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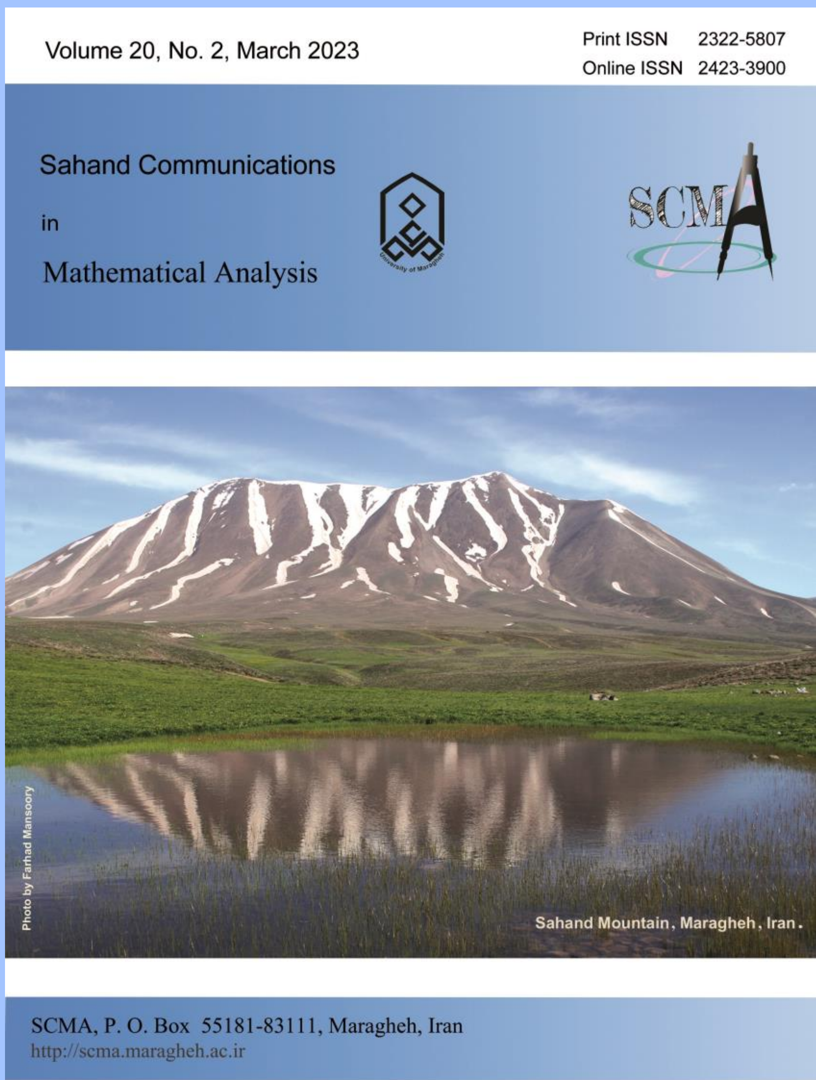

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Maps Completely Preserving the Quadratic Operators

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a bijective map. In this paper, we show that ϕ is completely preserving quadratic operator in both directions if and only if ϕ is 2-quadratic preserving operator in both directions and if and only if ϕ is either an isomorphism or (in the complex case) a conjugate isomorphism.

1. Introduction

The study of maps on operator algebras preserving certain properties or subsets is a subject that has attracted the attention of many mathematicians. To get acquainted with this topic and as an introduction to it, you can see reference [8] and references inside it. Recently, some of these problems have been related to the completely preserving of certain properties or subsets of operators. For example see [2–7].

Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a Banach space $\mathcal{X}$. Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is quadratic if there exist two complex numbers $a, b \in \mathbb{C}$, such that $(T - aI)(T - bI) = 0$.

Let $\phi : \mathcal{M} \to \mathcal{N}$ be a map, where $\mathcal{M}$ and $\mathcal{N}$ are linear spaces. Define for each $n \in \mathbb{N}$, a map $\phi_n : \mathcal{M} \otimes M_n \to \mathcal{N} \otimes M_n$ by

$$\phi_n \left((a_{ij})_{n\times n} \right) = \left(\phi(a_{ij}) \right)_{n\times n} .$$

Then ϕ is said to be n-quadratic preserving if ϕ_n preserves quadratic operators. ϕ is said to be completely quadratic preserving if ϕ_n preserves quadratic operators for each $n \in \mathbb{N}$.

