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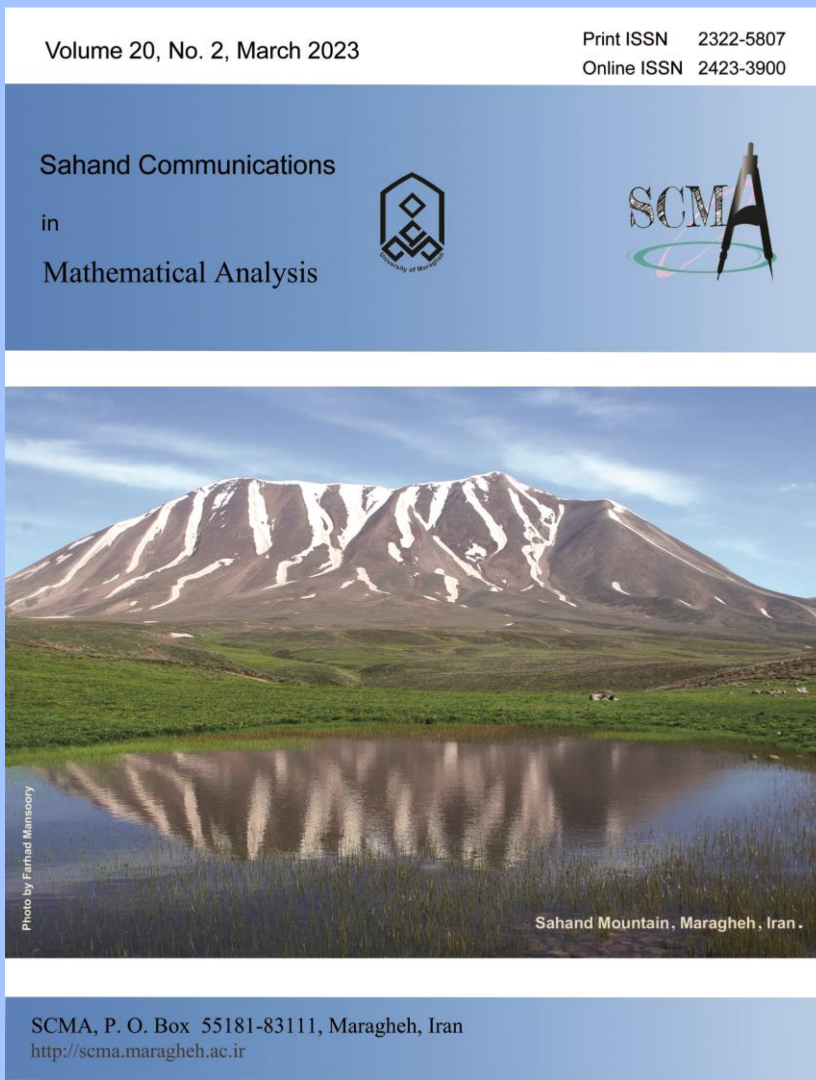

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New Iteration Algorithms for Solving Equilibrium Problems and Fixed Point Problems of Two Finite Families of Asymptotically Demicontractive Multivalued Mappings

Imo Kalu Agwu[1*] and Donatus Ikechi Igbokwe[2]

Abstract. ln this paper, we introduce a new class of mapping called asymptotically demicontractive multivalued mapping in the setting of a real Hilbert space. Furthermore, a new iteration scheme was constructed and it was proved that our algorithm converges strongly to the common element of solutions of an equilibrium problem and the set of common fixed points of two finite families of type-one asymptotically demicontractive multivalued mappings without any sum conditions imposed on the finite family of the control sequences. Also, we provided a numerical example to demonstrate the implementablity of our proposed iteration technique. Our results improve, extend and generalize many recently announced results in the current literature.

1. Introduction

Many physical phenomena of the type

$$\Gamma(\omega) = \xi,$$

arising from physical formulations can equivalently be transformed into a fixed point problem of the form

$$\Gamma(\omega) = \omega. \tag{1.1}$$

The solution of (1.1) can be achieved using approximate fixed point theorem. In 1922, Banach demonstrated a remarkable conclusion for fixed point theory on the metric space, which was later called Banach

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contraction principle. Subsequently, several notable works have been published about fixed point theory for different kinds of contractions on different spaces, with real life applications cutting across different disciplines; see, for example, [51–54] and the references contained in them.

Since fixed point theory for multivalued mappings has invaluable applications in different fields, it is natural to extend the known results for single-valued mappings to the setup of multivalued mappings.

In 1969, Nadler [24] introduced multivalued contraction mapping and proved corresponding convergence theorem. Later on, a lot of very interesting results have been obtained in this direction: Markin [22] introduce the idea of using the Hausdorff metric to study the fixed points for multivalued contraction and nonexpansive mappings, Hu et al [38] proved the convergence theorem for finding common fixed points of two multivalued nonexpansive mappings satisfying certain contractive conditions, Bunyawat and suntain [34] introduced an iterative scheme for finding common fixed point of a countable family of multivalued quasi-nonexpansive mappings in uniformly convex Babcn space, Isogugu [17] introduced type-one multivalued mapping which guarantees strong convergence without the condition that the fixed point set is stirct, Agwu and Igbokwe [55] proved convergence theorem for finding common element of solution for minimization problems and fixed point problems of asymptotically quasi-nonexpansive multivalued mappings etc. But what captures the interest of the authors, basically because of the introduction of a new scheme that solves the problem of 'sum condition' (i.e., for any given nonnegative sequence $\left\{\{\alpha_{in}\}_{i=1}^{N}\right\}_{n=1}^{\infty}$ of real numbers, $\sum\limits_{i=1}^{m}\alpha_{in}=1$, where $m\in N$), is the following theorems:

Theorem 1.1 ([40])**.** *Let K be a nonempty convex and closed subset of a real Hilbert space H. Suppose that $\{S_i\}_{i=1}^{\mathbb{N}}, \mathbb{N}\geq 2$ is a countable family of type-one demicontractive mappings $S_i : K\to\mathcal{P}(K)$ from K into the family of all proximinal subsets of K with contractive coeficient $\lambda_i\in[0,1)$ for each i. Suppose that $\cap_{i=1}^{\mathbb{N}}F(S_i)\neq\emptyset$ and for each i, $(I-S_i)$ is weakly demiclosed at zero; then, unsder appropriate conditions on the control sequence, the sequence $\{x_n\}$ defined by*

$$x_{n+1}=c_{n,1}x_n+\sum_{i=1}^{\mathbb{N}}c_{n,i}\prod_{j=1}^{i-1}(1-c_{n,j})\,y_{n,i-1}+\prod_{j=1}^{\mathbb{N}}(1-c_{n,j})\,y_{n,\mathbb{N}},\quad n\geq 1,$$

where $y_{n,i}\in S_ix_n$ for each i, converges weakly to $q\in\cap_{i=1}^{\mathbb{N}}F(S_i)$. Also, if in addition, S_i is L-Lipschitizian and satisfies condition (I) (see Definition 2.8) for each i, then $\{x_n\}$ converges strongly to $q\in\cap_{i=1}^{\mathbb{N}}F(S_i)$.

Theorem 1.2 ([40]). *Let C be a nonempty convex and closed subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ a bifunction satisfying $(B1) - (B4)$ and $\{T_i\}_{i=1}^{\mathbb{N}}$ be such that $T_{i:C\to\mathcal{P}(C)}$ is type-one λ_i-strictly pseudocontractive-type mappings and $(I - T_i)$ is weakly demiclosed at zero for each $i = 1, 2, \ldots, \mathbb{N}$. Suppose $F = \cap_{i=1}^{\mathbb{N}} F_s(T_i) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from arbitrary $x_0 \in C$ as follows:*

$$\begin{cases} y_n = c_{n,1}x_n + A_n, \\ u_n \in K : F(u_n, y) + B_n, \\ x_{n+1} = \frac{1}{2}(u_n + x_n), \end{cases} \tag{1.2}$$

where

$$A_n = \sum_{i=1}^{\mathbb{N}} c_{n,i} \prod_{j=1}^{i-1} (1 - c_{n,j})\, y_{n,i-1} + \prod_{j=1}^{\mathbb{N}} (1 - c_{n,j})\, y_{n,\mathbb{N}},$$

$$B_n = \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K.$$

Let T_i satisfies condition ???? and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, under appropriate conditions on the control sequences, $\{x_n\}$ converges strongly to $p \in F$.

The iteration schemes defined by (1.2) and (1.2) were recently introduced by Isogogu, Izuchukwu and Okeke [40].

Further studies, in line with constructing iteration schemes for the case in which more than one auxiliary maps are used, of the modified version of the lshikawa iterative scheme are developed; and it has been noted (see, for example, [32] for details) that those iteration schemes for which more than one multivalued mapping is used as auxiliary mapping are more robust against certain numerical errors than the ones that involve only one auxiliary mapping. Thus, the following natural question arises:

Question 1.3. Is it possible to construct an iteration scheme with more than one auxiliary mappings that would guarantee strong convergence for multivalued mappings larger than the one considered in [40]?

Moltivated and inspired by the above works, the purpose of this paper if of three folds:

- (a) To introduce a new class of multivalued mapping called strict asymptotically demicontractive multivalued mapping in the setting of a real Hilbert space;
- (b) To give an affirmative answer to Question 1.3 above and

(c) To prove strong convergence theorem for the solution of equilibrium problems and fixed point problems for a finite family of asymptotically demicontractive multivalued type-one mappings in the setting of a real Hilbert space.

2. Relevant Preliminaries

In the sequel, we shall need the following concepts and known results: Let K be a convex, closed and nonempty subset of a real Hilbert space H with the inner product $\langle , . , \rangle$ and the norm $\|.\|$. In this paper, the set of natural numbers shall be denoted as $\mathbb{N}$, the set of real numbers shall be denoted as $\mathbb{R}$ and if $\{x_n\}_{n=1}^{\infty}$ is a sequence in H, then weak and strong convergence of $\{x_n\}_{n=1}^{\infty}$ shall be denoted as $\rightharpoonup$ and $\rightarrow$, respectively.

Let $\Gamma, G : K \rightarrow K$ be two nonlinear mappings.The set of fixed points of Γ and G shall be denoted by $F(\Gamma)$ and $F(G)$, respectivly. We shall denote the set of common fixed point of Γ and G by $\mathcal{F} = \{y \in K : y \in \Gamma \cap G\}$.

Definition 2.1. Recall that Γ is called:

(a) k-strictly asymptotically pseudocontractive if there exist sequence $\{k_n\}_{n=0}^{\infty} \subseteq [1, \infty)$ with $\lim\limits_{x\to\infty} k_n = 1$ and a constant $k \in [0, 1)$ such that for all $x, y \in K$, we have

$$\|\Gamma^n x - \Gamma^n y\|^2 \leq k_n^2 \|x - y\|^2 + k \|(I - \Gamma^n)\, x - (I - \Gamma^n)\, y\|^2 , \quad n \geq 1. \tag{2.1}$$

The class of mapping defined by (2.1) properly contains the class of asymptotically nonexpansive mapping(where Γ is called asymptotically nonexpansive if for all $x, y \in K$, there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim\limits_{x\to\infty} k_n = 1$ such that $\|\Gamma^n x - \Gamma^n y\| \leq k_n \|x - y\|, \forall n \geq 1)$ introduced by Goebel and Kirk [1]; in fact, every asymptotically nonexpansive mapping is 0-strictly asymptotically pseudocontractive.

