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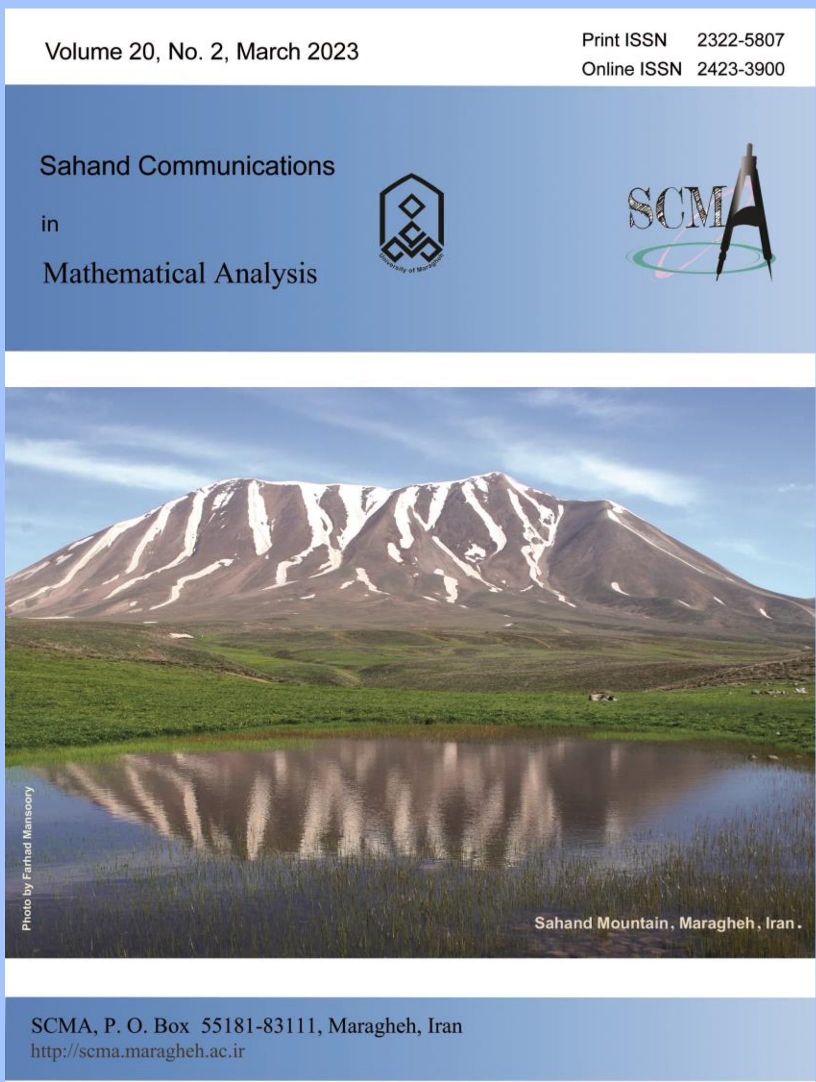

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Some Basic Results on Fuzzy Strong ϕ-b-Normed Linear Spaces

Abhishikta Das[1], Tarapada Bag[2*] and Shayani Chatterjee[3]

Abstract. Following the definition of fuzzy normed linear space which was introduced by Bag and Samanta in general t-norm settings, in this paper, definition of fuzzy strong ϕ-b-normed linear space is given. Here the scalar function $|c|$ is replaced by a general function $\phi(c)$ where ϕ satisfies some properties. Some basic results on finite dimensional fuzzy strong ϕ-b-normed linear space are studied.

1. Introduction

The concept of a fuzzy set was introduced initially by Zadeh [23] in 1965. Since then, many authors have expansively developed the theory of fuzzy sets. Osmo Kaleva [12], Kramosil and Michalek [17], Georage and Veeramani [11] et al. have introduced the concept of fuzzy metric spaces in different approaches. On the other hand, Katsaras [13], Felbin [8], Cheng and Mordeson [6], Bag and Samanta [1] have given the definition of fuzzy normed linear spaces in different way.

Recently different types of generalized metric as well as norm (viz. 2-metric [9], b-metric [7], strong b-metric [15], G-metric [19], 2-norm [10], G-norm [14], etc.) and consequently generalized fuzzy metric and fuzzy norm (viz. fuzzy b-metric [20], strong fuzzy b-metric [22], fuzzy cone metric [21], fuzzy cone norm [4], G-fuzzy norm [5], etc.) have introduced in different approaches.