In [2], authors characterized completely rank-nonincreasing linear maps and then later extended on [3]. Completely invertibility preserving maps

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were characterized in [5]. Subsequently, in [6] completely idempotent and completely square-zero preserving maps and in [7] completely commutativity and completely Jordan zero product preserving maps were discussed.

Recall that a standard operator algebra on $\mathcal{X}$ is a norm closed subalgebra of $\mathcal{B}(\mathcal{X})$ which contains the identity and all finite rank operators. Our main results is as follows.

Theorem 1.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be infinite dimensional Banach spaces and $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a bijective map. Then the following statements are equivalent:*

(1) *ϕ is completely preserving quadratic operator in both directions.*
(2) *ϕ is 2-quadratic preserving operators in both directions.*
(3) *There exist a bounded invertible linear or (in the complex case) conjugate- linear operator $A : \mathcal{X} \to \mathcal{Y}$ and a scalar λ such that*

$$\phi(T) = \lambda ATA^{-1}, \quad \text{for all } T \in \mathcal{A}.$$

Theorem 1.2. *Let $\phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ $(n \geq 3)$ be a bijective map, where $\mathbb{F}$ is the real or complex field. Then the following statements are equivalent:*

(1) *ϕ is completely preserving quadratic operator in both directions.*
(2) *ϕ is 2-quadratic preserving operator in both directions.*
(3) *There exist an invertible matrix $A \in \mathcal{M}_n$, a scalar λ and an automorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that*

$$\phi(T) = \lambda AT_\tau A^{-1}, \quad \text{for all } T \in \mathcal{M}_n(\mathbb{F}).$$

Here $T_\tau = (\tau(t_{ij}))$ for $T = (t_{ij})$.

2. Proofs

Denote by $\mathcal{X}^*$ the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. The rank one operator $x \otimes f$ is idempotent (resp. nilpotent) if and only if $f(x) = 1$ (resp. $f(x) = 0$). Let P and Q be two idempotent operators. We say that P and Q are orthogonal if and only if $PQ = QP = 0$. Let $\mathcal{A}$ be a standard operator algebra on $\mathcal{X}$. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of all idempotent in $\mathcal{A}$. In order to prove the main results, we need some auxiliary lemmas.

Lemma 2.1. *$\phi(0) = 0$ and $\phi(\mathbb{C}I) = \mathbb{C}I$, specially $\phi(I) = \lambda I$ for some constant λ.*

Proof. Let $c \in \mathbb{C}$ be a nonzero number and T be an arbitrary operator. Set

$$A = \begin{pmatrix} cI & T \\ 0 & 0 \end{pmatrix}.$$

It is clear that $A\left(A - \begin{pmatrix} cI & 0 \\ 0 & cI \end{pmatrix}\right) = 0$ and then there exist constants a, b such that $\left(\phi_2(A) - \begin{pmatrix} aI & 0 \\ 0 & aI \end{pmatrix}\right)\left(\phi_2(A) - \begin{pmatrix} bI & 0 \\ 0 & bI \end{pmatrix}\right) = 0$. This implies

$$\begin{pmatrix} \phi(cI) - aI & \phi(T) \\ \phi(0) & \phi(0) - aI \end{pmatrix}\begin{pmatrix} \phi(cI) - bI & \phi(T) \\ \phi(0) & \phi(0) - bI \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and hence