(b) β-strictly asymptotically pseudononspreading [46] if there exists a sequence $\{k_n\}_{n=0}^{\infty} \subseteq [1, \infty)$ with $\lim\limits_{x\to\infty} k_n = 1$ such that for all $x, y \in K$ and for some $\beta \in [0, 1)$, we have

$$\begin{aligned} \|\Gamma^n x - \Gamma^n y\|^2 \leq k_n^2 \|x - y\|^2 &+ \beta \|(I - \Gamma^n)\, x - (I - \Gamma^n)\, y\|^2 \\ &+ 2 \langle x - \Gamma^n x, y - \Gamma^n y \rangle , \quad n \geq 1. \end{aligned} \tag{2.2}$$

The class of asymptotically nonspreading type (where Γ is said to be asymptotically nonspreading type if $\forall x, y \in K$, we have $\|\Gamma^n x - \Gamma^n y\|^2 \leq \|x - y\|^2 + 2 \langle x - \Gamma^n x, y - \Gamma^n y \rangle , n \geq 1)$ is a subclass of the class mapping defined by (2.2). In [48] and [49], Osilike and Chima [48] (respectively Osilike et al [49]) showed that the classes of β-strictly pseudononspreading (where a nonlinear map Γ is called β-strictly pseudononspreading if for all $x, y \in K$, there exists some $\beta \in [0,1)$ such that $\|\Gamma x - \Gamma y\|^2 \leq \|x-y\|^2 + \beta \|(I-\Gamma)x - (I-\Gamma)y\|^2 + 2\langle x - \Gamma x, y - \Gamma y\rangle$) and β-strictly asymptotically pseudononspreading (respectively k-strictly pseudocontractive and k-strictly asymptotically pseudocontractive, where a nonlinear map Γ is called k-strictly pseudocontractive if for all $x, y \in K$, there exists some $k \in [0,1)$ such that $\|\Gamma x - \Gamma y\|^2 \leq \|x-y\|^2 + k\|(I-\Gamma)x - (I-\Gamma)y\|^2$) mappings are independent.

Remark 2.2. Note that if $F(\Gamma) \neq \emptyset$, then (2.1) and (2.2) coincide. The coincidence gives birth to a map called asymptotically demicontractive mapping and is defined as follows:

Definition 2.3. Let K and H retain their usual meaning. A mapping Γ is called asymptotically demicontractive if, for any $(x \times q) \in K \times F(\Gamma)$, there exist a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and a constant $\delta \in [0,1)$ such that

$$\|\Gamma^n x - \Gamma^n q\|^2 \leq k_n^2 \|x-q\|^2 + \delta\|x - \Gamma^n x\|, \quad n \geq 1. \tag{2.3}$$

This class of map was introduced by Qihou [47], contains the class of mapping studied in [12] and has been extensively studied in literature (see, e.g,. [32, 39] and the reference therein).

Let $\Psi : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem for Ψ (for short EP) is to find $\omega \in K$ such that

$$\Psi(\omega, y) \geq 0, \quad \forall y \in K. \tag{2.4}$$

A point $z \in K$ solving problem (2.4) is called equilibrium problem. The set of solution of problem (2.4) is denoted by EP(Ψ); that is,

$$EP(\Psi) = \{\omega \in K : \Psi(\omega, y) \geq 0, \forall y \in K\}. \tag{2.5}$$

Due to the indispensable nature of equilibrium problems, different techniques and algorithms have been employed to analyze the existence and approximation of a solution to problem (2.4); see [25]. In about 40 years or so, many researchers studied the problems of finding a common element of solution of equilibrium problems and the common element of the set of fixed points of nonexpansive mappings and some of their generalizations in the setting of a real Hilbert space; see for example, [25–28, 33–35, 40, 43, 45] and the reference therein.

Let B be a strong positive bounded linear operator on a real Hilbert space; that is, there exists a constant $\bar{\alpha} > 0$ such that

$$\langle By, y\rangle \geq \bar{\alpha}\|y\|^2, \quad \forall y \in H.$$

The problem of interest is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping τ on a real Hilbert space:

$$\min_{s\in F(\tau)} \frac{1}{2}\left\langle Bs, s\right\rangle - \left\langle s, b\right\rangle,$$

where b is a given point in H.

Moltivated by the work of Xu [29], Marino and Xu [6] studied the following iteration scheme for solving a fixed point problem of nonexpansive mapping using viscosity iteration scheme due to Moudafi [7]:

$$s_{n+1} = \alpha_n \gamma g(s_n + (1-\alpha_n A)\tau s_n, \tag{2.6}$$

where g is a contraction and τ is a nonexpansive mapping. They showed that under mild condions on the control sequences, $\{s_n\}$ converges to the unique solution of the variational inequality:

$$\langle (B-\gamma g)s^\star, s - s^\star\rangle \geq 0, \quad \forall s \in F(\tau), \tag{2.7}$$

which is the optimality condition for the minimization problem:

$$\min_{s\in F(\tau)} \frac{1}{2}\left\langle Bs, s\right\rangle - \ell(s),$$

where ℓ is a potential function for γg (i.e., $\ell'(s) = \gamma g(s), \forall s \in H$).

In all, it is not difficult to see that approximation of fixed points for nonlinear single-valued mappings is easier to handle as compared to the corresponding multivalued nonlinear mappings. However, many authors have intensively studied the fixed point theorems for multivalued mappings possibly because of their invaluable applications in the areas of control theory, economics, convex optimisation, variational inequalities and differential inclusions (see [19–24] and the reference therein for details).

Let Z be a normed space. A subset K of Z is called proximinal if for each $x \in Z$, there exists a point $t \in K$ such that

$$\begin{aligned} \rho(x,K) &= \inf\{\|x-y\| : y \in K\} \\ &= \rho(x,t). \end{aligned} \tag{2.8}$$

It is well known that a closed and convex subset of a uniformly convex Banach space and a weakly compact convex subset of a Banach space is proximinal.

We denote $CB(Z)$, $C(K)$ and $\mathcal{P}(K)$ as the family of nonempty bounded closed subsets of Z, the family of nonempty compact subsets of K and the family of nonempty bounded proximinal subsets of k, respectively. The Hausdorff metric induced by the metric ρ of Z for all

$A, B \in CB(Z)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A}\rho(x,B), \sup_{y\in B}\rho(y,A)\right\}, \tag{2.9}$$

where $\rho(x,B) = \inf\{\|x-y\| : y \in B\}$ is the distance from the point x to the subset B. A point $q \in K$ is called a fixed point of the mulrivalued mapping Γ if $q \in \Gamma q$. The set of fixed points of Γ is denoted by $F(\Gamma) = \{x \in K : x \in \Gamma x\}$.

Definition 2.4. Let $\Gamma : D(\Gamma) \subseteq Z \to 2^Z$ be a multivalued mapping. Then Γ is said to be:

(1) uniformly η-Lipschitizian if there exists $\eta \geq 0$ such that

$$H(\Gamma^n x, \Gamma^n y) \leq \eta \|x-y\|, \quad \forall x,y \in D(\Gamma). \tag{2.10}$$

If $\eta = k_n$ in (2.10), where $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{n\to\infty} = 1$, then Γ is called asymptotically nonexpansive multivalued mapping.

(2) type-one [17] if given a pair $x, y \in D(\Gamma)$, then

$$\|u-v\| \leq H(\Gamma x, \Gamma y), \quad \forall u \in P_\Gamma x, v \in P_\Gamma y, \tag{2.11}$$

where $P_\Gamma x = \{u \in \Gamma x : \|x-u\| = \rho(x, \Gamma x)\}$.

(3) μ-strictly asymptotically pseudocontractive in the sense of Qihou [47] for single-valued mapping if there exist a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim\limits_{x\to\infty} k_n = 1$ and a constant $k \in [0,1)$ such that given any pair $x, y \in D(\Gamma)$ and $u \in \Gamma^n x$, there exists $v \in \Gamma^n y$ satisfying $\|u-v\| \leq H(\Gamma^n x, \Gamma^n y)$ and

$$H(\Gamma^n x, \Gamma^n y)^2 \leq k_n \|x-y\| + \mu\|x-u-(y-v)\|^2. \tag{2.12}$$

If $k = 1$ in (2.12), then Γ is called asymptotically pseudocontractive; whereas Γ is 0-strictly pseudocontractive (or asymptotically nonexpansive) if $k = 0$ in (2.12).

Definition 2.5. Let $G : D(G) \subseteq Z \to 2^Z$ be a multivalued mapping. Then G is said to be:

(a) asymptotically β-nonspreading if there exists $\beta > 0$ such that

$$H(G^n x, G^n y)^2 \leq \beta\left(\rho(G^n x, y)^2 + \rho(x, G^n y)^2\right), \quad \forall x, y \in K.$$

Note that G is called asymptotically nonspreading-type if $\beta = \frac{1}{2}$; that is,

$$2H(G^n x, GT^n y)^2 \leq \rho(G^n x, y)^2 + \rho(x, G^n y)^2, \quad \forall x, y \in K.$$

It is easy to see that if G is an asymptotically nonspreading-type and $F(G) \neq \emptyset$, then T is asymptotically quasi-nonexpansive mapping.

(b) β-strictly asymptotically pseudononspreading in the sense of Zhaoli and Wang [46] for single-valued mapping if there exist a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{x\to\infty} k_n = 1$ and a constant $\beta \in [0,1)$ such that given any pair $x, y \in D(G)$ and $u \in G^n x$, there exists $v \in G^n y$ satisfying $\|u - v\| \leq H(G^n x, G^n y)$ and

(2.13)
$$H(G^n x, G^n y)^2 \leq k_n \|x - y\| + \beta \|x - u - (y - v)\|^2 + 2\langle x - u, y - v\rangle .$$

If $\beta = 1$ in (2.13), then G is called asymptotically pseudononspreading; whereas G is 0-strictly pseudononspreading if $\beta = 0$ in (2.13).

Definition 2.6 ([17])**.** Let X be a Banach space and $S : D(S) \subseteq X \to 2^X$ be a multivalued mapping. $I - S$ is weakly demiclosed at zero if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(S)$ such that $\{x_n\}$ converges weakly to q and a sequence y_n with $y_n \in Sx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ strongly converges to zero. Then, $q \in Sq(i.e., 0 \in (I - S)q)$.

Definition 2.7 ([17])**.** Let X be a normed space and $S : D(S) \subseteq X \to 2^X$ be a multivalued map. S is of type-one is given any $x, y \in D(S)$, then $\|u - v\| \leq H(Sx, Sy), \forall u \in P_S x, v \in P_S y$.

Definition 2.8. [40] A multivalued mapping $\Gamma : K \longrightarrow P(K)$ is said to satisfy Condition (I) (see, for example, [40]) if there exists a nondecreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in [0,\infty)$ such that $d(x, \Gamma x) \geq f(x, F(\Gamma))$, $\forall x \in K$.

Lemma 2.9 ([6])**.** *Asumme that A is a strongly positive self adjoint bounded linear operator on H with coeficient $\bar{\alpha} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|1 - \rho A\| \leq 1 - \rho\bar{\alpha}$.*

Lemma 2.10 ([39])**.** *Let H be a real Hilbert space. Then*
$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle , \quad \forall x, y \in H.$$

Lemma 2.11 (see [29])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers with $a_{n+1} = (1 - \alpha_n)a_n + b_n, n \geq 0$, where α_n is a sequence in $(0,1)$ and b_n is a sequence in $\mathbb{R}$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim\sup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.12 ([30])**.** *For each $x_1, x_2, \dots, x_m$ and $\alpha_1, \alpha_2, \dots, \alpha_m \in [0,1]$ with $\sum_{i=1}^{m} \alpha_i = 1$, we have*

(2.14)
$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{0\leq i<j\leq m} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.13 ([31])**.** *Let $\{w_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{w_{n_j}\}$ of w_n such that $w_{n_j} < w_{n_j+1}$ for all $j \geq 0$. For every $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as $\tau(n) = \max\{k \leq n : w_n < w_{n+1}\}$. Then, $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0, \max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}$.*

For solving the equilirium problem, we assume that the ifunction $\Psi :$ $K \times K$ satisfies the following conditions:

$(B1)$ $\Psi(x,x) = 0, \quad \forall x \in K$;
$(B2)$ Ψ is monotone, i.e, $\Psi(x,y) + \Psi(y,x) \leq 0, \quad \forall x,y \in K$;
$(B3)$ Ψ is upper hemicontinous, i.e., for each $x,y,z \in K$,

$$\limsup_{t \to 0^+} \Psi(tz + (1-t)x, y) \leq \Psi(x,y);$$

$(B4)$ $\Psi(x,.)$ is convex and lower semicontinuoua for each $x \in K$.

