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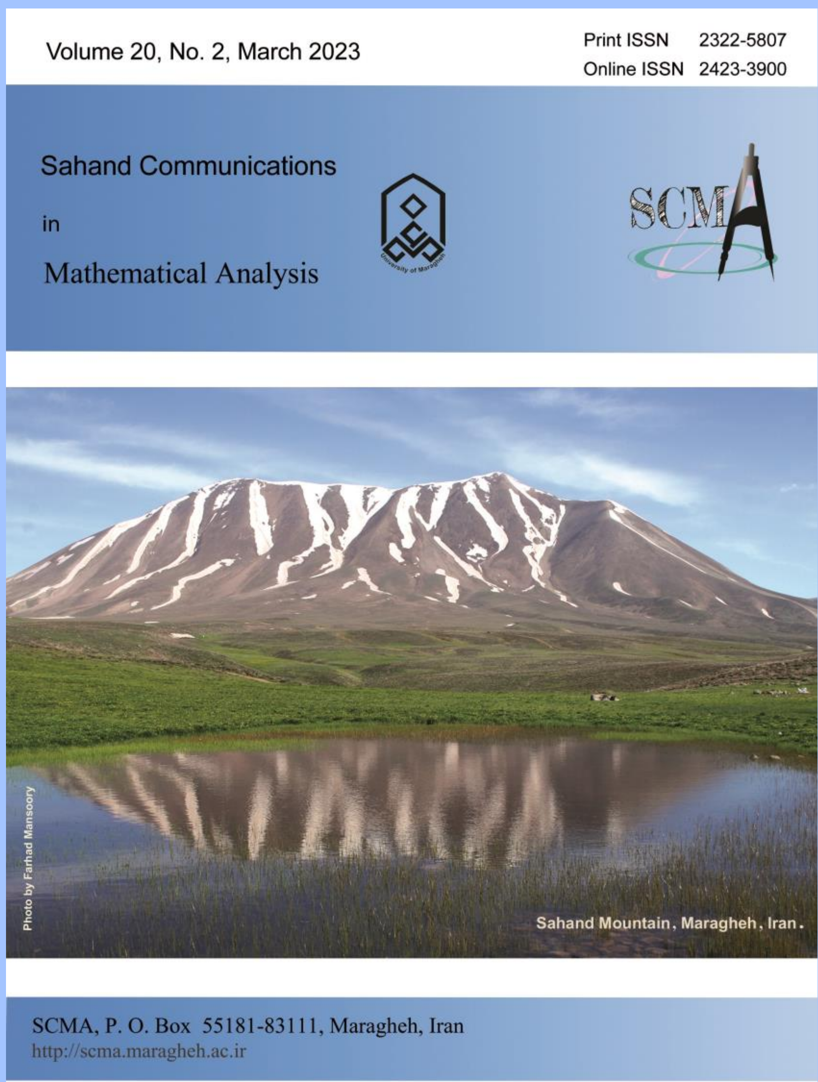

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A New Iterative Method for Solving Constrained Minimization, Variational Inequality and Split Feasibility Problems in the Framework of Banach Spaces

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Abstract. In this paper, we introduce a new type of modified generalized α-nonexpansive mapping and establish some fixed point properties and demiclosedness principle for this class of mappings in the framework of uniformly convex Banach spaces. We further propose a new iterative method for approximating a common fixed point of two modified generalized α-nonexpansive mappings and present some weak and strong convergence theorems for these mappings in uniformly convex Banach spaces. In addition, we apply our result to solve a convex-constrained minimization problem, variational inequality and split feasibility problem and present some numerical experiments in infinite dimensional spaces to establish the applicability and efficiency of our proposed algorithm. The obtained results in this paper improve and extend some related results in the literature.

