Lambda_{\ln}^{\alpha}(\xi)\right|d\xi,$$

where the notation $\Lambda_{\ln}^{\alpha}$ *is defined as in* (2.3).

Theorem 4.5. *Suppose that the mapping* ρ *is defined as in Theorem 2.1 and let* $\alpha>0$. *If* $|F'|^q$, $q>1$, *is a convex function on* $[\kappa_1,\kappa_2]$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following midpoint type inequalities including generalized fractional integrals hold:*

(4.8)

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2\Lambda_{\rho}^{\alpha}(1)}\left[I_{\kappa_1+;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_2-;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}\left(\int_0^1\left|\Lambda_{\rho}^{\alpha}(1)-\Lambda_{\rho}^{\alpha}(\xi)\right|^p d\xi\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3\left|F'(\kappa_1)\right|^q+\left|F'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|F'(\kappa_1)\right|^q+3\left|F'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}\left(4\int_0^1\left|\Lambda_{\rho}^{\alpha}(1)-\Lambda_{\rho}^{\alpha}(\xi)\right|^p d\xi\right)^{\frac{1}{p}}\left[\left|F'(\kappa_1)\right|+\left|F'(\kappa_2)\right|\right],$$

where the mappings Ψ *and* Λ_{ρ}^{α} *are defined as in* (1.5) *and* (2.1), *respectively.*

Proof. Taking absolute value of both sides of (4.1) and later using Hölder's inequality, we conclude that

$$\left|F\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2\Lambda_{\rho}^{\alpha}(1)}\left[I_{\kappa_1+;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+I_{\kappa_2-;\rho}^{\alpha}\Psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq\frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}\left(\int_0^1\left|\Lambda_{\rho}^{\alpha}(1)-\Lambda_{\rho}^{\alpha}(\xi)\right|^p d\xi\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|F'\left(\frac{1+\xi}{2}\kappa_1+\frac{1-\xi}{2}\kappa_2\right)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$+\frac{\kappa_2-\kappa_1}{4\Lambda_{\rho}^{\alpha}(1)}\left(\int_0^1\left|\Lambda_{\rho}^{\alpha}(1)-\Lambda_{\rho}^{\alpha}(\xi)\right|^p d\xi\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&\times \left(\int_0^1 \left| F' \left(\frac{1-\xi}{2}\kappa_1 + \frac{1+\xi}{2}\kappa_2 \right) \right|^q d\xi \right)^{\frac{1}{q}} \\
&\leq \frac{\kappa_2-\kappa_1}{4\Lambda_\rho^\alpha(1)} \left(\int_0^1 \left| \Lambda_\rho^\alpha(1) - \Lambda_\rho^\alpha(\xi) \right|^p d\xi \right)^{\frac{1}{p}} \\
&\times \left[\left(\frac{3\left|F'(\kappa_1)\right|^q + \left|F'(\kappa_2)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left|F'(\kappa_1)\right|^q + 3\left|F'(\kappa_2)\right|^q}{4} \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

The proof of the first inequality in (4.8) is thus completed.

Using the inequality (3.12) for $\kappa_{11} = 3\left|F'(\kappa_1)\right|^q$, $\kappa_{21} = \left|F'(\kappa_2)\right|^q$, $\kappa_{12} = \left|F'(\kappa_1)\right|^q$ and $\kappa_{22} = 3\left|F'(\kappa_2)\right|^q$, form $1 + 3^{\frac{1}{q}} \leq 4$, the second inequality in (4.8) can be readily deduced. The proof is thus completed. $\square$

Remark 4.6. Under the same assumptions of Theorem 4.5 with $\rho(\xi) = \xi$, the following inequalities involving Hadamard fractional integrals hold:

(4.9)

$$
\begin{aligned}
&\left| F\left(\frac{\kappa_1+\kappa_2}{2} \right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha} \left[J_{\kappa_1+}^\alpha F\left(\frac{\kappa_1+\kappa_2}{2} \right) + J_{\kappa_2-}^\alpha F\left(\frac{\kappa_1+\kappa_2}{2} \right) \right] \right| \\
&\leq \frac{\kappa_2-\kappa_1}{4} \left(\int_0^1 (1-\xi^\alpha)^p d\xi \right)^{\frac{1}{p}} \\
&\times \left[\left(\frac{3\left|F'(\kappa_1)\right|^q + \left|F'(\kappa_2)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left|F'(\kappa_1)\right|^q + 3\left|F'(\kappa_2)\right|^q}{4} \right)^{\frac{1}{q}} \right] \\
&\leq \frac{\kappa_2-\kappa_1}{4^{\frac{1}{q}}} \left(\int_0^1 (1-\xi^\alpha)^p d\xi \right)^{\frac{1}{p}} \left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_2)\right| \right].
\end{aligned}
$$

Corollary 4.7. *Choosing $\rho(\xi) = \ln \xi$ in (4.8), then one has the inequalities*