Lemma 2.14 ([44])**.** *Let K be a nonempty closed and convex subset of a real Hilbert space H and let Ψ be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying $(B1) - (B4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that*

$$\Psi(x,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K. \tag{2.15}$$

Lemma 2.15 ([25])**.** *Assume that $\Psi : K \times K$ satisfies $(B1)-(B4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ in the following way:*

$$T_r x = \left\{ z \in K : \Psi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K \right\}.$$

Then, we have

(i) T_r is single-valued;
(ii) T_r is firmly nonexpansive, i.e., for any $x,y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(\Psi)$;

Proposition 2.16 ([40])**.** *Let $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where k is a fixed nonnegative integer and $N \in \mathbb{N}$ is any integer with $k+1 \leq N$. Then, the following holds:*

$$\alpha_k + \sum_{i=k+1}^{N} \left(\alpha_i \prod_{j=k}^{i-1} (1-\alpha_j) \right) + \prod_{j=k}^{N} (1-\alpha_j) = 1. \tag{2.16}$$

Proposition 2.17 ([40])**.** *Let t, u and v be arbitrary elements of a real Hilbert space H. Let k be any fixed nonnegetive integer and $N \in \mathbb{N}$ be* *such that $k+1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0,1]$ be a countable finite subset of H and $\mathbb{R}$, respectively. Define*

$$y = \alpha_k + \sum_{i=k+1}^{N} \left(\alpha_i \prod_{j=k}^{i-1} (1-\alpha_j) v_{i-1} \right) + \prod_{j=k}^{N} (1-\alpha_j) v.$$

Then,

$$\begin{aligned} \|y-u\|^2 = \alpha_k \|t-u\|^2 &+ \sum_{i=k+1}^{N} \left(\alpha_i \prod_{j=k}^{i-1} (1-\alpha_j) \|v_{i-1}-u\|^2 \right) \\ &+ \prod_{j=k}^{N} (1-\alpha_j) \|v-u\|^2 \\ &- \alpha_k \left[\sum_{i=k+1}^{N} \left(\alpha_i \prod_{j=k}^{i-1} (1-\alpha_j) \|t-v_{i-1}\|^2 \right) + \prod_{j=k}^{i-1} (1-\alpha_j) \|t-v\|^2 \right] \\ &- (1-\alpha_k) \left[\sum_{i=k+1}^{N} \left(\alpha_i \prod_{j=k}^{i-1} (1-\alpha_j) \|v_{i-1} - (\alpha_{i+1} + w_{i+1})\|^2 \right) \right. \\ &\left. + \alpha_N \prod_{j=k}^{i-1} (1-\alpha_j) \|v - v_{N-1}\|^2 \right], \end{aligned}$$

where

$$w_k = \sum_{i=k+1}^{N} \left(\alpha_i \prod_{j=k}^{i-1} (1-\alpha_j) v_{i-1} \right) + \prod_{j=k}^{i-1} (1-\alpha_j) v, \quad k = 1, 2, \ldots, N$$

and $w_n = (1-c_n)v$.

3. Results

In this section, we prove our main results. In the sequel, we provide the following definition:

Definition 3.1. Let Z be a normed space and $\Gamma : D(\Gamma) \subseteq Z \to 2^Z$ be a multivalued mapping. Then, Γ is called asymptotically demicontrctive if $F(\Gamma) \neq \emptyset$ and Definition 2.4 (3) and Definition 2.5 (b) hold; that is, Γ is asymptotically demicontractive in the sense of Qihou [47] for single-valued mapping if $F(\Gamma) \neq \emptyset$ and for all $(x \times q) \in D(\Gamma) \times F(\Gamma)$ and $\mu \in [0,1)$, there exist a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{x\to\infty} k_n = 1$ and $u \in \Gamma^n x$ such that

$$H(\Gamma^n x, \Gamma^n q)^2 \leq k_n^2 \|x-q\|^2 + \mu \|x-u\|^2. \tag{3.1}$$

Remark 3.2. If $F(\Gamma) \neq \emptyset$ and $F(G) \neq \emptyset$ in Definition 2.4 and Definition 2.5, then (3) in Definition 2.4 and (b) in Definition 2.5 reduce to Definition 3.1. Thus, the class of asymptotically demicontractive mapping is larger than the classes of k-strictly asymptotically pseudocotractive multivalued mapping and β-strictly pseudononspreading multivalued mapping.

Example 3.3 ([41])**.** Let $X = \mathbb{R}$ be endowed with the usual metric. Define $\Gamma : [-9, 3) \to \mathcal{P}(Z)$ by

$$\Gamma x = \begin{cases} \left[-\frac{1}{3}x, -\frac{1}{2}x\right], & x \in (-3, 0], \\ \left[-\frac{1}{2}x, -\frac{1}{3}x\right], & x \in [0, 3). \end{cases}$$

Then, Γ is uniformly $\dfrac{1}{2}-$Lipschitizian and asymptotically demicontractive mapping with $F(\Gamma) = \{0\}$. Indeed, for all $x \in (-3, 0]$, we have

$$H(\Gamma^n x, 0)^2 = \max\left\{\left|\frac{1}{3^n}x\right|^2, \left|\frac{1}{2^n}x\right|^2\right\} \tag{3.2}$$

$$= \left|\frac{1}{2}x\right|^2$$

$$\leq \left(1 + \frac{1}{2^n}\right)|x - 0|^2 + |x - 0|^2.$$

Again, for $u_n \in \Gamma^n x, u_n = -\delta^n x,\ \textit{where}\ \dfrac{1}{3^n} \leq \delta^n \leq \dfrac{1}{2^n}$, we have

$$|x - u_n|^2 = |x + \delta x|^2 = (1 + \delta^n)^2 |x - 0|^2. \tag{3.3}$$

(3.2) and (3.3) imply that

$$\begin{aligned} H(\Gamma^n x, 0)^2 &\leq \left(1 + \frac{1}{2^n}\right)|x - 0|^2 + \frac{1}{(1 + \delta^n)^2}|x - u_n|^2 \\ &\leq \left(1 + \frac{1}{2^n}\right)|x - 0|^2 + \frac{1}{\left(1 + \frac{1}{3}\right)^2}|x - u_n|^2 \\ &= \left(1 + \frac{1}{2^n}\right)|x - 0|^2 + \frac{9}{16}|x - u_n|^2. \end{aligned}$$

For $x \in [0, 3)$, the result follows as in Example 4.1 in [41] with $i = 1$. Hence, Γ is an asymptotically demicontractive multivalued mapping.

Now, we show that Γ is uniformly L-Lipschitizian. Indeed, for $x, y \in (-3, 0]$, we have

$$H(\Gamma^n x, \Gamma^n y) = \max\left\{\left|-\frac{1}{3^n}x + \frac{1}{3^n}\right|, \left|-\frac{1}{2^n}x + \frac{1}{2^n}\right|\right\}$$

$$
\begin{aligned}
&= \frac{1}{2^n}\,|x-y| \\
&\le \frac{1}{2}\,|x-y|\,;
\end{aligned}
$$

For $x \in (-3, 0]$ and $y \in [0, 3)$, we have

$$
\begin{aligned}
H(\Gamma^n x, \Gamma^n y) &= \max\left\{\left|-\frac{1}{3^n}x + \frac{1}{2^n}\right|, \left|-\frac{1}{2^n}x + \frac{1}{3^n}\right|\right\} \\
&= \left|-\left(\frac{x}{2} - \frac{y}{3}\right)\right| \\
&= \left|\frac{x}{2} - \frac{y}{3}\right| \\
&\le \left|\frac{x}{2} - \frac{y}{2}\right| \\
&= \frac{1}{2}\,|x-y|\,.
\end{aligned}
$$

and for $x, y \in [0, 3)$, we have

$$
\begin{aligned}
H(\Gamma^n x, \Gamma^n y) &= \max\left\{\left|-\frac{1}{2^n}x + \frac{1}{2^n}\right|, \left|-\frac{1}{3^n}x + \frac{1}{3^n}\right|\right\} \\
&= \frac{1}{2^n}|x-y| \\
&\le \frac{1}{2}|x-y|.
\end{aligned}
$$

In all cases, Γ is uniformly $\frac{1}{2}-$ Lipschitizian.

Theorem 3.4. *Let K be a convex, closed and nonempty subset of a real Hilbert space H and let Ψ be a bifunction of $K \times K$ into $\mathbb{R}$. Suppose that $\{\Gamma_i\}_{i=1}^{\mathbb{N}}$ and $\{G_i\}_{i=1}^{\mathbb{N}}$,$\mathbb{N} \ge 2$ are two finite families of type-one and L_i-uniformly Lipschitizian asymptotically demicontractive multivalued mappings $\Gamma_i : K \to \mathcal{P}(K)$ and $G_i : K \to \mathcal{P}(K)$, respectively from K into the family of all proximinal subsets of K with contractive coeficient $\mu_i^{(1)}, \mu_i^{(2)} \in [0.1)$ for each i. Assume that $\mathcal{F} = \left(\cap_{i=1}^{\mathbb{N}} F(\Gamma_i)\right) \cap \left(\cap_{i=1}^{\mathbb{N}} F(G_i)\right) \cap EP(\Theta) \neq \emptyset$ and for each i, $(I - \Gamma_i)$ and $(I - G_i)$ are weakly demiclosed at zero. let g_i be a contraction of K into itself with constant $\rho \in (0, 1)$ and A be a strong positive self adjoint bounded linear operator on H with coeficient $\bar{\alpha}$ such that $0 < \rho\gamma + 2\epsilon < \bar{\alpha}$.*

Let the sequence $\{x_n\}$ *be given iteratively as follows:*

$$(3.4)\qquad \begin{cases} u_n \ni \Psi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K; \\ \omega_n = \gamma_{n,1}u_n + \sum\limits_{i=2}^{\mathbb{N}} \gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\nu_{n,i-1}D_{n1}; \\ s_n = \beta_{n,1}\omega_n + \sum\limits_{i=2}^{\mathbb{N}} \beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})z_{n,i-1} + D_{n2}; \\ x_{n+1} = P_K\left(\alpha_n\gamma g(x_n) + \delta_n s_n + ((1-\delta_n)I - \alpha_n A)s_n\right), \end{cases}$$

where

$$D_{n1} = +\prod_{j=1}^{\mathbb{N}}(1-\gamma_{n,j})\nu_{n,\mathbb{N}}, \qquad D_{n2} = \prod_{j=1}^{\mathbb{N}}(1-\beta_{n,j})z_{n,\mathbb{N}},$$

$\nu_{n,i} \in \Gamma_i^n u_n$ *and* $z_{n,i} \in G_i^n\omega_n$ *for each* i, $\{\alpha_n\}, \{\delta_n\} \in [0,1], \left\{\{\beta_{n,i}\}_{n=1}^{\infty}\right\}_{i=1}^{\mathbb{N}}$ *and* $\left\{\{\gamma_{n,i}\}_{n=1}^{\infty}\right\}_{i=1}^{\mathbb{N}}$ *are countably finite families of real sequences in* $[0,1]$. *Suppose the following conditions are satisfied:*

(*i*) $\beta_{n,1} \geq \gamma_{n,1} > \max\{\mu_i\}_{i=1}^{\mathbb{N}} : \beta_{n,i} \leq \gamma_{n,i} < \gamma \leq \beta < 1$, *for each* i;

(*ii*) $\lim\inf\limits_{n\to\infty} \beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})(\beta_{n,1}-\mu_{i-1}) > 0$ *and* $\lim\inf\limits_{n\to\infty}\prod_{j=1}^{i-1}(1-\beta_{n,j})(\beta_{n,1} - \mu_{\mathbb{N}} > 0$;

(*iii*) $\lim\inf\limits_{n\to\infty} \gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})(\gamma_{n,1}-\mu_{i-1}) > 0$ *and* $\lim\inf\limits_{n\to\infty}\prod_{j=1}^{i-1}(1-\gamma_{n,j})(\gamma_{n,1} - \mu_{\mathbb{N}} > 0$;

(*iv*) $\lim\limits_{x\to\infty}\alpha_n = 0, \sum\limits_{n=1}^{\infty}\alpha_n = \infty$ *and* $\lim\limits_{x\to\infty}\dfrac{(1-\beta_{n,j})\left(k_n^2-1\right)}{\alpha_n} = 0$;

(*v*) $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$.