$$(\phi(cI) - aI)(\phi(cI) - bI) + \phi(T)\phi(0) = 0, \tag{2.1}$$

$$(\phi(cI) - aI)\phi(T) + \phi(T)(\phi(cI) - bI) = 0. \tag{2.2}$$

Surjectivity of ϕ yields that there exist $T_0, T_1 \in \mathcal{A}$ such that $\phi(T_0) = 0$ and $\phi(T_1) = I$. Taking $T = T_0$ in Equation (2.1) yields that $(\phi(cI) - aI)(\phi(cI) - bI) = 0$ and then $\phi(T)\phi(0) = 0$. Taking $T = T_1$ in last equation yields $\phi(0) = 0$. Again taking $T = T_1$ in Equation (2.2) implies $\phi(cI) - aI + \phi(cI) - bI = 0$ and therefore $\phi(cI) = \frac{a+b}{2}I$. Thus $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$. Since ϕ is bijective and ϕ^{-1} has all properties of ϕ, then we can conclude that $\phi(\mathbb{C}I) = \mathbb{C}I$. □

Remark 2.2. It is clear that $\lambda^{-1}\phi$ satisfies assumptions on ϕ and also by previous lemma is unital. Thus without loss of generality, in the following lemmas, we assume that $\phi(I) = I$.

Lemma 2.3. *ϕ preserves the idempotent operators in both directions.*

Proof. Let P be an idempotent. If $P = 0$ or I, then by previous lemma and remark, the assertion is true. So let P be a non-trivial idempotent. It is clear that $R = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$ is idempotent. So by assumption and Lemma 2.1, there exist constants a_1, b_1 such that

$$\left(\phi_2(R) - \begin{pmatrix} a_1 I & 0 \\ 0 & a_1 I \end{pmatrix}\right)\left(\phi_2(R) - \begin{pmatrix} b_1 I & 0 \\ 0 & b_1 I \end{pmatrix}\right) = 0.$$

Then

$$\begin{pmatrix} \phi(P) - a_1 I & 0 \\ 0 & -a_1 I \end{pmatrix}\begin{pmatrix} \phi(P) - b_1 I & 0 \\ 0 & -b_1 I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies

$$(\phi(P) - a_1 I)(\phi(P) - b_1 I) = 0,$$

$$a_1 b_1 I = 0.$$

These two relations imply

$$\phi(P)^2 = (a_1 + b_1)\phi(P). \tag{2.3}$$

It is clear that $S = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$ is idempotent. So by assumption and Lemma 2.1, there exist constants a_2, b_2 such that

$$\left(\phi_2(S) - \begin{pmatrix} a_2 I & 0 \\ 0 & a_2 I \end{pmatrix}\right)\left(\phi_2(S) - \begin{pmatrix} b_2 I & 0 \\ 0 & b_2 I \end{pmatrix}\right) = 0.$$

Then

$$\begin{pmatrix} \phi(P) - a_2 I & 0 \\ 0 & (1-a_2)I \end{pmatrix}\begin{pmatrix} \phi(P) - b_2 I & 0 \\ 0 & (1-b_2)I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies

$$(\phi(P) - a_1 I)(\phi(P) - b_1 I) = 0, \tag{2.4}$$

$$(1-a_2)(1-b_2)I = 0. \tag{2.5}$$

(2.4) together with (2.3) implies

$$(a_2 + b_2 - a_1 - b_1)\phi(P) = a_2 b_2 I. \tag{2.6}$$

We assert that $a_2+b_2-a_1-b_1 = 0$. Otherwise, $\phi(P)$ is a nonzero multiple of identity which Lemma 2.1 yields that P is a nonzero multiple of identity. This is a contradiction, because P is a non-trivial idempotent.

Thus $a_2 + b_2 - a_1 - b_1 = 0$ and then by (2.6), $a_2 b_2 = 0$. So from (2.5), $a_2 + b_2 = 1$ which implies $a_1 + b_1 = 1$. This and (2.3) follow the idempotency of $\phi(P)$. Since ϕ is bijective and ϕ^{-1} has all properties of ϕ, then we can conclude that ϕ preserves the idempotent operators in both directions. □

Lemma 2.4. *ϕ preserves the orthogonality of idempotent operators in both directions.*