In 2018, Oner [22] introduced the concept of fuzzy strong b-metric spaces and developed some topological results in such spaces. Following

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this definition of fuzzy strong b-metric spaces, in this paper, we give a definition of fuzzy strong ϕ-b-normed linear space whose induced fuzzy metric is Oner type. In fuzzy normed linear space, scalar multiplication is given by $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$. But in our definition of fuzzy strong ϕ-b-norm, scalar multiplication is given by $N(cx,t) = N\left(x, \frac{t}{\phi(c)}\right)$ where ϕ is a non-negative real valued function satisfying some properties. We study some results on finite dimensional fuzzy strong ϕ-b-normed linear spaces.

The organization of the paper is in the following.

Section 2 consists some preliminary results. In Section 3, we introduce a definition of fuzzy strong ϕ-b-norm by using a special function ϕ in general t-norm settings and illustrate by examples. Some basic results of finite dimensional fuzzy strong ϕ-b-normed linear spaces are established in Section 4.

2. Preliminaries

In this section, some definitions and results are collected which are used in this paper.

Definition 2.1 ([16])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *t*-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $\alpha * 1 = \alpha,\ \forall \alpha \in [0,1]$;
(iii) $\alpha * \gamma \leq \beta * \delta$ whenever $\alpha \leq \beta$ and $\gamma \leq \delta,\ \forall \alpha,\beta,\gamma,\delta \in [0,1]$.

If $*$ is continuous, then it is called continuous *t*-norm.

The following are examples of some t-norms.

(i) Standard intersection: $\alpha * \beta = \min\{\alpha, \beta\}$.
(ii) Algebraic product: $\alpha * \beta = \alpha\beta$.
(iii) Bounded difference: $\alpha * \beta = \max\{0, \alpha + \beta - 1\}$.

Definition 2.2 ([3])**.** Let X be a linear space over a field $\mathbb{F}$. A fuzzy subset N of $X \times \mathbb{R}$ is called fuzzy norm on X if for all $x, y \in X$ the following conditions hold:

(N1) $\forall t \in \mathbb{R}$ with $t \leq 0,\ N(x,t) = 0$;
(N2) $(\forall t \in \mathbb{R},\ t > 0,\ N(x,t) = 1) \quad \Leftrightarrow \quad x = \theta$;
(N3) $\forall t \in \mathbb{R}$, and $c \in \mathbb{R} \setminus \{0\},\ t > 0,\ N(cx,t) = N\left(x, \frac{t}{|c|}\right)$;
(N4) $\forall s,t \in \mathbb{R},\ N(x+y, s+t) \geq N(x,s) * N(y,t)$;
(N5) $N(x,\cdot)$ is a non-decreasing function of t and $\lim_{t\to\infty} N(x,t) = 1$.

Then the pair (X, N) is called fuzzy normed linear space.

Definition 2.3 ([1])**.** Let (X, N) be a fuzzy normed linear space.

(i) A sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. Then x is called the limit of the sequence $\{x_n\}$ and denoted by $\lim_{n\to\infty} x_n$.

(ii) A sequence $\{x_n\}$ in a fuzzy normed linear space (X, N) is said to be Cauchy if $\lim_{n\to\infty} N(x_{n+p} - x_n, t) = 1$ for all $t > 0$ and $p = 1, 2, \ldots$.

(iii) $A \subseteq X$ is said to be closed if for any sequence $\{x_n\}$ in A converges to $x \in A$.

(iv) $A \subseteq X$ is said to be the closure of A, denoted by $\overline{A}$ if for any $x \in \overline{A}$, if there is a sequence $\{x_n\} \subseteq A$ such that $\{x_n\}$ converges to x.

(v) $A \subseteq X$ is said to be compact if any sequence $\{x_n\} \subseteq A$ has a subsequence converging to an element of A.

Definition 2.4 ([2])**.** Let (X, N) be a fuzzy normed linear space.

(i) A set $B(x, \alpha, t)$, $0 < \alpha < 1$ is defined as

$$B(x, \alpha, t) = \{y : N(x - y, t) > 1 - \alpha\}.$$

(ii) $\tau = \{G \subseteq X : x \in G, \ \exists t > 0, \ 0 < \alpha < 1 \text{ such that } B(x, \alpha, t) \subset G\}$ is a topology on (X, N).

(iii) Members of τ are called open sets in (X, N).