Then, the sequence defined by (3.4) *converges strongly to* $q \in \mathcal{F}$, *which solves the variational inequality* $\langle (A-\gamma g)q, x-q\rangle \geq 0, \forall x \in \mathcal{F}$.

Proof. Firstly, we show that the map $P_{\mathcal{F}}[\delta I + (\alpha\gamma g + ((1-\delta)I - \alpha A))]$ is a contraction of K into itself. Now, for any two points $\alpha, \delta \in \left(0, \min\left\{1, \dfrac{1}{\bar\alpha}\right\}\right)$ and for all $x, y \in H$, using Lemma 2.13 with $Q = P_{\mathcal{F}}[\delta I + (\alpha\gamma g + ((1-\delta)I - \alpha A))]x$ and $R = P_{\mathcal{F}}[\delta I + (\alpha\gamma g + ((1-\delta)I - \alpha A))]y$, we have

$$\begin{aligned}\|P_{\mathcal{F}}Q - P_{\mathcal{F}}R\| &\leq \|P_{\mathcal{F}}[\delta I + (\alpha\gamma g + ((1-\delta)I - \alpha A))]x \\ &\quad - P_{\mathcal{F}}[\delta I + (\alpha\gamma g + ((1-\delta)I - \alpha A))]y\| \\ &\leq \alpha\gamma\|g(x) - g(y)\| + \delta_n\|x-y\| + \|((1-\delta_n)I - \alpha A)(x-y)\| \\ &\leq \alpha\gamma\rho\|x-y\| + \delta_n\|x-y\| + ((1-\delta_n)I - \alpha\bar\alpha)\|x-y\| \\ &= (1 - (\bar\alpha - \gamma\rho)\alpha))\|x-y\|.\end{aligned}$$

Hence, there exists a unique point $q \in K$ such that $q = P_{\mathcal{F}}[\delta I + (\alpha\gamma g + ((1-\delta)I - \alpha A))]q$, which is equivalent to

$$\langle (I - A + \gamma g)q - q, q - p\rangle \geq 0, \quad \forall p \in \mathcal{F}$$

Now, since $\lim_{n\to\infty} \alpha_n = 0$, we may assume that $\alpha_n \in \left(0, \frac{1}{\|A\|}\right)$ for all $n \geq 0$. By condition (iv), there exists a constant ϵ with $0 < \epsilon < 1 - \delta$ and $\sum_{i=1}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2 - 1\right) < \epsilon\alpha_n$ for each i. Also, by Lemma 2.9, we get $\|(1-\delta_n)I - \alpha_n A\| \leq (1-\delta_n)I - \alpha_n\bar{\alpha}$.

Let $q \in \mathcal{F}$ and $k_n = \max\left\{k_{n,i}^{(1)}, k_{n,i}^{(2)}\right\}, i = 1, 2, \ldots, m$. Since $u_n = T_{r_n}x_n$ and $q = T_{r_n}q$, we obtain (using Lemma 2.15) that

$$\|u_n - q\| \leq \|T_{r_n}x_n - T_{r_n}q\| \leq \|x_n - q\|. \tag{3.5}$$

Next, we show that the sequence $\{x_n\}$ is bounded. Since $\Gamma_i : K \to \mathcal{P}(K)$ and $G_i : K \to \mathcal{P}(K)$ are both asymptotically demicontractive, $F(\Gamma) \neq \emptyset$ and $F(G) \neq \emptyset$. Thus, there exist two sequences $\left\{\left\{k_{n,i}^{(1)}\right\}_{i=1}^{m}\right\}_{n=1}^{\infty}$, $\left\{\left\{k_{n,i}^{(2)}\right\}_{i=1}^{m}\right\}_{n=1}^{\infty} \subseteq [1,\infty) : k_{n,i}^{(1)}, k_{n,i}^{(2)} \to 1,\ \mathit{for\ each}\ i,\ \mathit{as}\ n \to \infty$ and real positive constants $\mu_i^{(1)}, \mu_i^{(2)} \in [0,1)$ such that for any $(x \times q) \in K \times \mathcal{F}$, we have

$$\|v_{n,i} - q\|^2 \leq k_{n,i}^{(1)}\|x_n - q\|^2 + \mu_i^{(1)}\|\omega_n - v_{n,i}\|^2, \quad v_{n,i} \in G_i^n\omega_n, \tag{3.6}$$

and

$$\|z_{n,i} - q\|^2 \leq k_{n,i}^{(2)}\|x_n - q\|^2 + \mu_i^{(2)}\|u_n - z_{n,i}\|^2, \quad z_{n,i} \in G_i^n\omega_n. \tag{3.7}$$

Using (3.4) and Proposition 2.16 with $s_n = y, \omega_n = t, q = u, k = 1$ and $z_{n,\mathbb{N}} \in G_i^n\omega_n$, we get

$$\begin{aligned} \|s_n - q\|^2 \leq{}& \beta_{n,1}\|\omega_n - q\|^2 + \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j})\|z_{n,i-1} - q\|^2 \\ &+ \prod_{j=1}^{i-1}(1-\beta_{n,j})\|z_{n,\mathbb{N}} - q\|^2 \\ &- \beta_{n,1}\left[\sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j})\|\omega_n - z_{n,i-1}\|^2 \right. \\ &\left. + \prod_{j=1}^{i-1}(1-\beta_{n,j})\|\omega_n - z_{n,\mathbb{N}}\|^2\right]. \end{aligned} \tag{3.8}$$

Since G_i is type-one asymptotically demicontractive, it follows from (3.8) that

$$
\begin{aligned}
\|s_n - q\|^2 \leq{}& \beta_{n,1}\|\omega_n - q\|^2 + \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) \Big[\left(k_{n,i}^{(1)}\right)^2 \|\omega_n - q\|^2 \\
&+ \mu_{i-1}^{(1)} \|\omega_n - z_{n,i-1}\|^2 \Big] \\
&+ \prod_{j=1}^{i-1}(1-\beta_{n,j}) \left[\left(k_{n,i}^{(1)}\right)^2 \|\omega_n - q\|^2 + \mu_{n,\mathbb{N}} \|\omega_n - z_{n,\mathbb{N}}\|^2 \right] \\
&- \beta_{n,1} \left[\sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) \|\omega_n - z_{n,i-1}\|^2 \right. \\
&\left. + \prod_{j=1}^{i-1}(1-\beta_{n,j}) \|\omega_n - z_{n,\mathbb{N}}\|^2 \right] \\
\leq{}& \left(\beta_{n,1} + \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) + \prod_{j=1}^{i-1}(1-\beta_{n,j}) \right) \|\omega_n - q\|^2 \\
&+ \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) \left(\left(k_{n,i}^{(1)}\right)^2 - 1 \right) \|\omega_n - q\|^2 \\
&+ \prod_{j=1}^{i-1}(1-\beta_{n,j}) \left(\left(k_{n,i}^{(1)}\right)^2 - 1 \right) \|\omega_n - q\|^2 \\
&- \left(\beta_{n,1} - \mu_{i-1}^{(1)} \right) \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) \|\omega_n - z_{n,i-1}\|^2 \\
&- \left(\beta_{n,1} - \mu_{n,\mathbb{N}}^{(1)} \right) \prod_{j=1}^{i-1}(1-\beta_{n,j}) \|\omega_n - z_{n,\mathbb{N}}\|^2.
\end{aligned}
$$

With $k = 1$ in Proposition 2.17, we have

(3.9)

$$
\begin{aligned}
\|s_n - q\|^2 \leq{}& \|\omega_n - q\|^2 + \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) \left(\left(k_{n,i}^{(1)}\right)^2 - 1 \right) \|\omega_n - q\|^2 \\
&+ \prod_{j=1}^{i-1}(1-\beta_{n,j}) \left(\left(k_{n,i}^{(1)}\right)^2 - 1 \right) \|\omega_n - q\|^2
\end{aligned}
$$

$$- \left[\sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1} (1-\beta_{n,j}) \left(\beta_{n,1} - \mu_{i-1}^{(1)}\right) \|\omega_n - z_{n,i-1}\|^2 \right.$$
$$\left. + \prod_{j=1}^{i-1} (1-\beta_{n,j}) \left(\beta_{n,1} - \mu_{n,\mathbb{N}}^{(1)}\right) \|\omega_n - z_{n,\mathbb{N}}\|^2 \right].$$

Similarly, from (3.4) and Proposition 2.17 with $\omega_n = y, u_n = t, q = u, k = 1$ and $\nu_{n,i} \in \Gamma_i^n u_n$, it is easy to see (using the same argument as above) that

(3.10)

$$\|w_n - q\|^2 \leq \|u_n - q\|^2 + \sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(\left(k_{n,i}^{(2)}\right)^2 - 1 \right) \|u_n - q\|^2$$
$$+ \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(\left(k_{n,i}^{(2)}\right)^2 - 1 \right) \|u_n - q\|^2$$
$$- \left[\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(\gamma_{n,1} - \mu_{i-1}^{(2)}\right) \|u_n - \nu_{n,i-1}\|^2 \right.$$
$$\left. + \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(\gamma_{n,1} - \mu_{n,\mathbb{N}}^{(2)}\right) \|\omega_n - z_{n,\mathbb{N}}\|^2 \right],$$

(3.9) and (3.10) imply that

(3.11)

$$\|s_n - q\|^2 \leq \left[1 + \sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1} (1-\beta_{n,j}) \left(k_n^2 - 1\right) + \prod_{j=1}^{i-1} (1-\beta_{n,j}) \left(k_n^2 - 1\right) \right]$$
$$\times \left\{ \left[1 + \sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(k_n^2 - 1\right) \right. \right.$$
$$\left. + \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(k_n^2 - 1\right) \right] \|x_n - q\|^2$$
$$- \left[\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1} (1-\gamma_{n,j}) \left(\gamma_{n,1} - \mu_{i-1}^{(2)}\right) \|u_n - \nu_{n,i-1}\|^2 \right.$$

$$
\begin{aligned}
&\left.\left. + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1}-\mu_{n,\mathbb{N}}^{(2)}\right)\|\omega_n - z_{n,\mathbb{N}}\|^2\right]\right\} \\
&- \left[\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{i-1}^{(1)}\right)\|\omega_n - z_{n,i-1}\|^2\right. \\
&\left. + \prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{n,\mathbb{N}}^{(1)}\right)\|\omega_n - z_{n,\mathbb{N}}\|^2\right] \\
\leq &\left\{1 + \left(\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j}) + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\left(k_n^2-1\right)\right. \\
&+ \left(\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j}) + \prod_{j=1}^{i-1}(1-\beta_{n,j})\right) \\
&\times \left[1 + \left(\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right.\right. \\
&\left.\left.\left. + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\right]\left(k_n^2-1\right)\right\}\|x_n - q\|^2 \\
&- \left[\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1}-\mu_{i-1}^{(2)}\right)\|u_n - \nu_{n,i-1}\|^2\right. \\
&\left. + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1}-\mu_{n,\mathbb{N}}^{(2)}\right)\|\omega_n - z_{n,\mathbb{N}}\|^2\right] \\
&- \left[\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{i-1}^{(1)}\right)\|\omega_n - z_{n,i-1}\|^2\right. \\
&\left. + \prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{n,\mathbb{N}}^{(1)}\right)\|\omega_n - z_{n,\mathbb{N}}\|^2\right].
\end{aligned}
$$

Furthermore, we obtain from (3.4) that

(3.12)

$$
\begin{aligned}
\|x_{n+1} - q\| &\leq \|\alpha_n\gamma g(x_n) + \delta_n s_n + ((1-\delta_n)I - \alpha_n A)s_n - q\| \\
&\leq \alpha_n\|\gamma g(x_n) - Aq\| + \delta_n\|s_n - q\| + \|(1-\delta_n)I - \alpha_n A\|\|s_n - q\| \\
&\leq \alpha_n\gamma\|g(x_n) - g(q)\| + \alpha_n\|\gamma g(q) - Aq\| + \delta_n\|s_n - q\|
\end{aligned}
$$

$$+\left((1-\delta_n)I-\alpha_n\bar{\alpha}\right)\|s_n-q\|$$
$$\leq \alpha_n\gamma\rho\|x_n-q\|+\alpha_n\|\gamma g(q)-Aq\|+(1-\alpha_n\bar{\alpha})\|s_n-q\|.$$