Proof. Let P and Q be non-trivial orthogonal idempotents. By Lemmas 2.1 and 2.3, $\phi(P)$ and $\phi(Q)$ are non-trivial idempotents. So it is enough to show that $\phi(P)\phi(Q) = \phi(Q)\phi(P) = 0$. It is cleat that $R^2 - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 0$, where $R = \begin{pmatrix} I & P \\ Q & -I \end{pmatrix}$. By Lemma 2.1, there exist, an scalar α such that $\phi(-I) = \alpha$. So by assumption and Lemma 2.1, there exist constants a, b such that

$$\left(\phi_2(R) - \begin{pmatrix} aI & 0 \\ 0 & aI \end{pmatrix}\right)\left(\phi_2(R) - \begin{pmatrix} bI & 0 \\ 0 & bI \end{pmatrix}\right) = 0.$$

Then

$$\begin{pmatrix}(1-a)I & \phi(P)\\ \phi(Q) & (\alpha-a)I\end{pmatrix}\begin{pmatrix}(1-b)I & \phi(P)\\ \phi(Q) & (\alpha-b)I\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

which implies

$$(1-a)(1-b)I+\phi(P)\phi(Q)=0, \tag{2.7}$$

$$(\alpha-a)(\alpha-b)I+\phi(Q)\phi(P)=0. \tag{2.8}$$

Multiplying (2.7) from left by $\phi(P)$, we see that

$$(1-a)(1-b)\phi(P)+\phi(P)\phi(Q)=0.$$

This equation together with (2.7) entails

$$(1-a)(1-b)\phi(P)=(1-a)(1-b)I.$$

If $(1-a)(1-b)\neq 0$, then $\phi(P)=I$ which is a contradiction. Hence $(1-a)(1-b)=0$ and then by (2.7), $\phi(P)\phi(Q)=0$. Using the same method on equation (2.8), follows $\phi(Q)\phi(P)=0$. Thus ϕ preserves the orthogonality of idempotents. Since ϕ is bijective and ϕ^{-1} has all properties of ϕ, we can conclude that ϕ preserves the orthogonality of idempotents in both directions. □

Lemma 2.5. *$\phi(P)=APA^{-1}$, for every $P\in\mathcal{P}_{\mathcal{A}}$, where $A:X\longrightarrow Y$ is a bijective bounded linear operator.*

Proof. Lemma 2.4 implies that ϕ is a bijection preserving the orthogonality of idempotents in both directions. It follows from lemma 3.1 in [10] that there exists a bijective bounded linear or (in the complex case) conjugate linear operator $A:X\longrightarrow Y$ such that

$$\phi(P)=APA^{-1},\quad (P\in\mathcal{P}_{\mathcal{A}})$$

or a bijective bounded linear or (in the complex case) conjugate linear operator $A:X'\longrightarrow Y$ such that

$$\phi(P)=AP^{*}A^{-1},\quad (P\in\mathcal{P}_{\mathcal{A}}).$$

We show that the second case can not occur. Assume on the contrary that $\phi(P)=AP^{*}A^{-1}$ for all $P\in\mathcal{P}_{\mathcal{A}}$. Let $x,y\in X$ be linearly independent vectors. So there exist $f,g\in X'$ such that $f(x)=1$,$f(y)=-1$ and $g(x)=g(y)=1$. Hence $R^2-\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}=0$, where $R=\begin{pmatrix}I-x\otimes f & x\otimes g\\ -y\otimes f & I-y\otimes g\end{pmatrix}$. Thus there exist constants a,b such that

$$\left(\phi_2(R)-\begin{pmatrix}aI & 0\\ 0 & aI\end{pmatrix}\right)\left(\phi_2(R)-\begin{pmatrix}bI & 0\\ 0 & bI\end{pmatrix}\right)=0.$$