Definition 2.5 ([3])**.** A subset B of a fuzzy normed linear space (X, N) is said to be fuzzy bounded if for each r, $0 < r < 1$, $\exists t > 0$ such that

$$N(x, t) > 1 - r, \quad \text{for all } x \in B.$$

Lemma 2.6 ([3])**.** *Let (X, N) be a fuzzy normed linear space and $N(x, \cdot)(x \neq 0)$. If the set $M = \{x : N(x, 1) > 0\}$ is compact, then X is finite dimensional.*

3. Fuzzy Strong ϕ-b-Normed Linear Space

In this section, we give the definition of fuzzy normed linear space in a new approach.

Definition 3.1. Let ϕ be a function defined on $\mathbb{R}$ to $\mathbb{R}^+$ with the following properties

(ϕ1) $\phi(-t) = \phi(t)$, for all $t \in \mathbb{R}$;

(ϕ2) $\phi(1) = 1$;

(ϕ3) ϕ is strictly increasing and continuous on $(0, \infty)$;

(ϕ4) $\lim_{\alpha\to 0} \phi(\alpha) = 0$ and $\lim_{\alpha\to\infty} \phi(\alpha) = \infty$.

The followings are examples of such functions.

(i) $\phi(\alpha) = |\alpha|$ for all $\alpha \in \mathbb{R}$.

(ii) $\phi(\alpha) = |\alpha|^p$ for all $\alpha \in \mathbb{R}$, $p \in \mathbb{R}^+$.

(iii) $\phi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$ for all $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$.

Definition 3.2. Let X be a linear space over the field $\mathbb{R}$ and $b \geq 1$ be a given real number. A fuzzy subset N of $X \times \mathbb{R}$ is called fuzzy strong ϕ-b-norm on X if for all $x, y \in X$ the following conditions hold:

(bN1) $\forall t \in \mathbb{R}$ with $t \leq 0,\ N(x,t) = 0$;
(bN2) $(\forall t \in \mathbb{R},\ t > 0,\ N(x,t) = 1)$ iff $x = \theta$;
(bN3) $\forall t \in \mathbb{R},\ t > 0,\ N(cx,t) = N\left(x, \frac{t}{\phi(c)}\right)$ if $\phi(c) \neq 0$;
(bN4) $\forall s, t \in \mathbb{R},\ N(x+y, s+bt) \geq N(x,s) * N(y,t)$;
(bN5) $N(x,\cdot)$ is a non-decreasing function of t and $\lim_{t\to\infty} N(x,t) = 1$.

Then $(X, N, \phi, b, *)$ is called fuzzy strong ϕ-b-normed linear space.

Remark 3.3. If $b = 1$ and $\phi(\alpha) = |\alpha|$ then (X, N) is a fuzzy normed linear space in the sense of Definition 2.2.

Example 3.4. Consider the linear space $\mathbb{R}$ and a fuzzy subset N of $\mathbb{R} \times \mathbb{R}$ by

$$N(x,t) = \begin{cases} \frac{t}{t+|x|^p} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$ and $0 < p \leq 1$.

Consider the t-norm $*$ by $\alpha * \beta = \min\{\alpha, \beta\}$ for all $\alpha, \beta \in [0,1]$. We show that N is a fuzzy strong ϕ-b-norm on $\mathbb{R}$. For,

(i) $\forall t \in \mathbb{R}$ with $t \leq 0$, by definition we have, $N(x,t) = 0$. Thus, (bN1) holds.
(ii) $(\forall t \in \mathbb{R}, t > 0,\ N(x,t) = 1) \Leftrightarrow \frac{t}{t+|x|^p} = 1 \Leftrightarrow |x|^p = 0 \Leftrightarrow x = 0$
Therefore (bN2) holds.
(iii) $\forall t > 0$ and $c \in \mathbb{R}\setminus\{0\}$, $N(cx,t) = \frac{t}{t+|cx|^p} = \frac{\frac{t}{|c|^p}}{\frac{t}{|c|^p}+|x|^p} = N\left(x, \frac{t}{\phi(c)}\right)$
where $\phi(c) = |c|^p,\ c \in \mathbb{R}$ and clearly ϕ satisfies all the conditions of Definition 3.2. Thus, (bN3) holds.
(iv) $\forall s, t > 0$ and $x, y \in \mathbb{R}$, $N(x+y, bs+t) = \frac{bs+t}{bs+t+|x+y|^p}$ and

$$\begin{aligned} N(x,s) * N(y,t) &= \min\{N(x,s), N(y,t)\} \\ &= \min\left\{\frac{s}{s+|x|^p}, \frac{t}{t+|y|^p}\right\}. \end{aligned}$$