From (3.11) and conditions [(i) and (iv)], we get

(3.13)

$$\begin{aligned}\|s_n-q\| &\leq \left[1+\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\right]\\ &\quad\times\|x_n-q\|\\ &\leq (1+2\epsilon\alpha_n)\|x_n-q\|,\end{aligned}$$

(3.12) and (3.13) imply that

$$\begin{aligned}\|x_{n+1}-q\| &\leq \alpha_n\gamma\rho\|x_n-q\|+\alpha_n\|\gamma g(q)-Aq\|\\ &\quad+(1-\alpha_n\bar{\alpha})(1+2\epsilon\alpha_n)\|x_n-q\|\\ &\leq \alpha_n\gamma\rho\|x_n-q\|+\alpha_n\|\gamma g(q)-Aq\|\\ &\quad+(1+2\epsilon\alpha_n-\alpha_n\bar{\alpha})\|x_n-q\|\\ &=[1-(\bar{\alpha}-2\epsilon-\gamma\rho)\alpha_n]\|x_n-q\|+\alpha_n\|\gamma g(q)-Aq\|.\end{aligned}$$

Using mathematical inductional argument, we get

$$\|x_n-q\|\leq \max\left\{\|x_0-q\|,\frac{\|\gamma g(q)-Aq\|}{\bar{\alpha}-2\epsilon-\gamma\rho}\right\},\quad \forall n\in\mathbb{N},$$

which implies that the sequence $\{x_n\}$ is bounded; and from which we conclude that the following sequences: $\{u_n\},\{s_n\}$ and $\{g(x_n)\}$ are as well bounded.

Next, for each i, we show that $\|\omega_n-z_{n,i}\|\to 0$ and $\|u_n-\nu_{n,i}\|\to 0$ as $n\to\infty$. From Lemma 2.9, (3.11) and (3.13), we get

(3.14)

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \|\alpha_n\gamma g(x_n)+\delta_n s_n+((1-\delta_n)I-\alpha_n A)s_n-q\|^2\\ &\leq \|\alpha_n(\gamma g(x_n)-Aq)-\delta_n(q-s_n)\\ &\quad+((1-\delta_n)I-\alpha_n A)(s_n-q)\|^2\\ &=\|\alpha_n(\gamma g(x_n)-Aq)-\delta_n(q-s_n)\|^2\\ &\quad+\|((1-\delta_n)I-\alpha_n A)(s_n-q)\|^2\\ &\quad+2\alpha_n\|\gamma g(x_n)-Aq\|\|((1-\delta_n)I-\alpha_n A)(s_n-q)\|\\ &\leq \alpha_n^2\|\gamma g(x_n)-Aq\|^2\\ &\quad+\delta_n^2\|q-s_n\|^2+((1-\delta_n)I-\alpha_n\bar{\alpha})^2\|s_n-q\|^2\\ &\quad+2\alpha_n((1-\delta_n)I-\alpha_n\bar{\alpha})\|\gamma g(x_n)-Aq\|\|s_n-q\|\\ &\leq \left(1-2\delta_n+\delta_n^2-2(1-\delta_n)\alpha_n\bar{\alpha}+\alpha_n^2\bar{\alpha}^2\right)\end{aligned}$$

$$
\begin{aligned}
&\times \|s_n - q\|^2 + \delta_n^2 \|q - s_n\|^2 + \alpha_n^2 \|\gamma g(x_n) - Aq\|^2 \\
&+ 2\alpha_n((1-\delta_n)I - \alpha_n \bar{\alpha})\|\gamma g(x_n) - Aq\|\|s_n - q\| \\
= &\,(1 - 2\delta_n(\delta_n - 1) + \alpha_n \bar{\alpha}(\alpha_n \alpha - 2) + 2\delta_n \alpha_n \bar{\alpha})\|s_n - q\|^2 \\
&+ \alpha_n^2 \|\gamma g(x_n) - Aq\|^2 \\
&+ 2\alpha_n((1-\delta_n)I - \alpha_n \bar{\alpha})\|\gamma g(x_n) - Aq\|\|s_n - q\| \\
\leq &\,(1 + 2\delta_n \alpha_n \bar{\alpha})\|s_n - q\|^2 + \alpha_n^2 \|\gamma g(x_n) - Aq\|^2 \\
&+ 2\alpha_n((1-\delta_n)I - \alpha_n \bar{\alpha})\|\gamma g(x_n) - Aq\|\|s_n - q\| \\
\leq &\left\{1 + \left(\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j}) + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\left(k_n^2 - 1\right)\right. \\
&+ \left(\sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) + \prod_{j=1}^{i-1}(1-\beta_{n,j})\right) \\
&\left.\times \left[1 + \left(\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j}) + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\right]\left(k_n^2 - 1\right)\right\} \\
&\times \|x_n - q\|^2 - \left[\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1} - \mu_{i-1}^{(2)}\right)\right. \\
&\left.\times \|u_n - \nu_{n,i-1}\|^2 + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1} - \mu_{n,\mathbb{N}}^{(2)}\right)\|\omega_n - z_{n,\mathbb{N}}\|^2\right] \\
&- \left[\sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1} - \mu_{i-1}^{(1)}\right)\|\omega_n - z_{n,i-1}\|^2\right. \\
&\left.+ \prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1} - \mu_{n,\mathbb{N}}^{(1)}\right)\|\omega_n - z_{n,\mathbb{N}}\|^2\right] \\
&+ 2\delta_n \alpha_n \bar{\alpha}\left\{1 + \left(\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j}) + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\right. \\
&\times \left(k_n^2 - 1\right) + \left(\sum_{i=2}^{\mathbb{N}} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j}) + \prod_{j=1}^{i-1}(1-\beta_{n,j})\right) \\
&\left.\times \left[1 + \left(\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j}) + \prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\right]\left(k_n^2 - 1\right)\right\} \\
&\times \|x_n - q\|^2 - \left[\sum_{i=2}^{\mathbb{N}} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1} - \mu_{i-1}^{(2)}\right)\right.
\end{aligned}
$$

$$\times\|u_n-\nu_{n,i-1}\|^2+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1}-\mu_{n,\mathbb{N}}^{(2)}\right)\|\omega_n-z_{n,\mathbb{N}}\|^2\Bigg]$$
$$-\left[\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{i-1}^{(1)}\right)\|\omega_n-z_{n,i-1}\|^2\right.$$
$$\left.+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{n,\mathbb{N}}^{(1)}\right)\|\omega_n-z_{n,\mathbb{N}}\|^2\right]$$
$$+\alpha_n^2\|\gamma g(x_n)-Aq\|^2+2\alpha_n(1+2\epsilon\alpha_n)((1-\delta_n)I-\alpha_n\bar{\alpha})$$
$$\times\|\gamma g(x_n)-Aq\|\|x_n-q\|.$$

Set

$$M=\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1}-\mu_{i-1}^{(2)}\right)\|u_n-\nu_{n,i-1}\|^2$$
$$+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(\gamma_{n,1}-\mu_{n,\mathbb{N}}^{(2)}\right)\|\omega_n-z_{n,\mathbb{N}}\|^2.$$

Then, we obtain from the last inequality that

(3.15)

$$M\le\|x_n-q\|^2-\|x_{n+1}-q\|^2$$
$$+\left\{\left\{\left(\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\left(k_n^2-1\right)\right.\right.$$
$$+\left(\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right)\left[1+\left(\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right.\right.$$
$$\left.\left.\left.+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\right]\left(k_n^2-1\right)\right\}\|x_n-q\|^2$$
$$-\left[\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{i-1}^{(1)}\right)\|\omega_n-z_{n,i-1}\|^2\right.$$
$$\left.+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{n,\mathbb{N}}^{(1)}\right)\|\omega_n-z_{n,\mathbb{N}}\|^2\right]$$
$$+2\delta_n\alpha_n\bar{\alpha}\left\{1+\left(\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\left(k_n^2-1\right)\right.$$

$$+\left(\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right)\left[1+\left(\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right.\right.$$
$$\left.\left.+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)\right]\left(k_n^2-1\right)\Bigg\}\|x_n-q\|^2$$
$$-\left[\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{i-1}^{(1)}\right)\|\omega_n-z_{n,i-1}\|^2\right.$$
$$\left.\left.+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(\beta_{n,1}-\mu_{n,\mathbb{N}}^{(1)}\right)\|\omega_n-z_{n,\mathbb{N}}\|^2\right]\right\}+\alpha_n^2\|\gamma g(x_n)-Aq\|^2$$
$$+2\alpha_n(1+2\epsilon\alpha_n)((1-\delta_n)I-\alpha_n\bar{\alpha})\|\gamma g(x_n)-Aq\|\|x_n-q\|.$$

Now, to show $x_n\to q$ as $n\to\infty$, we consider the following cases:
Case A: Suppose the sequence $\{\|x_n-q\|\}$ is monotonically decreasing. Then, $\{\|x_n-q\|\}$ is convergent. Hence,

$$\lim_{n\to\infty}[\|x_n-q\|-\|x_{n+1}-q\|]=0. \tag{3.16}$$

Consequently, from (3.15), (3.16), conditions [(i), (ii) and (iv)] and the fact that $\lim_{n\to\infty}k_n=1$, we obtain

$$\lim_{n\to\infty}\|u_n-\nu_{n,i}\|=0. \tag{3.17}$$

Since $\nu_{n,i}\in\Gamma_i^n u_n$, using (3.17), we have

$$\lim_{n\to\infty}\|u_n-\Gamma_i^n u_n\|=0. \tag{3.18}$$

Using the same argument as above (considering (3.15), (3.16), conditions [(i), (iii) and (iv)] and the fact that $\lim_{n\to\infty}k_n=1$), we easily see that

$$\lim_{n\to\infty}\|\omega_n-z_{n,i}\|=0. \tag{3.19}$$

Since $z_{n,i}\in G_i^n\omega_n$, using (3.19), we get

$$\|\omega_n-G_i^n\omega_n\|=0. \tag{3.20}$$

Next, we show that $\lim_{n\to\infty}\|x_n-u_n\|=0$. For any $q\in\mathcal{F}$, we have

$$\begin{aligned}\|u_n-q\|^2&\le\|Tr_nx_n-Tr_nq\|^2\\&\le\langle Tr_nx_n-Tr_nq,x_n-q\rangle\\&=\langle u_n-q,x_n-q\rangle\\&=\frac{1}{2}\left(\|u_n-q\|^2+\|x_n-q\|^2-\|x_n-u_n\|^2\right),\end{aligned}$$

and hence

$$\|u_n-q\|^2\le\|x_n-q\|^2-\|x_n-u_n\|^2. \tag{3.21}$$

(3.9), (3.10) and (3.13) imply that

$$\begin{aligned}\|x_{n+1}-q\|^2 \le{}& \|\omega_n-q\|^2+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\|\omega_n-q\|^2\\ &+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\|\omega_n-q\|^2+2\alpha_n[\delta_n\bar{\alpha}\|s_n-q\|^2\\ &+((1-\delta_n)I-\alpha_n\bar{\alpha})\|\gamma g(x_n)-Aq\|\|s_n-q\|]\\ \le{}& \|u_n-q\|^2+\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)\|u_n-q\|^2\\ &+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)\|u_n-q\|^2\\ &+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})(k_n-1)\|\omega_n-q\|^2\\ &+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\|\omega_n-q\|^2+2\alpha_n[\delta_n\bar{\alpha}\|s_n-q\|^2\\ &+((1-\delta_n)I-\alpha_n\bar{\alpha})\|\gamma g(x_n)-Aq\|\|s_n-q\|],\end{aligned}$$

which by (3.21) yields

(3.22)

$$\begin{aligned}\|x_n-u_n\|^2 \le{}& \|x_n-q\|^2-\|x_{n+1}-1\|^2+\left[\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)\right.\\ &+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\\ &\left.+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\right]Q\\ &+2\alpha_n\left[\delta_n\bar{\alpha}Q+((1-\delta_n)I-\alpha_n\bar{\alpha})\|\gamma g(x_n)-Aq\|\|s_n-q\|\right],\end{aligned}$$

where $Q^\star=\sup\|u_n-q\|^2$,$Q^{\star\star}=\sup\|\omega_n-q\|^2$ and $Q=\max\{Q^\star,Q^{\star\star}\}$. From (3.16), condition (iv) and the fact that $\lim_{n\to\infty}k_n=1$, we obtain from the last inequality that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.23}$$