(4.10)

$$
\begin{aligned}
&\left| F\left(\frac{\kappa_1+\kappa_2}{2} \right) - \frac{\Gamma(\alpha+1)}{2\Lambda_{\ln}^\alpha(1)} \left[\mathbf{J}_{\kappa_1+}^\alpha \Psi\left(\frac{\kappa_1+\kappa_2}{2} \right) + \mathbf{J}_{\kappa_2-}^\alpha \Psi\left(\frac{\kappa_1+\kappa_2}{2} \right) \right] \right| \\
&\leq \frac{\kappa_2-\kappa_1}{4\Lambda_{\ln}^\alpha(1)} \left(\int_0^1 \left| \Lambda_{\ln}^\alpha(1) - \Lambda_{\ln}^\alpha(\xi) \right|^p d\xi \right)^{\frac{1}{p}} \\
&\times \left[\left(\frac{3\left|F'(\kappa_1)\right|^q + \left|F'(\kappa_2)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left|F'(\kappa_1)\right|^q + 3\left|F'(\kappa_2)\right|^q}{4} \right)^{\frac{1}{q}} \right]
\end{aligned}
$$

$$\leq \frac{\kappa_2 - \kappa_1}{4\Lambda_{\ln}^{\alpha}(1)} \left(4 \int_0^1 \left| \Lambda_{\ln}^{\alpha}(1) - \Lambda_{\ln}^{\alpha}(\xi) \right|^p d\xi \right)^{\frac{1}{p}} \left[\left| F'(\kappa_1) \right| + \left| F'(\kappa_2) \right| \right]$$

where the notation $\Lambda_{\ln}^{\alpha}$ *is defined as in* (2.3).

5. Conclusions

In this paper, we present several generalized inequalities for convex functions via generalized fractional integrals. It is also shown that the results given here are the strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study and obtain similar inequalities for different kinds of convexity in their next works.

Acknowledgment. The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

1. G.A. Anastassiou, *General fractional Hermite–Hadamard inequalities using m-convexity and (s, m)-convexity*, Front. Time Scal. Ineq., 1 (2016), pp. 237-255.
2. M.A. Ardıc, A.O. Akdemir and E. Set, *On New Integral Inequalities via Geometric-Arithmetic Convex Functions with Applications*, Sahand Commun. Math. Anal. 19(2) (2022), pp. 1-14.
3. H. Budak and M.Z. Sarikaya, *Hermite-Hadamard type inequalities for s-convex mappings via fractional integrals of a function with respect to another function*, Fasc. Math., 27 (2016) pp.25-36, 2016.
4. H. Budak, *On Fejér type inequalities for convex mappings utilizing fractional integrals of a function with respect to another function*, Result. Math., 74(1) (2019), 29.
5. H. Budak, *New Hermite-Hadamard type inequalities for convex mappings utilizing generalized fractional integrals*, Filomat, 33(8) (2019), pp. 2329-2344.
6. H. Budak, H. Kara, R Kapucu, *New midpoint type inequalities for generalized fractional integral*, Comput. Methods Differ. Equ., 10(1) (2022), pp. 93-108.
7. H. Budak, C.C. Bilişik, M.Z. Sarikaya, *On Some New Extensions of Inequalities of Hermite-Hadamard Type for Generalized Fractional Integrals*, Sahand Commun. Math. Anal., 19(2) (2022), pp. 65-79.
8. S.I. Butt, M. Umar, S. Rashid, A. O. Akdemir and Y. M. Chu, *New Hermite–Jensen–Mercer-type inequalities via k-fractional integrals*, Adv. Difference Equ., 1 (2020), pp. 1-24.

9. B. Celik, M.E. Ozdemir, A.O. Akdemir and E. Set, Integral Inequalities for Some Convexity Classes via Atangana-Baleanu Integral Operators, TJOI, 5(2) (2021), 82-92.
10. H. Chen and U.N. Katugampola, *Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for generalized fractional integrals*,J. Math. Anal. Appl. 446 (2017), pp. 1274-1291
11. Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1 (2010), pp. 51-58.
12. B. Daraby, . *Generalizations of Some Inequalities for Sugino Integrals*, Sahand Commun. Math. Anal., 19(3) (2022), pp. 141-168.
13. J. Deng and J. Wang, *Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions*. J. Inequal. Appl. 2013 (2013), art. 364.
14. S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monog., Vic. Univ., 2000.
15. S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., 11(5) (1998), pp. 91-95.
16. S.S. Dragomir, *Some inequalities of Hermite-Hadamard type for symmetrized convex functions and Riemann-Liouville fractional integrals,* RGMIA Res. Rep. Coll., 20 (2017), 15.
17. S. Erden, *Weighted inequalities involving conformable integrals and its applications for random variable and numerical integration*, Filomat 34(8) 2020, pp. 2785-2796.
18. A. Ekinci, M. Ozdemir, *Some new integral inequalities via Riemann-Liouville integral operators*, Appl. Comput. Math., 18(3) (2019), pp. 288-295.
19. G. Farid, A. ur Rehman and M. Zahra, *On Hadamard type inequalities for k-fractional integrals*, Konuralp J. Math., 4(2) 2016, 79-86.
20. R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.
21. J. Hadamard, *Etude sur les proprietes des fonctions entieres en particulier d'une fonction consideree par Riemann*, J. Math. Pures Appl. 58 (1893), pp. 171-215.
22. M. Iqbal, M.I. Bhatti and K. Nazeer, *Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc., 52(3) (2015), pp. 707-716.