We only prove the inequality for $s, t > 0$. Let $N(x,s) * N(y,t) = \min\{N(x,s), N(y,t)\} = N(x,s)$. Then $N(y,t) \geq N(x,s) \Rightarrow \frac{t}{t+|y|^p} \geq \frac{s}{s+|x|^p} \Rightarrow t|x|^p \geq s|y|^p$. Again $x, y \in \mathbb{R}$ and $0 < p \leq 1$, $|x+y|^p \leq 2^p|x|^p + |y|^p$. If we take $b = 2^p$, then

$$N\left(x+y, 2^p s + t\right) - N(x,s) = \frac{2^p s + t}{2^p s + t + |x+y|^p} - \frac{s}{s+|x|^p}$$

$$\begin{aligned}&\geq \frac{2^p s+t}{2^p s+t+2^p|x|^p+|y|^p}-\frac{s}{s+|x|^p}\\&=\frac{t|x|^p-s|y|^p}{(2^p s+t+2^p|x|^p+|y|^p)(s+|x|^p)}\\&\geq 0\end{aligned}$$

Hence $N\left(x+y, 2^p s+t\right) \geq N(x, s)=\min\{N(x, s), N(y, t)\}$. Similarly, it can be shown that if $\min\{N(x, s), N(y, t)\}=N(y, t)$, then $N\left(x+y, 2^p s+t\right) \geq N(y, t)=\min\{N(x, s), N(y, t)\}$.
Therefore, (bN4) holds.

(v) From the definition of $N(x,t)$, it is clear that $N(x,.)$ is a non-decreasing function of t and $\lim\limits_{t\to\infty} N(x,t)=1$.

Hence $(X, N, \phi, b, *)$ is a fuzzy strong ϕ-b-normed linear space where $b=2^p(>1)$ and $\phi(\alpha)=|\alpha|^p,\ \forall\alpha\in\mathbb{R},\ 0<p\leq 1$.

Example 3.5. Consider the linear space $\mathbb{R}$ and a fuzzy subset N of $\mathbb{R}\times\mathbb{R}$ by

$$N(x,t)=\begin{cases}\exp(-\frac{|x|^p}{t}) & t>0\\ 0 & t\leq 0\end{cases}$$

for all $x\in\mathbb{R}$ and $0<p\leq 1$ and consider the t-norm $*$ by
$\alpha*\beta=\alpha\beta\ \forall\alpha,\beta\in[0,1]$. Now,

(i) Clearly (bN1) holds from the definition.

(ii) $(\forall t\in\mathbb{R}, t>0,\ N(x,t)=1) \quad\Leftrightarrow\quad \exp(-\frac{|x|^p}{t})=1 \quad\Leftrightarrow$ $|x|^p=0 \quad\Leftrightarrow\quad x=0$. Therefore (bN2) holds.

(iii) $\forall t>0$ and $c\in\mathbb{R}\setminus\{0\}$,

$$\begin{aligned}N(cx,t)&=\exp\left(-\frac{|cx|^p}{t}\right)\\&=\exp\left(-\frac{|x|^p}{\frac{t}{|c|^p}}\right)\\&=N\left(x,\frac{t}{\phi(c)}\right)\end{aligned}$$

where $\phi(c)=|c|^p,\ c\in\mathbb{R}$ and clearly ϕ satisfies all the conditions of Definition 3.2. Thus, (bN3) holds.

(iv) For $s,t>0$ and $x,y\in\mathbb{R}$, $N(x+y,bs+t)=\exp\left(-\frac{|x+y|^p}{bs+t}\right)$ and

$$\begin{aligned}N(x,s)*N(y,t)&=N(x,s)\cdot N(y,t)\\&=\exp\left(-\frac{|x|^p}{s}\right)\cdot\exp\left(-\frac{|y|^p}{t}\right).\end{aligned}$$

Using the inequality, $|x+y|^p \leq 2^p|x|^p+|y|^p,\ x,y \in \mathbb{R}$ and $0<p\leq 1$ and taking $b=2^p$, we obtain

$$\begin{aligned} -\frac{|x+y|^p}{2^p s+t} &\geq -\frac{2^p|x|^p+|y|^p}{2^p s+t} \\ &\geq -\frac{2^p|x|^p}{2^p s+t}-\frac{|y|^p}{2^p s+t} \\ &\geq -\frac{|x|^p}{s}-\frac{|y|^p}{t} \end{aligned}$$

which implies $N\left(x+y, 2^p s+t\right) \geq N(x,s)\cdot N(y,t)$. Thus (bN4): $N(x+y,2^p s+t) \geq N(x,s) * N(y,t)$ holds $\forall s,t \in \mathbb{R}$ and $\forall x,y \in \mathbb{R}$.