Moreover, from (3.4), we have the following estimates:

(3.24)
$$\begin{aligned}\|x_{n+1}-s_n\| &\le \|\alpha_n\alpha g(x_n)+\delta_n s_n+((1-\delta_n)I-A)s_n-s_n\|\\ &=\alpha_n\|\alpha g(x_n)-As_n\|\to 0 \text{ as } n\to\infty \text{ (by condition (iv))},\end{aligned}$$

(3.25)
$$\begin{aligned}\|s_n-\omega_n\| &= \Bigg\|\left(\beta_{n,1}+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right)\omega_n\\ &\quad+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})(z_{n,i-1}-\omega_n)\\ &\quad+\prod_{j=1}^{i-1}(1-\beta_{n,j})(z_{n,\mathbb{N}}-\omega_n)-\omega_n\Bigg\|\\ &\le \sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\|z_{n,i-1}-\omega_n\|\\ &\quad+\prod_{j=1}^{i-1}(1-\beta_{n,j})\|z_{n,\mathbb{N}}-\omega_n\|\to 0,\end{aligned}$$

as $n\to\infty$ [by (3.15) and (3.16)], and by following the argument as was used for (3.15), we have (using (3.14) and (3.16)) that

(3.26)
$$\begin{aligned}\|\omega_n-u_n\| &= \Bigg\|\left(\gamma_{n,1}+\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\right)u_n\\ &\quad+\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})(\nu_{n,i-1}-u_n)\\ &\quad+\prod_{j=1}^{i-1}(1-\gamma_{n,j})(\nu_{n,\mathbb{N}}-u_n)-\omega_n\Bigg\|\\ &\le \sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\,\|\nu_{n,i-1}-u_n\|\\ &\quad+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\,\|\nu_{n,\mathbb{N}}-u_n\|\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Observe rhat

$$\|x_{n+1}-x_n\| \leq \|x_{n+1}-s_n\| + \|s_n-\omega_n\| + \|\omega_n-u_n\| + \|u_n-x_n\|$$

(3.23), (3.24), (3.25), (3.26) and the last inequality imply

$$\lim_{x\to\infty} \|x_{n+1}-x_n\| = 0. \tag{3.27}$$

Also, observe that

(3.28)

$$\begin{aligned}
\|\omega_n - G_i\omega_n\| &\leq \|\omega_n - G_i^n\omega_n\| + \|G_i^n\omega_n - G_i^n u_n\| \\
&\quad + \left\|G_i(G_i^{n-1}u_n) - G_i(G^{n-1}\omega_{n-1}\right\| \\
&\quad + \left\|G_i(G_i^{n-1}\omega_{n-1}) - G_i\omega_n)\right\| \\
&\leq \|\omega_n - G_i^n\omega_n\| + L\|\omega_n - u_n\| + L\|u_n - \omega_{n-1}\| \\
&\quad + L\left\|G_i^{n-1}\omega_{n-1} - \omega_n\right\| \\
&\leq \|\omega_n - G_i^n\omega_n\| + L\|\omega_n - u_n\| + L[\|u_n - x_n\| \\
&\quad + \|x_n - s_{n-1}\| + \|s_{n-1} - \omega_{n-1}\|] + L\left[\left\|G_i^{n-1}\omega_{n-1} - \omega_{n-1}\right\|\right. \\
&\quad + \|\omega_{n-1} - s_{n-1}\| + \|s_{n-1} - x_n\| + \|x_n - u_n\| + \|u_n - \omega_n\|] \\
&\leq \|\omega_n - G_i^n\omega_n\| + L\left\|G_i^{n-1}\omega_{n-1} - \omega_{n-1}\right\| + 2L[\|\omega_n - u_n\| \\
&\quad + \|u_n - x_n\| + \|x_n - s_{n-1}\| + \|s_{n-1} - \omega_{n-1}\|],
\end{aligned}$$

(3.20), (3.23), (3.24), (3.25), (3.26) and (3.28) imply that

$$\lim_{n\to\infty} \|\omega_n - G_i\omega_n\| = 0. \tag{3.29}$$

Again, observe that

(3.30)

$$\begin{aligned}
\|u_n - \Gamma_i u_n\| &\leq \|u_n - \Gamma_i^n u_n\| + \|\Gamma_i^n u_n - \Gamma_i^n x_n\| \\
&\quad + \left\|\Gamma_i(\Gamma_i^{n-1}x_n) - \Gamma_i(\Gamma_i^n u_{n-1})\right\| + \|\Gamma_i(\Gamma_i^n u_{n-1}) - \Gamma_i u_n\| \\
&\leq \|u_n - \Gamma_i^n u_n\| + L\|u_n - x_n\| + L\|x_n - u_{n-1}\| \\
&\quad + L\|\Gamma_i^n u_{n-1} - u_n\| \\
&\leq \|u_n - \Gamma_i^n u_n\| + L\|u_n - x_n\| + L[\|x_n - s_{n-1}\| \\
&\quad + \|s_{n-1} - \omega_{n-1}\| + \|\omega_{n-1} - u_{n-1}\|] \\
&\quad + L[\|\Gamma_i^n u_{n-1} - u_{n-1}\| + \|u_{n-1} - \omega_{n-1}\| \\
&\quad + \|\omega_{n-1} - s_{n-1}\| + \|s_{n-1} - x_n\| + \|x_n - u_n\|] \\
&\leq \|u_n - \Gamma_i^n u_n\| + L\|\Gamma_i^n u_{n-1} - u_{n-1}\| + 2L[\|u_n - x_n\| \\
&\quad + \|x_n - s_{n-1}\| + \|s_{n-1} - \omega_{n-1}\| + \|\omega_{n-1} - u_{n-1}\|].
\end{aligned}$$

(3.18), (3.23), (3.24), (3.25), (3.26) and (3.30) imply that

$$\lim_{n\to\infty} \|u_n - \Gamma_i u_n\| = 0. \tag{3.31}$$

Next, we show that

$$\limsup_{n\to\infty} \langle A - \gamma g)q, q - x_n\rangle \leq 0, \quad \forall n \in \mathbb{N}, \tag{3.32}$$

where $q = P_{\mathcal{F}}(1 - A + \gamma g)q$ is a unique solution of the variational inequality (2.7). To do this, choose a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{i\to\infty} \langle A - \gamma g)q, q - x_{n_k}\rangle = \lim_{n\to\infty} \langle A - \gamma g)q, q - x_n\rangle . \tag{3.33}$$

Now, since the sequence is bounded (as shown above),there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ that converges weakly to $x^\star \in \mathcal{F}$. We assume (without loss of generality) that $x_{n_k} \rightharpoonup x^\star$ *as* $n \to \infty$. Since $\|u_n - x_n\| \to 0$ *as* $n \to \infty$, we have $u_{n_k \rightharpoonup} \to x^\star$ *as* $k \to \infty$. We show that $x^\star \in \mathcal{F}$. To do this, we proceed as follows:
First, we show that $x^\star \in EP(\Psi)$. By $u_n = Tr_n x_n$, we get

$$\Psi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K.$$

From (B2), we also get

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \geq \Psi(y, u_n), \quad \forall y \in K,$$

consequently,

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}} \right\rangle \geq \Psi(y, u_{n_k}).$$

Since $\frac{u_{n_k} - x_{n_k}}{r_{n_k}} \to 0$ and $u_{n_k} \rightharpoonup x^\star$ *as* $k \to \infty$, from (B4), we have

$$0 \geq \Psi(y, x^\star), \quad \forall y \in K.$$

Let $y_t = ty + (1-t)x^\star$, where $y \in K$ and $0 < t \leq 1$. Since $y, x^\star \in K$ and K is convex, we have $y_t \in K$ and $\Psi(y_t, x^\star) \leq 0$. Hence, from (B1)and (B4), we obtain

$$\begin{aligned} 0 &= \Psi(y_t, y_t) \\ &\leq t\Psi(y_t, y) + (1-t)\Psi(y_t, x^\star) \\ &\leq t\Psi(y_t, y), \end{aligned}$$

which yields $\Psi(y_t, y) \geq 0$. From (B3), we get $\Psi(x^\star, y) \geq 0, \forall y \in K$. Thus, $x^\star \in EP(\Psi)$.

Also, from $u_{n_k} \rightharpoonup x^\star$, the fact that $\lim_{n\to\infty} \|u_n - \Gamma_i u_n\| = 0$ and demiclosedness of $I - \Gamma_i$ at zero for each i, following standard argument, we conclude that $x^\star \in \cap_{i=1}^m F(\Gamma_i)$. Again, since $\lim_{n\to\infty} \|\omega_n - G_i\omega_n\| = 0$

and $\lim_{n\to\infty}\|u_n - s_n\| = 0$, by our assumption that $I - G_i$ is demiclosed at zero for each i, we obtain $x^\star \in \cap_{i=1}^m F(G_i)$. Hence, $x^\star \in \mathcal{F}$. Since $q = P_{\mathcal{F}}(i - A + \gamma g)q$ and $x^\star \in \mathcal{F}$, it follows from (3.31) that

$$\begin{aligned}\limsup_{n\to\infty}\langle A-\gamma g)q, q - x_n\rangle &= \lim_{i\to\infty}\left\langle A-\gamma g)q, q - x_{n_{k_i}}\right\rangle \\ &= \lim_{i\to\infty}\langle A-\gamma g)q, q - x^\star\rangle \\ &\leq 0,\end{aligned} \tag{3.34}$$

as required.

Now, using (3.4) and Lemma 2.10, we get

(3.35)

$$\begin{aligned}&\|x_{n+1} - q\|^2 \\ &\quad\leq \|\alpha_n\gamma g(x_n) + \delta_n s_n + ((1-\delta_n)I - \alpha_n A)s_n - q\|^2 \\ &\quad= \|\alpha_n(\gamma g(x_n) - Aq) + \delta_n(s_n - q) + ((1-\delta_n)I - \alpha_n A)(s_n - q)\|^2 \\ &\quad\leq \|((1-\delta_n) - \alpha_n A)(s_n - q) - \delta_n(q - s_n)\|^2 \\ &\qquad + 2\alpha_n\langle\gamma g(x_n) - Aq, x_{n+1} - q\rangle \\ &\quad\leq ((1-\delta_n)I - \alpha_n\alpha)^2\|s_n - q\|^2 + \delta_n^2\|s_n - q)\|^2 \\ &\qquad + 2\alpha_n\gamma\langle g(x_n) - g(q), x_{n+1} - q\rangle \\ &\qquad + 2\alpha_n\langle(A-\gamma g)q, q - x_{n+1}\rangle \\ &\quad\leq [1 - 2\delta_n + \delta_n^2 - 2(1-\delta_n)\alpha_n\alpha + \alpha_n^2\alpha^2]\|s_n - q\|^2 \\ &\qquad + \delta_n^2\|s_n - q)\|^2 + 2\alpha_n\gamma\rho\|x_n - q\|\|x_{n+1} - q\| \\ &\qquad + 2\alpha_n\langle(A-\gamma g)q, q - x_{n+1}\rangle \\ &\quad= [1 + 2\delta_n(\delta_n - \alpha_n\alpha - 1) - 2\alpha_n\alpha + \alpha_n^2\alpha^2]\|s_n - q\|^2 \\ &\qquad + \alpha_n\gamma\rho[\|x_n - q\|^2 + \|x_{n+1} - q\|^2] \\ &\qquad + 2\alpha_n\langle(A-\gamma g)q, q - x_{n+1}\rangle \\ &\quad\leq (1 - 2\alpha_n\alpha)\|s_n - q\|^2 + \alpha_n^2\alpha^2\|s_n - q\|^2 + \alpha_n\gamma\rho[\|x_n - q\|^2 \\ &\qquad + \|x_{n+1} - q\|^2] + 2\alpha_n\langle(A-\gamma g)q, q - x_{n+1}\rangle.\end{aligned}$$