It is clear that $I - x\otimes f$, $x\otimes g$, $-y\otimes f$ and $I-y\otimes g$ are idempotents and so

$$\begin{aligned}&\begin{pmatrix}A(I-f\otimes x)A^{-1}-aI & A(g\otimes x)A^{-1}\\ -A(f\otimes y)A^{-1} & A(I-g\otimes y)A^{-1}-aI\end{pmatrix}\\ &\quad\times\begin{pmatrix}A(I-f\otimes x)A^{-1}-bI & A(g\otimes x)A^{-1}\\ -A(f\otimes y)A^{-1} & A(I-g\otimes y)A^{-1}-bI\end{pmatrix}\\ &=\begin{pmatrix}0&0\\0&0\end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}&\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}(1-a)I-f\otimes x & g\otimes x\\ -f\otimes y & (1-a)I-g\otimes y\end{pmatrix}\begin{pmatrix}A^{-1}&0\\0&A^{-1}\end{pmatrix}\\ &\quad\times\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}(1-b)I-f\otimes x & g\otimes x\\ -f\otimes y & (1-b)I-g\otimes y\end{pmatrix}\begin{pmatrix}A^{-1}&0\\0&A^{-1}\end{pmatrix}\\ &=\begin{pmatrix}0&0\\0&0\end{pmatrix}\end{aligned}$$

which implies

$$\begin{aligned}&\begin{pmatrix}(1-a)I-f\otimes x & g\otimes x\\ -f\otimes y & (1-a)I-g\otimes y\end{pmatrix}\\ &\quad\times\begin{pmatrix}(1-b)I-f\otimes x & g\otimes x\\ -f\otimes y & (1-b)I-g\otimes y\end{pmatrix}\\ &=\begin{pmatrix}0&0\\0&0\end{pmatrix}.\end{aligned}$$

One of the equations that can be obtained from this relation is the following equation:

$$((1-a)I-f\otimes x)((1-b)I-f\otimes x)-(g\otimes x)(f\otimes y)=0.$$

This implies

$$(a+b-1)f\otimes x+g\otimes y+(a-1)(b-1)I=0,$$

and then

$$\begin{aligned}&(a+b-1)x\otimes f+y\otimes g+(a-1)(b-1)I=0\\ \Rightarrow\quad&(a+b-1)(x\otimes f)x+(y\otimes g)x+(a-1)(b-1)x=0\\ \Rightarrow\quad&abx+y=0,\end{aligned}$$

which is a contradiction, because x and y are linearly independent. Therefore, the second case can not occur and the proof is completed. □

Lemma 2.6. *For every operator T, there exists a complex number λ_T such that $\phi(T)=\lambda_T T$.*

Proof. By Lemma 2.5, $\phi(P) = APA^{-1}$, for every idempotent operator P, where $A : X \longrightarrow Y$ is a bijective bounded linear operator. It is trivial that without loss of generality, we can suppose that $\phi(P) = P$ for all $P \in \mathcal{P}_{\mathcal{A}}$.

For every $T \in \mathcal{A}$ we have

$$\begin{pmatrix} T & I \\ I - T^2 & -T \end{pmatrix}^2 - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 0.$$

Thus there exist constants a, b such that

$$\begin{aligned} &\left(\phi_2\left(\begin{pmatrix} T & I \\ I - T^2 & -T \end{pmatrix}\right) - \begin{pmatrix} aI & 0 \\ 0 & aI \end{pmatrix}\right) \\ &\quad \times \left(\phi_2\left(\begin{pmatrix} T & I \\ I - T^2 & -T \end{pmatrix}\right) - \begin{pmatrix} bI & 0 \\ 0 & bI \end{pmatrix}\right) \\ &= 0. \end{aligned}$$

One of the equations that can be obtained from this relation is the following equation:

$$\phi(-T) + \phi(T) = (a + b)I, \tag{2.9}$$

which implies $a + b = 0$, because otherwise by Lemma 2.1, there exists a nonzero scalar t such that $\phi(tI) = (a + b)I$. By replacing tI instead T in equation (2.9), we get $\phi(-tI) = 0$ which implies $t = 0$, a contradiction. Therefore

$$\phi(-T) = -\phi(T). \tag{2.10}$$

On the other hand, for any $T, S \in \mathcal{A}$ we have

$$\begin{pmatrix} I - TS & -T \\ STS - 2S & ST - I \end{pmatrix}^2 - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 0.$$