(v) Clearly $N(x,\cdot)$ is a non-decreasing function of t and $\lim_{t\to\infty} N(x,t) = 1$.

Hence $(X,N,\phi,b,*)$ is a fuzzy strong ϕ-b-normed linear space where $b=2^p(>1)$ and $\phi(\alpha)=|\alpha|^p, \forall\alpha\in\mathbb{R},\ 0<p\leq 1$.

Remark 3.6. The notions of converges, Cauchy sequences, boundedness, etc. are same as definitions in Bag and Samanta type fuzzy normed linear space [1].

4. Finite Dimensional Fuzzy Strong ϕ-b-Normed Linear Spaces

In this section, some basic results on finite dimensional fuzzy strong ϕ-b-normed linear spaces are established.

Lemma 4.1. *Let $(X,N,\phi,b,*)$ be a fuzzy strong ϕ-b-normed linear space with the underlying t-norm $*$ continuous at $(1,1)$ and $\{x_1,x_2,\cdots x_n\}$ be a linearly independent set of vectors in X. Then there exists $c>0$ and $\delta\in(0,1)$ such that for any set of scalars $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ with $\sum_{i=1}^{n}|\alpha_i| \neq 0$,*

$$N\left(\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{n}|\alpha_i|}\right)}\right) < 1-\delta. \tag{4.1}$$

Proof. The relation (4.1) is equivalent to the relation

$$N\left(\beta_1x_1+\beta_2x_2+\cdots+\beta_nx_n, bc\right) < 1-\delta \tag{4.2}$$

for some $c>0$ and $\delta\in(0,1)$ and for all set of scalars $\{\beta_1,\beta_2,\cdots,\beta_n\}$ with $\sum_{i=1}^{n}|\beta_i|=1$.

If possible, suppose that (4.2) does not hold. Thus for each $c > 0$ and $\delta \in (0,1)$, there exists a set of scalars $\{\beta_1, \beta_2, \ldots, \beta_n\}$ with $\sum_{i=1}^{n} |\beta_i| = 1$ for which

$$N(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n, bc) \geq 1 - \delta.$$

Then for $c = \delta = \frac{1}{m}, m = 1, 2, \ldots,$ there exists a set of scalars $\left\{\beta_1^{(m)}, \beta_2^{(m)}, \ldots, \beta_n^{(m)}\right\}$ with $\sum_{i=1}^{n} \left|\beta_i^{(m)}\right| = 1$ such that $N\left(y_m, \frac{b}{m}\right) \geq 1 - \frac{1}{m}$ where $y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \cdots + \beta_n^{(m)} x_n$.

Since $\sum_{i=1}^{n} \left|\beta_i^{(m)}\right| = 1$, we have $0 \leq \left|\beta_i^{(m)}\right| \leq 1$ for $i = 1, 2, \ldots, n$. So for each fixed i, the sequence $\left\{\beta_i^{(m)}\right\}$ is bounded and hence $\left\{\beta_i^{(m)}\right\}$ has a convergent subsequence. Let β_1 denotes the limit of that subsequence and let $\{y_{1,m}\}$ denotes the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\left\{\beta_2^{(m)}\right\}$ converges to β_2. Continuing in this way, after n steps we obtain a subsequence $\{y_{n,m}\}$ where $y_{n,m} = \sum_{i=1}^{n} \gamma_i^{(m)} x_i$ with $\sum_{i=1}^{n} |\gamma_i^{(m)}| = 1$ and $\gamma_i^{(m)} \to \beta_i$ as $m \to \infty$ for each $i = 1, 2, \cdots, n$.