By (3.5), (3.9), (3.10) and (3.35), we have

(3.36)

$$\begin{aligned}&\|x_{n+1} - q\|^2 \\ &\quad\leq (1 - 2\alpha_n\alpha)\|\omega_n - q\|^2 + \sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2 - 1\right)\|\omega_n - q\|^2\end{aligned}$$

$$
\begin{aligned}
&+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\|\omega_n-q\|^2+\alpha_n^2\alpha^2\|s_n-q\|^2 \\
&+\alpha_n\gamma\rho[\|x_n-q\|^2+\|x_{n+1}-q\|^2]+2\alpha_n\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle \\
\leq{}& (1-2\alpha_n\alpha)\|x_n-q\|^2+\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)\|u_n-q\|^2 \\
&+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)\|u_n-q\|^2 \\
&+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\|\omega_n-q\|^2 \\
&+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)\|\omega_n-q\|^2+\alpha_n^2\alpha^2\|s_n-q\|^2 \\
&+\alpha_n\gamma\rho\left[\|x_n-q\|^2+\|x_{n+1}-q\|^2\right]+2\alpha_n\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle \\
\leq{}& [1-(2\bar{\alpha}-\gamma\rho)\alpha_n]\|x_n-q\|^2+\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)Q^\star \\
&+\prod_{j=1}^{i-1}(1-\gamma_{n,j})\left(k_n^2-1\right)Q^\star+\sum_{i=2}^{\mathbb{N}}\beta_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)Q^{\star\star} \\
&+\prod_{j=1}^{i-1}(1-\beta_{n,j})\left(k_n^2-1\right)Q^{\star\star}+\alpha_n^2\alpha^2Q^{\star\star}+\alpha_n\gamma\rho\|x_{n+1}-q\|^2 \\
&+2\alpha_n\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle \\
\leq{}& [1-(2\bar{\alpha}-\gamma\rho)\alpha_n]\|x_n-q\|^2 \\
&+\left\{\alpha_n^2\alpha^2+2\left(k_n^2-1\right)\left[\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right]Q\right\} \\
&+\alpha_n\gamma\rho\|x_{n+1}-q\|^2+2\alpha_n\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle,
\end{aligned}
$$

where $Q, Q^\star$ and $Q^{\star\star}$ still retain their usual meaning.

The last inequality implies that

(3.37)

$$\|x_{n+1}-q\|^2 \leq \left[1-\frac{2(\bar{\alpha}-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho}\right]\|x_n-q\|^2$$

$$+\frac{\alpha_n}{1-\alpha_n\gamma\rho}\left\{\left\{\alpha_n\alpha^2+\frac{2\left(k_n^2-1\right)}{\alpha_n}\left[\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})\right.\right.\right.$$
$$\left.\left.\left.+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right]\right\}Q+2\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle\right\}.$$

Set

$$\sigma_n=\frac{2(\bar{\alpha}-\gamma\rho)\alpha_n}{1-\alpha_n\gamma\rho},$$
$$\lambda_n=\frac{\alpha_n}{1-\alpha_n\gamma\rho}\left\{\left\{\alpha_n\alpha^2+\frac{2\left(k_n^2-1\right)}{\alpha_n}\left[\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right]Q\right\}\right.$$
$$\left.+2\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle\right\},$$

and

$$\ell_n=\frac{\lambda_n}{\sigma_n}$$
$$=\frac{1}{2(\bar{\alpha}-\gamma\rho)}\left\{\left\{\alpha_n\alpha^2+\frac{2\left(k_n^2-1\right)}{\alpha_n}\left[\sum_{i=2}^{\mathbb{N}}\gamma_{n,i}\prod_{j=1}^{i-1}(1-\beta_{n,j})+\prod_{j=1}^{i-1}(1-\beta_{n,j})\right]Q\right\}\right.$$
$$\left.+2\left\langle (A-\gamma g)q,q-x_{n+1}\right\rangle\right\}.$$

Then, we obtain from (3.37) that

$$b_{n+1}\leq(1-\sigma_n)b_n+\ell_n, \tag{3.38}$$

where $b_n=\|x_n-q\|^2$. It is easy to see, using condition (iv) and the fact that $\lim\limits_{x\to\infty}k_n=1$, that

$$\sigma_n\to 0\quad\text{as}\quad n\to\infty,\sum_{n=1}^{\infty}\sigma_n=\infty\quad\text{and}\quad\limsup_{n\to\infty}\ell_n\leq 0.$$

Therefore, by Lemma 2.11 and (3.38), the sequence $\{x_n\}$ converges strongly to $q\in\mathcal{F}$.

Case B: Assume that sequence $\{\|x_n-q\|\}$ is monotonicalyy inreasing. Then, we can define an integer sequence $\{\tau_n\}$ for all $n\geq 0$ (for some n_0 large enough) as follows:

$$\tau_n=\max\left\{k\in\mathbb{N}:k\leq n:\|x_k-q\|<\|x_{k+1}-q\|\right\}. \tag{3.39}$$

It is obvious that τ is a nondecreaing sequence such that $\tau_n\to\infty$ as $n\to\infty$ and for all $n\geq n_0$, we have

$$\|x_{\tau(n)}-q\|<\|x_{\tau(n)+1}-q\|. \tag{3.40}$$

From (3.15), (3.18), (3.23) and (3.31) with (n replaced by $\tau_{(n)}$), we have

$$\lim_{x\to\infty} \|u_{\tau(n)} - \Gamma_i u_{\tau(n)}\| = 0, \tag{3.41}$$

and

$$\lim_{x\to\infty} \|x_{\tau(n)} - u_{\tau(n)}\| = 0. \tag{3.42}$$

Following the same argument as in Case A, we obtain

$$b_{\tau(n)+1} \leq (1 - \sigma_{\tau(n)}) b_{\tau(n)} + \ell_{\tau(n)}, \tag{3.43}$$

$\sigma_{\tau(n)} \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \sigma_{\tau(n)} = \infty$ and $\lim \sup_{n\to\infty} \ell_{\tau(n)} \leq 0$. Thus, by Lemma 2.11, we have $\lim_{n\to\infty} \|x_{\tau(n)} - q\| = 0$ and $\lim_{n\to\infty} \|x_{\tau(n)+1} - q\| = 0$. Hence, by Lemma 2.13, we obtain

$$\begin{aligned} 0 &\leq \|x_n - q\| \\ &\leq \max\{\|x_n - q\|, \|x_{\tau(n)} - q\|\} \lim_{n\to\infty} \\ &\leq \|x_{\tau(n)+1} - q\|.t. \end{aligned}$$

Therefore, $\{x_n\}$ converges to $q = P_{\mathbb{F}}(1 - A + \gamma g)q$ and this completes the proof. □

Theorem 3.5. *Let K be a convex, closed and nonempty subset of a real Hilbert space H and let Ψ be a bifunction of $K \times K$ into $\mathbb{R}$. Suppose that $\{\Gamma_i\}_{i=1}^N$ and $\{G_i\}_{i=1}^N$,$N \geq 2$ are finite families of type-one and L_i-uniformly Lipschitizian asymptotically strictly pseudocontractive multivalued mapping $\Gamma_i : K \to \mathcal{P}(K)$and type-one and L_i-uniformly Lipschitizian asymptotically strictly pseudononpreading multivalued mapping $G_i : K \to \mathcal{P}(K)$, respectively from K into the family of all proximinal subsets of K with contractive coeficient $\mu_i^{(1)}, \mu_i^{(2)} \in [0.1)$ for each i. Assume that $\mathcal{F} = (\cap_{i=1}^N (F(\Gamma_i)) \cap (\cap_{i=1}^N F(G_i) \cap EP(\Theta) \neq \emptyset$ and for each i, $(I - \Gamma_i)$ and $(I - G_i)$ are weakly demiclosed at zero. let g_i be a contraction of K into itself with constant $\rho \in (0, 1)$ and A be a strong positive self adjoint bounded linear operator on H with coeficient $\bar{\alpha}$ such that $0 < \rho\gamma + 2\epsilon < \bar{\alpha}$. Let the sequence $\{x_n\}$ be given iteratively as follows:*

$$\begin{cases} u_n \ni \Psi(u_n, y) + Q_{n1}, \quad \forall y \in K; \\ \omega_n = \gamma_{n,1} u_n + Q_{n2}, \\ s_n = \beta_{n,1}\omega_n + Q_{n3}, \\ x_{n+1} = P_K(\alpha_n \gamma g(x_n) + \delta_n s_n + ((1 - \delta_n)I - \alpha_n A)s_n), \end{cases} \tag{3.44}$$

where

$$Q_{n1} = \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0,$$

$$Q_{n2} = \sum_{i=2}^{N} \gamma_{n,i} \prod_{j=1}^{i-1} (1-\gamma_{n,j})\nu_{n,i-1} + \prod_{j=1}^{N}(1-\gamma_{n,j})\nu_{n,N},$$

$$Q_{n3} = \sum_{i=2}^{N} \beta_{n,i} \prod_{j=1}^{i-1} (1-\beta_{n,j})z_{n,i-1} + \prod_{j=1}^{N}(1-\beta_{n,j})z_{n,N},$$

$\nu_{n,i} \in \Gamma_i^n u_n$ and $z_{n,i} \in G_i^n \omega_n$ for each i, $\{\alpha_n\}, \{\delta_n\} \in [0,1], \big\{\{\beta_{n,i}\}_{n=1}^{\infty}\big\}_{i=1}^{N}$ and $\big\{\{\gamma_{n,i}\}_{n=1}^{\infty}\big\}_{i=1}^{N}$ are countably finite families of real sequences in $[0,1]$. Suppose the following conditions are satisfied:

(i) $\beta_{n,1} \geq \gamma_{n,1} > max\{\mu_i\}_{i=1}^{N} : \beta_{n,i} \leq \gamma_{n,i} < \gamma \leq \beta < 1$, *for each i;*

(ii) $\lim \inf_{n\to\infty} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j})(\beta_{n,1}-\mu_{i-1}) > 0$ *and* $\lim \inf_{n\to\infty} \prod_{j=1}^{i-1}(1-\beta_{n,j})(\beta_{n,1}-\mu_N > 0;$

(iii) $\lim \inf_{n\to\infty} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j})(\gamma_{n,1}-\mu_{i-1}) > 0$ *and* $\lim \inf_{n\to\infty} \prod_{j=1}^{i-1}(1-\gamma_{n,j})(\gamma_{n,1}-\mu_N > 0;$

(iv) $\lim_{x\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{x\to\infty} \dfrac{(1-\beta_{n,j})(k_n-1)}{\alpha_n} = 0;$

(v) $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$.