Hence there exist constants c, d such that

$$\begin{aligned} &\left(\phi_2\left(\begin{pmatrix} I - TS & -T \\ STS - 2S & ST - I \end{pmatrix}\right) - \begin{pmatrix} cI & 0 \\ 0 & cI \end{pmatrix}\right) \\ &\quad \times \left(\phi_2\left(\begin{pmatrix} I - TS & -T \\ STS - 2S & ST - I \end{pmatrix}\right) - \begin{pmatrix} dI & 0 \\ 0 & dI \end{pmatrix}\right) \\ &= 0. \end{aligned}$$

One of the equations that can be obtained from this relation is the following equation:

$$[\phi(I - TS) - cI]\phi(-T) + \phi(-T)[\phi(ST - I) - dI] = 0.$$

Let T be an operator and x be an arbitrary nonzero vector of X such that $x \notin \ker T$. So there exists a nonzero function f such that $f(Tx) = 1$. If $S = x \otimes f$, then $I - TS$ is an idempotent and so $\phi(I - TS) = I - TS$

and also by (2.10), $\phi(TS - I) = TS - I$. Setting $S = x \otimes f$ in previous equation and using (2.10) yields

$$([(1-c)I - Tx \otimes f]\phi(T) + \phi(T)[x \otimes fT - (1+d)I] = 0.$$

Then

$$(c+d)\phi(T) + Tx \otimes f\phi(T) - \phi(T)x \otimes fT = 0,$$

which putting $T = I$ follows $c + d = 0$ and then

$$Tx \otimes f\phi(T) = \phi(T)x \otimes fT. \tag{2.11}$$

This implies that $\phi(T)x$ and Tx are linearly dependent for all $x \in X$ such that $x \notin \ker T$. So it is clear that $\phi(T)x$ and Tx are linearly dependent for all $x \in X$. Thus we can conclude from Theorem 2.3 in [1] that there exists either a complex number λ_T such that $\phi(T) = \lambda_T T$ for all non-rank one operator T or $h, g \in X'$ such that $T = x \otimes h$ and $\phi(T) = x \otimes g$. By placing in equation (2.11), we have

$$x \otimes g = g(x)f(x)x \otimes h,$$

which implies $g = g(x)f(x)h$ and then for all operator T, there exists a complex number λ_T such that $\phi(T) = \lambda_T T$. □

Proof of Theorem 1.1. By Lemma 2.6, it is enough to prove that $\lambda_T = 1$. If T is equal to 0, I or an idempotent, then by Lemmas 2.1 and 2.5, we know that $\lambda_T = 1$.

For any $T, S \in \mathcal{A}$ we have

$$\begin{pmatrix} TS & T \\ -STS & -ST \end{pmatrix}^2 = 0,$$

and then there exists a, b such that

$$\begin{pmatrix} \phi(TS) - aI & \phi(T) \\ \phi(-STS) & \phi(-ST) - aI \end{pmatrix} \begin{pmatrix} \phi(TS) - bI & \phi(T) \\ \phi(-STS) & \phi(-ST) - bI \end{pmatrix} = 0,$$

which by Lemma 2.6 and Equation (2.10)

$$\begin{pmatrix} \lambda_{TS}TS - aI & \lambda_T T \\ -\lambda_{STS}STS & -\lambda_{ST}ST - aI \end{pmatrix} \begin{pmatrix} \lambda_{TS}TS - bI & \lambda_T T \\ -\lambda_{STS}STS & -\lambda_{ST}ST - bI \end{pmatrix} = 0.$$

This implies

$$(\lambda_{TS}TS - aI)(\lambda_{TS}TS - bI) - \lambda_T\lambda_{STS}TSTS = 0, \tag{2.12}$$

and

$$\lambda_T(\lambda_{TS}TS - aI)T - \lambda_T T(\lambda_{ST}ST + bI) = 0. \tag{2.13}$$

Suppose $x \in \mathcal{X}$ and $f \in \mathcal{X}'$ are arbitrary elements. Hence there exists $y \in \mathcal{X}$ such that y is linearly independent of x and $f(y) = 1$ and also there exists $g \in \mathcal{X}'$ such that $g(x) = 1$ and $g(y) = 1$. Setting $T = x \otimes f$ and $g = y \otimes g$ in Equation (2.12), we get

$$(\lambda_{x\otimes g} x \otimes g - aI)(\lambda_{x\otimes g} x \otimes g - bI) - \lambda_{x\otimes f}\lambda_{y\otimes g} x \otimes g = 0.$$