Let $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$. Now,

$$\begin{aligned}
&N(y_{n,m} - y, t) \\
&= N\left(\sum_{j=1}^{n} \left(\gamma_j^{(m)} - \beta_j\right) x_j, t\right) \\
&= N\left(\left(\gamma_1^{(m)} - \beta_1\right) x_1 + \sum_{j=2}^{n} \left(\gamma_j^{(m)} - \beta_j\right) x_j, \frac{t}{n} + b(n-1)\frac{t}{nb}\right) \\
&\geq N\left(\left(\gamma_1^{(m)} - \beta_1\right) x_1, \frac{t}{n}\right) * N\left(\sum_{j=2}^{n} \left(\gamma_j^{(m)} - \beta_j\right) x_j, (n-1)\frac{t}{nb}\right) \\
&= N\left(\left(\gamma_1^{(m)} - \beta_1\right) x_1, \frac{t}{n}\right) \\
&\quad * N\left(\left(\gamma_2^{(m)} - \beta_2\right) x_2 + \sum_{j=3}^{n} \left(\gamma_j^{(m)} - \beta_j\right) x_j, \frac{t}{nb} + b\left(1 - \frac{2}{n}\right)\frac{t}{b^2}\right) \\
&\geq N\left(\left(\gamma_1^{(m)} - \beta_1\right) x_1, \frac{t}{n}\right) * N\left(\left(\gamma_2^{(m)} - \beta_2\right) x_2, \frac{t}{nb}\right)
\end{aligned}$$

$$* N\left(\sum_{j=3}^{n}\left(\gamma_j^{(m)}-\beta_j\right)x_j, \left(1-\frac{2}{n}\right)\frac{t}{b^2}\right)$$
$$\vdots$$
$$\geq N\left(\left(\gamma_1^{(m)}-\beta_1\right)x_1, \frac{t}{n}\right) * N\left(\left(\gamma_2^{(m)}-\beta_2\right)x_2, \frac{t}{nb}\right) * \cdots$$
$$* N\left(\left(\gamma_n^{(m)}-\beta_n\right)x_n, \frac{t}{nb^{n-1}}\right)$$
$$= N\left(x_1, \frac{t}{n\phi\left(\left(\gamma_1^{(m)}-\beta_1\right)\right)}\right) * \cdots * N\left(x_n, \frac{t}{nb^{n-1}\phi\left(\left(\gamma_n^{(m)}-\beta_n\right)\right)}\right).$$

Taking limit as $m \to \infty$ on both sides, we have

$$\lim_{m\to\infty} N\left(y_{n,m}-y, t\right) \geq 1 * 1 * \cdots * 1, \quad \forall t > 0$$

i.e

$$\lim_{m\to\infty} N\left(y_{n,m}-y, t\right) = 1, \quad \forall t > 0.$$

Now for $r > 0$, choose m such that $\frac{1}{m} < \frac{r}{b^2}$. We have

$$N\left(y_{n,m}, \frac{r}{b}\right) = N\left(y_{n,m}+\theta, \frac{b}{m}+b\left(\frac{r}{b^2}-\frac{1}{m}\right)\right)$$
$$\geq N\left(y_{n,m}, \frac{b}{m}\right) * N\left(\theta, \frac{r}{b^2}-\frac{1}{m}\right)$$
$$\geq \left(1-\frac{b}{m}\right) * 1$$

which implies $\lim\limits_{m\to\infty} N\left(y_{n,m}, \frac{r}{b}\right) \geq 1$ i.e $\lim\limits_{m\to\infty} N\left(y_{n,m}, \frac{r}{b}\right) = 1$. Again,

$$N(y, 2r) = N\left(y - y_{n,m} + y_{n,m}, r + b\cdot\frac{r}{b}\right)$$
$$\geq N\left(y-y_{n,m}, r\right) * N\left(y_{n,m}, \frac{r}{b}\right)$$

thus

$$N(y, 2r) \geq \lim_{m\to\infty} N\left(y-y_{n,m}, r\right) * \lim_{m\to\infty} N\left(y_{n,m}, \frac{r}{b}\right)$$
$$\Rightarrow \qquad N(y, 2r) \geq 1 * 1 = 1$$
$$\Rightarrow \qquad N(y, 2r) = 1.$$

Since $r > 0$ is arbitrary, so $y = \theta$.

Again since $\sum_{i=1}^{n}\left|\beta_i^{(m)}\right| = 1$ and $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set of vectors so $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \neq \theta$, thus we arrive at a contradiction. Hence (4.2) holds and Lemma is proved. □

Theorem 4.2. *Every finite dimensional fuzzy strong ϕ-b-normed linear space with the underlying t-norm $*$ continuous at $(1,1)$ is complete.*

Proof. Let $(X, N, \phi, b, *)$ be a fuzzy strong ϕ-b-normed linear space where $b(>1)$ is a real constant. Let $\dim X = r$ and $\{e_1, e_2, \ldots, e_r\}$ be a basis for X. Let $\{x_p\}$ be a Cauchy sequence in X. Then $x_n = \sum_{k=1}^{r} \beta_k^{(n)} e_k$ for suitable scalars $\beta_1^{(n)}, \beta_2^{(n)}, \ldots, \beta_r^{(n)}$. So,

$$\lim_{m,n\to\infty} N\left(x_m - x_n, t\right) = 1, \quad \forall t > 0 \tag{4.3}$$