1. Introduction

Let C be a nonempty set. A fixed point problem for a mapping $T : C \to C$ is obtaining $x \in C$ such that

$$Tx = x. \tag{1.1}$$

The set of all fixed points of T is denoted by $F(T)$. The theory of fixed point has progressively become an invaluable area of study as many problems in mathematics, engineering, physics, economics, game theory, etc can be transformed into a fixed point problem. In general,

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solving fixed point problems analytically is almost impossible and thus the need to consider iterative solutions for fixed point problems arises. Over the years researchers have developed several iterative schemes for solving fixed point problems for different operators but the research is still ongoing in order to develop faster and more efficient iterative algorithms.

The Picard iterative process

$$x_{n+1} = Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.2}$$

is one of the earliest iterative processes used to approximate the solution of Equation (1.1), where T is a contraction mapping. The Picard iterative process fails to approximate the solution of Equation (1.1) if T is a nonexpansive mapping even when the existence of the fixed point is guaranteed. Browder [7] established that the class of nonexpansive self-mappings on a closed and bounded subset of a uniformly convex Banach space has a fixed point. Thereafter, researchers in this area have developed different iterative processes to approximate fixed points of nonexpansive mappings and a host of other nonlinear mappings. In general, developing faster and more efficient iterative algorithms for approximating fixed points of nonlinear mappings is still an active area of research. The following are some well-known iterative algorithms in literature for approximating fixed points of nonlinear mappings. Among them are Mann [26], Ishikawa [20], Krasnosel'skii [25], Agarwal [4], Noor [30], etc. There are numerous papers dealing with the approximation of fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces (for example, see [1, 3, 4] and the references therein).

In 2011, Phuengrattana and Suantai [34] introduced the following SP-iterative process, as: Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n Ty_n, \qquad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

In 2017, Karakaya et al. [23] introduced a new iteration process, as: Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be an arbitrary nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$

in C is defined by

$$(1.4)\qquad \begin{cases} z_n = Tx_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = Ty_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. They proved that their iterative process converges faster than Picard, Mann [26], Ishikawa [20], Noor, Abass et al. [2], and some existing iterative methods in the literature.

In 2018, Ullah et al., [39] introduced a new iteration process called the M-iteration process, as: Let Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be an arbitrary nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$(1.5)\qquad \begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = Tz_n, \\ x_{n+1} = Ty_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. They proved that their iterative process converges faster than Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process and some existing iterative methods in the literature.

Remark 1.1. It was established in [1] that the iterative processes (1.4) and (1.5) have the same rate of convergence.

In 2020, Chuadchawna et al. [15] introduced a generalized M-iteration in the framework of hyperbolic spaces. The corresponding definition of generalized M-iteration is given in the framework of normed space as: Let Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$(1.6)\qquad \begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = \beta_n z_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)Ty_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. They established some fixed point results in the framework of hyperbolic spaces. They also stated clearly that for $\beta_n = \gamma_n = 0$, then iterative process (1.6) becomes (1.5). In addition, they established that the generalized M-iteration converges faster than the M-iteration. They gave a numerical example to justify this claim.

Remark 1.2.
(i) If $\alpha = \beta_n = \gamma_n = \frac{1}{2}$, then iterative processes (1.6) and (1.3) are the same.
(ii) If $\beta_n = \gamma_n = 1$, the SP iteration becomes the M-iteration.

Remark 1.3. Researchers have devised multiple iterative approaches for addressing fixed point problems for various operators over the years. However, researchers are still faced with the challenge of constructing quicker and more efficient iterative algorithms. In light of this, it is natural to ask if an iterative method can be developed with a better approximation and rate of convergence when compared with other existing iterative algorithms in the literature.

On the other hand, let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$, C be a nonempty closed convex subset of H and $A : H \to H$ be an operator. The classical Variational Inequality Problem (VIP) is formulated as: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.7}$$