Then

$$(1 - a - b - \lambda_{x\otimes f})x \otimes g + abI = 0,$$

which implies

$$ab = 0, \qquad \lambda_{x\otimes f} = 1 - a - b. \tag{2.14}$$

Now with the same placement, that is $T = x \otimes f,\ g = y \otimes g$, in Equation (2.13), we get

$$\lambda_{x\otimes f}(\lambda_{x\otimes g} x \otimes g - aI)x \otimes f - \lambda_{x\otimes f} x \otimes f(\lambda_{y\otimes f} y \otimes f + bI) = 0.$$

Then

$$[\lambda_{x\otimes f}(1 - a) - \lambda_{x\otimes f}(1 + b)]x \otimes f = 0,$$

which due to being non-zero of $\lambda_{x\otimes f}$, we have

$$a = -b. \tag{2.15}$$

This together with $ab = 0$ from (2.14) implies $a = b = 0$ and then again by using (2.14), we obtain $\lambda_T = 1$ for $T = x \otimes f$.

Suppose $0 \neq T \in \mathcal{A}$ and $x \in \mathcal{X}$ are arbitrary elements. Thus there exists a function f such that $f(Tx) = 1$. Setting $S = x \otimes f$ in Equation (2.12), we get

$$(Tx \otimes f - aI)(Tx \otimes f - bI) - \lambda_T Tx \otimes f = 0.$$

Then

$$(1 - a - b - \lambda_T)Tx \otimes f + abI = 0,$$

which implies

$$ab = 0, \qquad \lambda_{x\otimes f} = 1 - a - b. \tag{2.16}$$

Setting $S = x \otimes f$ in Equation (2.13), we get

$$\lambda_T(Tx \otimes f - aI)T - \lambda_T T(x \otimes fT + bI) = 0$$

$$\Rightarrow -\lambda_T(a + b)T = 0,$$

which due to being nonzero of λ_T, we have

$$a = -b. \tag{2.17}$$

This together with $ab = 0$ from (2.16) implies $a = b = 0$ and then again by using (2.16), we obtain $\lambda_T = 1$ for $T \neq 0$. The proof is complete.

Proof of Theorem 1.2. Using the result in [9] concerning characterizing maps on idempotent matrices, the assertion can be proved by a similar argument as in the proof of Theorem 1.2. □

References

1. M. Brešar and P. Šemrl, *On locally linearly dependent operators and derivation*, Trans. Am. Math. Soc., 351 (1999), pp. 1257-1275.
2. D. Hadwin and D. Larson, *Completely rank nonincreasing linear maps*, Ann. Funct. Anal., 199 (2003), pp. 263-277.
3. D. Hadwin, J. Hou and H. Yousefi, *Completely rank-nonincreasing linear maps on spaces of operators*, Linear Algebra Appl., 383 (2004), pp. 213-232.
4. R. Hosseinzadeh, I. Sharifi and A. Taghavi, *Maps completely preserving fixed points and maps completely preserving kernel of operators*, Anal. Math., 44 (2018), pp. 451-459.
5. J. Hou and L. Huang, *Characterizations of isomorphisms: maps completely preserving invertibility or spectrum*, J. Appl. Math. Anal. Appl., 359 (2009), pp. 81-87.
6. J. Hou and L. Huang, *Maps completely preserving idempotents and maps completely preserving square-zero operators*, Isr. J. Math., 176 (2010), pp. 363-380.
7. L. Huang, Y. Liu, *Maps completely preserving commutativity and maps completely preserving jordan-zero products*, Isr. J. Math., 462 (2014), pp. 233-249.
8. C.-K. Li, N.-K. Tsing, *Linear preserver problems: a brief introduction and some special techniques*, Linear Algebra Appl., 162-164 (1992) 217-235.
9. P. G. Ovchinnikov,, *Automorphisms of the poset of skew projections*, J. Funct. Anal., 115 (1993), 184-189.

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