Now from Lemma 4.1, it follows that $\exists\ c > 0$ and $\delta \in (0,1)$ such that

$$N\left(\sum_{i=1}^{r}\left(\beta_i^{(m)} - \beta_i^{(n)}\right)e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\beta_i^{(m)} - \beta_i^{(n)}\right|}\right)}\right) < 1 - \delta. \tag{4.4}$$

If $\sum_{i=1}^{r}\left|\beta_i^{(m)} - \beta_i^{(n)}\right| = 0$ then $\beta_i^{(m)} = \beta_i^{(n)}$ $\forall i$ implies $\{x_n\}$ is a constant sequence and hence follows the theorem. So we may assume $\sum_{i=1}^{r}\left|\beta_i^{(m)} - \beta_i^{(n)}\right| \neq 0$.

Again for $0 < \delta < 1$, from (4.3), it follows that there exists a positive integer $n_0(\delta, t)$ such that

$$N\left(\sum_{i=1}^{r}\left(\beta_i^{(m)} - \beta_i^{(n)}\right)e_i, t\right) > 1 - \delta, \quad \forall m, n \geq n_0(\delta, t). \tag{4.5}$$

Now from (4.4) and (4.5), $\forall m, n \geq n_0(\delta, t)$ we have,

$$N\left(\sum_{i=1}^{r}\left(\beta_i^{(m)} - \beta_i^{(n)}\right)e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\beta_i^{(m)} - \beta_i^{(n)}\right|}\right)}\right) < N\left(\sum_{i=1}^{r}\left(\beta_i^{(m)} - \beta_i^{(n)}\right)e_i, t\right)$$

thus

$$\frac{bc}{\phi\left(\frac{1}{\sum\limits_{i=1}^{r}\left|\beta_i^{(m)}-\beta_i^{(n)}\right|}\right)} < t \text{ (Since } N(x,t) \text{ is non-decreasing w.r.t } t)$$

Since $t > 0$ is arbitrary, thus

$$\lim_{m,n\to\infty} \frac{bc}{\phi\left(\frac{1}{\sum\limits_{i=1}^{r}\left|\beta_i^{(m)}-\beta_i^{(n)}\right|}\right)} = 0$$

then

$$\lim_{m,n\to\infty} \phi\left(\frac{1}{\sum\limits_{i=1}^{r}\left|\beta_i^{(m)}-\beta_i^{(n)}\right|}\right) = \infty$$

and thus

$$\phi\left(\frac{1}{\lim\limits_{m,n\to\infty}\sum\limits_{i=1}^{r}\left|\beta_i^{(m)}-\beta_i^{(n)}\right|}\right) = \infty \text{ (Since } \phi \text{ is continuous)}$$

then

$$\lim_{m,n\to\infty}\sum_{i=1}^{r}\left|\beta_i^{(m)}-\beta_i^{(n)}\right| = 0 \text{ (Since } \lim_{\alpha\to\infty}\phi(\alpha)=\infty)$$

Therefore, $\left\{\beta_i^{(m)}\right\}$ is a Cauchy sequence of scalars for each $i = 1, 2, \ldots, r$. So each sequence $\left\{\beta_i^{(m)}\right\}$ converges. Let $\lim\limits_{n\to\infty} \beta_i^{(n)} = \beta_i$ for $i = 1, 2, \ldots, r$. Define $x = \sum\limits_{i=1}^{r} \beta_i e_i$. Then clearly $x \in X$. By similar calculation as in Lemma 4.1, it can be shown that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1, \forall t > 0$. Hence X is complete. □

Theorem 4.3. *Let $(X, N, \phi, b, *)$ be a finite dimensional fuzzy strong ϕ-b-normed linear space in which the underlying t-norm $*$ continuous at $(1,1)$. Then a subset A of X is compact iff A is closed and bounded.*

Proof. First we suppose that A is compact. We have to show that A is closed and bounded. For, let $x \in \bar{A}$. Then there exist a sequence $\{x_n\}$ in A such that $\lim\limits_{n\to\infty} x_n = x$.

Since A is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a point in A. Again $x_n \to x$ so $x_{n_k} \to x$ and hence $x \in A$.

So A is closed. If possible suppose that A is not bounded. Then there exists r_0, $0 < r_0 < 1$, such that for each positive integer n, there exists $x_n \in A$ for which $N(x_n, n) \leq 1 - r_0$.