Then, the sequence defined by (3.44) *converges strongly to $q \in \mathcal{F}$, which solves the variational inequality $\langle (A-\gamma g)q, x-q\rangle \geq 0, \forall x \in \mathcal{F}$.*

Proof. Since $\mathcal{F} = (\cap_{i=1}^{N}(F(\Gamma_i)) \cap (\cap_{i=1}^{N} F(G_i) \cap EP(\Theta) \neq \emptyset$, it follows that both Γ_i and G_i are type-one and L_i-uniformly Lipschitizian asymptotically demicontractive multivalued mappings. Consequently the results of Theorem 3.5 follows immediately from Theorem 3.4. □

Theorem 3.6. *Let K be a convex, closed and nonempty subset of a real Hilbert space H . Let $\{\Gamma_i\}_{i=1}^{N}$ and $\{G_i\}_{i=1}^{N}$, $N \geq 2$ be finite families of type-one and L_i-uniformly Lipschitizian asymptotically strictly pseudocontractive multivalued mapping $\Gamma_i : K \to \mathcal{P}(K)$and type-one and L_i-uniformly Lipschitizian asymptotically strictly pseudononpreading multivalued mapping $G_i : K \to \mathcal{P}(K)$, respectively from K into the family of all proximinal subsets of K with contractive coeficient $\mu_i^{(1)}, \mu_i^{(2)} \in [0.1)$ for each i. Assume that $\mathcal{F} = (\cap_{i=1}^{N}(F(\Gamma_i)) \cap (\cap_{i=1}^{N} F(G_i) \neq \emptyset$ and for each i, $(I-\Gamma_i)$ and $(I-G_i)$ are weakly demiclosed at zero. let g_i be a contraction of K into itself with constant $\rho \in (0,1)$ and A be a strong positive self adjoint bounded linear operator on H with coeficient $\bar{\alpha}$ such that $0 < \rho\gamma + 2\epsilon < \bar{\alpha}$. Let the sequence $\{x_n\}$ be given iteratively as follows:*

$$\begin{cases} \omega_n = \gamma_{n,1}u_n + W_{n1}, \\ s_n = \beta_{n,1}\omega_n + W_{n2}, \\ x_{n+1} = \alpha_n v + ((1-\delta_n)s_n, \end{cases} \tag{3.45}$$

where

$$W_{n1} = \sum_{i=2}^{N} \gamma_{n,i} \prod_{j=1}^{i-1} (1-\gamma_{n,j})\nu_{n,i-1} + \prod_{j=1}^{N}(1-\gamma_{n,j})\nu_{n,N},$$

$$W_{n2} = \sum_{i=2}^{N} \beta_{n,i} \prod_{j=1}^{i-1} (1-\beta_{n,j})z_{n,i-1} + \prod_{j=1}^{N}(1-\beta_{n,j})z_{n,N},$$

$\nu_{n,i} \in \Gamma_i^n u_n$ *and* $z_{n,i} \in G_i^n \omega_n$ *for each* i, $\{\alpha_n\}, \{\delta_n\} \in [0,1], \big\{\{\beta_{n,i}\}_{n=1}^{\infty}\big\}_{i=1}^{\mathbb{N}}$ *and* $\big\{\{\gamma_{n,i}\}_{n=1}^{\infty}\big\}_{i=1}^{N}$ *are countably finite families of real sequences in* $[0,1]$. *Suppose the following conditions are satisfied:*

(*i*) $\beta_{n,1} \geq \gamma_{n,1} > \max\{\mu_i\}_{i=1}^{N} : \beta_{n,i} \leq \gamma_{n,i} < \gamma \leq \beta < 1$, *for each* i*;*

(*ii*) $\lim\inf_{n\to\infty} \beta_{n,i} \prod_{j=1}^{i-1}(1-\beta_{n,j})(\beta_{n,1}-\mu_{i-1}) > 0$ *and* $\lim\inf_{n\to\infty} \prod_{j=1}^{i-1}(1-\beta_{n,j})(\beta_{n,1}-\mu_N > 0$;

(*iii*) $\lim\inf_{n\to\infty} \gamma_{n,i} \prod_{j=1}^{i-1}(1-\gamma_{n,j})(\gamma_{n,1}-\mu_{i-1}) > 0$ *and* $\lim\inf_{n\to\infty} \prod_{j=1}^{i-1}(1-\gamma_{n,j})(\gamma_{n,1}-\mu_N > 0$;

(*iv*) $\lim_{x\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{x\to\infty} \dfrac{(1-\beta_{n,j})(k_n-1)}{\alpha_n} = 0.$

Then, the sequence defined by (3.45) *converges strongly to* $q \in \mathcal{F}$, *which solves the variational inequality* $\langle (A-\gamma g)q, x-q\rangle \geq 0, \forall x \in \mathcal{F}$.

Proof. If $\Psi(x,y) = 0$ for all $x, y \in K$, $r = 1$ for all $n \geq 0$ then $u_n = x_n$. Hence, with $g(x) = v$ and $A = I$ (where I is an identity map on H), the results follows immediately from Theorem 3.4. □

4. Numerical Example

Now, we give an example to illustrate that our proposed iteration scheme is implementable.

Example 4.1. Consider the nonempty closed convex subset $k = (-3,3)$ of a Hilbert space $\mathbb{R}$. Define the mappings $\Gamma_i : K \to \mathcal{P}K$ and $G_i : K \to \mathcal{P}K$, for $i = 1,2,3$, as follows:

$$\Gamma_i x = \begin{cases} \left[-\frac{1}{3^i}x, -\frac{1}{2^i}x\right], & x \in (-3,0], \\ \left[-\frac{1}{2^i}x, -\frac{1}{3^i}x\right], & x \in [0,3), \end{cases} \tag{4.1}$$

and

$$G_i x = \begin{cases} \left[-\frac{1}{3^{i+1}}x, -\frac{1}{2^{i+1}}x\right], & x \in (-3,0], \\ \left[-\frac{1}{2^{i+1}}x, -\frac{1}{3^{i+1}}x\right], & x \in [0,3). \end{cases} \tag{4.2}$$

Then,

$$\Gamma_i^n x = \begin{cases} \left[-\frac{1}{3^{in}}x, -\frac{1}{2^{in}}x\right], & x \in (-3, 0], \\ \left[-\frac{1}{2^{in}}x, -\frac{1}{3^{in}}x\right], & x \in [0, 3). \end{cases} \tag{4.3}$$

and

$$G_i^n x = \begin{cases} \left[-\frac{1}{3^{(i+1)n}}x, -\frac{1}{2^{(i+1)n}}x\right], & x \in (-3, 0], \\ \left[-\frac{1}{2^{(i+1)n}}x, -\frac{1}{3^{(i+1)n}}x\right], & x \in [0, 3). \end{cases} \tag{4.4}$$

Note that, using the same argument as in Example 4.1 in [41] and Example **??** above, it is easy to see that both maps are uniformly L-Lipschitizian and asymptotically demicontractive multivalued mappings.

Now, define the bifunction Ψ as follows:

$$\begin{cases} \Psi : K \times K \to \mathbb{R} \\ \Psi(x, y) = y^2 + xy - 2x^2. \end{cases} \tag{4.5}$$

It is not hard to see that Ψ satisfies conditions $(B1) - (B4)$. If we set $r_n = 1$, then $u_n = T_{r_n}(x_n) = \dfrac{x_n}{3r_n + 1} = \dfrac{x_n}{4}$, (*see* [50] *for details*). For $N = 3$, (3.4) becomes

$$\begin{cases} u_n \ni \Psi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in K, \\ \eta_n = \gamma_{n,1}u_n + Y_n, \\ s_n = \beta_{n,1}\omega_n + W_n, \\ x_{n+1} = P_K(\alpha_n\gamma g(x_n) + \delta_n s_n + ((1 - \delta_n)I - \alpha_n A)). \end{cases}$$

where

$$\begin{aligned} Y_n &= (1 - \gamma_{n,1})\gamma_{n,2}\nu_{n,1} + (1 - \gamma_{n,1})(1 - \gamma_{n,2})\gamma_{n,3}\nu_{n,2} \\ &\quad + (1 - \gamma_{n,1})(1 - \gamma_{n,2})(1 - \gamma_{n,3})\nu_{n,3}, \\ W_n &= (1 - \beta_{n,1})\beta_{n,2}z_{n,1} + (1 - \beta_{n,1})(1 - \beta_{n,2})\beta_{n,3}z_{n,2} \\ &\quad + (1 - \beta_{n,1})(1 - \beta_{n,2})(1 - \beta_{n,3})z_{n,3}. \end{aligned}$$

Put $g(x) = \dfrac{x}{2}, A = 1, \gamma = 1, \gamma_{n,1} = \gamma_{n,2} = \gamma_{n,3} = \dfrac{1}{4}, \beta_{n,1} = \beta_{n,2} = \beta_{n,3} = \dfrac{1}{3}$ and $\alpha_n = \dfrac{n}{n+1}$. Then, for arbitrary $x_0 \in K$, the above iteration scheme yields:

$$(4.6)\qquad \begin{cases} \omega_n = \dfrac{x_n}{4} + \dfrac{3}{16}\nu_{n,1} + \dfrac{3}{64}\nu_{n,2} + \dfrac{1}{64}\nu_{n,3}, \\ s_n = \dfrac{\omega_n}{3} + \dfrac{2}{9}z_{n,1} + \dfrac{2}{27}z_{n,2} + \dfrac{1}{27}z_{n.3}, \\ x_{n+1} = \dfrac{x_n}{2(n+1)} + \dfrac{n}{n+1}s_n, \end{cases}$$

where

$$\nu_{n,1} \in \left[-\frac{1}{2^n}x_n, -\frac{1}{3^n}x_n\right],$$
$$\nu_{n,2} \in \left[-\frac{1}{4^n}x_n, -\frac{1}{9^n}x_n\right],$$
$$\nu_{n,3} \in \left[-\frac{1}{8^n}x_n, -\frac{1}{27^n}x_n\right],$$

for $x_n \in (-3, 0]$ and

$$\nu_{n,1} \in \left[-\frac{1}{3^n}x_n, -\frac{1}{2^n}x_n\right],$$
$$\nu_{n,2} \in \left[-\frac{1}{9^n}x_n, -\frac{1}{4^n}x_n\right],$$
$$\nu_{n,3} \in \left[-\frac{1}{27^n}x_n, -\frac{1}{8^n}x_n\right],$$

for $x_n \in [0, 3)$ whereas

$$z_{n,1} \in \left[-\frac{1}{4^n}x_n, -\frac{1}{9^n}x_n\right],$$
$$z_{n,2} \in \left[-\frac{1}{8^n}x_n, -\frac{1}{27^n}x_n\right],$$
$$z_{n,3} \in \left[-\frac{1}{16^n}x_n, -\frac{1}{81^n}x_n\right],$$

for $x_n \in (-3, 0]$ and

$$z_{n,1} \in \left[-\frac{1}{9^n}x_n, -\frac{1}{4^n}x_n\right],$$
$$z_{n,2} \in \left[-\frac{1}{27^n}x_n, -\frac{1}{8^n}x_n\right],$$
$$z_{n,3} \in \left[-\frac{1}{81^n}x_n, -\frac{1}{16^n}x_n\right],$$

for $x_n \in [0, 3)$.

Observe that the sequence $\{x_n\}$ converges to zero ad $n \to \infty$. That is, $\mathcal{F} = \left(\cap_{i=1}^{3} \left(F\left(\Gamma_i\right)\right)\right) \cap \left(\cap_{i=1}^{3} F\left(G_i\right)\right) \neq \emptyset \cap EP(\Psi) = \{0\}$.

Remark 4.2. Since every asymptotically quasi-nonexpansive multivalued mapping is asymptotically demicontractive multivalued mapping with the constant $k = 0$, Theorem 3.4 holds true for this class of mapping.

Remark 4.3. Theorem 3.4 improves results of Isogugu, lzuchukwu and Okeke [40]. Indeed, Isogugu, lzuchukwu and Okeke considered a horizontal iteration scheme with one auxiliary multivalued (demicontractive) mapping, but here we presented a modified horizontal iteration scheme with two auxiliary multivalued (asymptotically demicontractive) mapping.

Remark 4.4. Theorem 3.4 generalises the results of Ali and Umar [35] and Osilike and Aniagbosor [39] from single-valued multivalued quasi-nonexpansive and single-valued asymptotically demicontractive mapping to finite family of asymptotically demicontractive multivalued mapping, respectively.

Remark 4.5. Theorem 3.5 generalises the results of Zhaoli and Wang [46] from single-valued asymptotically strictly pseudononspreading mapping to finite families of two asymptotically pseudononspreading mapping.

5. Conclusion

In this paper, we have introduced and studied a new class of asymptotically demicontractive multivalued mapping and proved convergence results for equilibrium problems and fixed point problems of type-one asymptotically demicontractive multivalued mapping without an imposition of any sum conditions on the control parameters, in the setup of a real Hilbert space. Also, we provided a numerical example to demonstrate the implementablity of our proposed iteration technique. The results obtained in this paper extend, improve and generalise several known results so far obtained in this direction.

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