23. İ. İşcan and S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Compt., 238 (2014), pp. 237-244.
24. I. Iscan, *Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals*, Stud. Univ. Babeș-Bolyai, Math., 60(3) (2015), pp. 355-366
25. M. Jleli and B. Samet, *On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function*, J. Nonlinear Sci. Appl., 9 (2016), pp. 1252-1260.
26. U.N. Katugampola, *New approach to a generalized fractional integrals*, Appl. Math. Comput., 218(4) (2011), pp. 860-865.
27. M.A. Khan, T.U. Khan, *Parameterized Hermite-Hadamard Type Inequalities For Fractional Integrals*, TJOI, 1(1), 2017, 26–37.
28. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
29. U.S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput., 147(5) (2004), pp. 137-146.
30. A.R. Khan, H. Nasir and S.S. Shirazi, *Weighted Cebysev Type Inequalities for Double Integrals and Application*, Sahand Commun. Math. Anal., 18(4) (2021), pp. 59-72.
31. M. Kunt, İ. İşcan, *Fractional Hermite–Hadamard–Fejer type inequalities for GA-convex functions*, TJOI, 2(1), 1–20, 2018.
32. S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, pp.2.
33. S. Mubeen, S. Iqbal and M. Tomar, *On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function and k-parameter*, J. Inequal. Math. Appl., 1 (2016), pp. 1-9.
34. M.A. Noor and M.U. Awan, *Some integral inequalities for two kinds of convexities via fractional integrals*, TJMM, 5(2) (2013), pp. 129-136.
35. M.E. Ozdemir, *New Refinements of Hadamard Integral inequlaity via k-Fractional Integrals for p-convex function*, Turkish J. Science, 6(1) (2021), pp. 1-5.
36. M.E. Özdemir, M. Avcı-Ardınç and H. Kavurmacı-Önalan, *Hermite-Hadamard type inequalities fors-convex ands-concave functions via fractional integrals*, Turkish J. Science, 1 (2016), pp. 28-40.
37. J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press,

Boston, 1992.

38. I. Podlubni, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
39. S. Peng, W. Wei and J-R. Wang, *On the Hermite-Hadamard inequalities for convex functions via Hadamard fractional integrals*, Facta Univ., Ser. Math. Inf., 29(1) (2014), pp. 55-75.
40. M.Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes, 7(2) (2016), pp. 1049-1059.
41. M.Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, *Hermite - Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Modelling, 57 (2013), pp. 2403-2407.
42. M.Z. Sarikaya and H. Budak, *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat, 30(5) (2016), pp. 1315-1326.
43. M.Z. Sarikaya, A. Akkurt , H. Budak, M.E. Yildirim and H. Yildirim, *Hermite-hadamard's inequalities for conformable fractional integrals*. RGMIA Res. Rep. Col., 19(83), (2016).
44. E. Set, M.Z. Sarikaya, M.E. Ozdemir and H. Yildirim, *The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results*, JAMSI, 10(2) (2014), pp. 69-83.
45. E. Set, M.Z. Sarikaya, M.E. Ozdemir and H. Yildirim, *The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results*, JAMSI, 10(2) (2014).
46. E. Set, J. Choi and B. Çelik, *New Hermite-Hadamard type inequalities for product of different convex functions involving certain fractional integral operators,* J. Math. and Comp. Sci., 18(1) (2018), pp. 29-36
47. E. Set, A. Gozpnar, A. Ekinci, *Hermite-Hadamard type inequalities via confortable fractional integrals,* Acta Math. Univ. Comen., 86 (2017), art. 309320.
48. J. Wang, X. Li, M. Fečkan, Y. Zhou, *Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals via two kinds of convexity*, Appl. Anal., 92(11) (2012), pp. 2241–2253.
49. J.R. Wang, X. Li, C. Zhu, *Refinements of Hermite-Hadamard type inequalities involving fractional integrals*, Bull. Belg. Math. Soc. Simon Stevin, 20 (2013), pp. 655-666.
50. J.R. Wang, C. Zhu, Y. Zhou, *New generalized Hermite–Hadamard type inequalities and applications to special means*, J. Inequal. Appl. 2013 (2013), art. 325.

51. Y. Zhang and J. Wang, *On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals,* J. Inequal. Appl. 2013 (2013), art. 220.
52. Z. Zhang, W. Wei, J. Wang, *Generalization of Hermite-Hadamard inequalities involving Hadamard fractional integrals*, Filomat, 29(7) (2015), pp. 1515-1524.
53. Z. Zhang, J.R. Wang and J.H. Deng, *Applying GG-convex function to Hermite-Hadamard inequalities involving Hadamard fractional integrals*, Int. J. Math. Comput. Sci., 2014 (2014), art. 136035.

---

[1] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Türkiye.
Email address: hasan64kara@gmail.com

[2] Department of Mathematics, Faculty of Science, Bartin-Türkiye.
Email address: erdensmt@gmail.com

[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Türkiye.
Email address: hsyn.budak@gmail.com