Since A is compact, there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ converging to some element $x \in A$. Thus $\lim_{p\to\infty} N(x_{n_p} - x, t) = 1, \forall t > 0$. Again, $N(x_{n_p}, n_p) \leq 1 - r_0$. Therefore,

$$\begin{aligned} 1 - r_0 &\geq N(x_{n_p}, n_p) \\ &= N\left(x_{n_p} - x + x, \frac{t}{b} + b\left(\frac{n_p}{b} - \frac{t}{b^2}\right)\right) \\ &\geq N\left(x_{n_p} - x, \frac{t}{b}\right) * N\left(x, \left(\frac{n_p}{b} - \frac{t}{b^2}\right)\right). \end{aligned}$$

On above inequality, taking limit as $p \to \infty$, we obtain

$$1 - r_0 \geq \lim_{p\to\infty} N\left(x_{n_p} - x, \frac{t}{b}\right) * \lim_{p\to\infty} N\left(x, \left(\frac{n_p}{b} - \frac{t}{b^2}\right)\right)$$

thus $1 - r_0 \geq 1 * 1 = 1$ and then $r_0 \leq 0$. This is a contradiction. Hence A is bounded.

Conversely suppose that A is closed and bounded and we have to show that A is compact. Let $\dim X = r$ and $\{e_1, e_2, \ldots, e_r\}$ be a basis for X. Let us choose a sequence $\{x_p\}$ in A and suppose $x_p = \beta_1^{(p)} e_1 + \beta_2^{(p)} e_2 + \cdots + \beta_r^{(p)} e_r$ for suitable scalars $\beta_1^{(p)}, \beta_2^{(p)}, \ldots, \beta_r^{(p)}$. Now from Lemma 4.1, there exists $c > 0$ and $\delta \in (0,1)$ such that

$$N\left(\sum_{i=1}^{r} \beta_i^{(p)} e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\beta_i^{(p)}\right|}\right)}\right) < 1 - \delta. \tag{4.6}$$

Again since A is bounded, for $\delta \in (0,1)$, there exists $t_0 > 0$ such that $N(x, t_0) > 1 - \delta \ \forall x \in A$. So

$$N\left(\sum_{i=1}^{r} \beta_i^{(p)} e_i, t_0\right) > 1 - \delta. \tag{4.7}$$

From (4.6) and (4.7) we get,

$$N\left(\sum_{i=1}^{r}\beta_i^{(p)}e_i, \frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\beta_i^{(p)}\right|}\right)}\right) < N\left(\sum_{i=1}^{r}\beta_i^{(p)}e_i, t_0\right)$$

then

$$\frac{bc}{\phi\left(\frac{1}{\sum_{i=1}^{r}\left|\beta_i^{(p)}\right|}\right)} < t_0 \quad \text{(Since } N(x,t) \text{ is non-decreasing w.r.t } t\text{).}$$

Without loss of generality we may assume that $\sum_{i=1}^{r}\left|\beta_i^{(p)}\right| \neq 0$.

If $\sum_{i=1}^{r}\left|\beta_i^{(p)}\right| = 0$, then $\beta_i^{(p)} = 0$ for $i = 1, 2, \ldots, r$ and $p = 1, 2, \ldots$. In this case $\{x_p\}$ is a constant sequence and the theorem follows.

Since c, b, t_0 are three fixed positive real numbers, it follows that $0 < \sum_{i=1}^{r}\left|\beta_i^{(p)}\right| < \infty$. Therefore the sequence of scalars $\left\{\beta_i^{(p)}\right\}$ is bounded for each $i = 1, 2, \ldots, r$. So by Bolzano-Weierstrass theorem, there exists a convergent subsequence of $\left\{\beta_i^{(p)}\right\}$. Now, we follow the techniques of Lemma 4.1 to show that, there exists a subsequence of $\{x_p\}$ that converges to some point in A. Thus A is compact and this proves the theorem. □

Conclusion

Recently different types of generalized fuzzy metric spaces as well as generalized fuzzy normed linear spaces have been developed by several authors. Following the definition of fuzzy strong b-metric spaces, we introduced the idea of fuzzy strong ϕ-b-normed linear spaces and studied some results in finite finite dimensional fuzzy strong ϕ-b-normed linear spaces. We think there is a huge scope of research to develop fuzzy strong ϕ-b-normed linear spaces. Results on completeness and compactness, operator norms etc. are the open problems in such spaces.

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