We denote the solution set of (1.7) by $VI(C, A)$. The notion of variational inequalities VIP (1.7) was introduced and studied by Stampachhia in [36], thereafter, a lot of researchers in this area of research have explored the notion of $VI(C, A)$ for detailed results on VIP (1.7) the reader should (see [12, 17, 24, 27–29, 32, 36, 41] and the references therein). It has been established over the years that the existence and approximation of a VIP (1.7) is equivalent to finding the fixed point problem

$$x^* \in C \text{ such that } x^* = P_C(I - \eta A)x^*, \tag{1.8}$$

where $\eta > 0$ and P_C is the metric projection onto C. It has been established in the literature that if A is L-Lipschitzian and v-strongly monotone, then the operator $P_C(I - \eta A)$ is a contraction on C provided that $0 < \eta < \frac{2v}{L^2}$. In light of this fact, the Banach contraction principle guarantees the existence and uniqueness of an approximate solution for a VIP (1.7). The well-known Picard iterative process takes the form:

$$x_{n+1} = P_C(I - \eta A)x_n. \tag{1.9}$$

This approach of approximating the solution of a VIP (1.7) is called Projected Gradient Method (PGM). It is well-known that

$$x^* \in VI(C, A) \text{ if and only if } x^* = P_C(x^* - \eta A x^*).$$

Remark 1.4. It is logical to ask whether the above mentioned iterative process can be applied to the VIP (1.7), split feasibility problem and constrained optimization variation inequality problem.

Based on Remark 1.3 and Remark 1.4 and the research in this direction, the purpose of this paper is to introduce a modified generalized α-nonexpansive mapping, an iterative method for approximating common fixed points of two generalized α- nonexpansive mappings and obtain some convergence results in the framework of uniformly convex Banach space. In addition, we apply our results to solve a convex-constrained minimization problem, variational inequality and split feasibility problem and present some numerical experiments in infinite dimensional spaces to establish the applicability and efficiency of our proposed algorithm. The results obtained in this paper improve, extend and unify some related results in the literature.

2. Preliminaries

Let X be a Banach space with a dimension greater than or equal to 2. The function $\delta_X(\epsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = 1; \|y\| = 1, \epsilon = \|x - y\| \right\},$$

is called the modulus of convexity of X. If $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, then X is called uniformly convex. Let X be a Banach space, its dual be X^* and $S(X) = \{x \in X : \|x\| = 1\}$. We have the value of $f \in X^*$ at $x \in X$ is defined by $\langle x, f \rangle$.

Definition 2.1. (i) The multivalued mapping $J : X \to 2^{X^*}$ defined by $J(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}$ is called normalized duality mapping.

(ii) A Banach space X is smooth if the limit $\lim\limits_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in S(X)$. In this case, the norm of X is called Gateaux differentiable. It is known that J is single-valued if X is smooth.

(iii) A Banach space X is Frechet differentiable norm, if for each $x \in S(X)$ the limit above exists and is attained uniformly for $y \in S(X)$. In this case for all $x, h \in X$

$$\langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x + h\|^2 \leq \langle h, J(x) \rangle + \frac{1}{2}\|x\|^2 + b(\|h\|),$$

where $J(x)$ is the Frechet derivative of the functional $\frac{1}{2}\| \cdot \|$ at $x \in X$ and b is an increasing function defined on $[0, \infty)$ such that $\lim\limits_{t \downarrow 0} \frac{b(t)}{t} = 0$.

(iv) A Banach space X is said to have Opial property [31] if for every weakly convergent sequence $\{x_n\}$ in X with weak limit y, we have

$$\liminf_{n \to \infty} \|x_n - y\| < \liminf_{n \to \infty} \|x_n - z\|, \quad \forall\, z \in X,$$

with $y \neq z$.

Let C be a nonempty subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. For all $x \in X$, we define

(i) asymptotic radius of $\{x_n\}$ at x by $r(x, \{x_n\}) = \limsup\limits_{n \to \infty} \|x_n - x\|$;

(ii) asymptotic radius of $\{x_n\}$ relative to C by

$$r(C,\{x_n\}) = \inf\{r,(x,\{x_n\}) : x \in C\};$$

(iii) asymptotic centre of $\{x_n\}$ relative to C by

$$A(C,\{x_n\}) = \{r(x,\{x_n\}) = r(C,\{x_n\}) : x \in C\}.$$

We note that $A(C,\{x_n\})$ is not empty, in particular if X is uniformly convex, then $A(C,\{x_n\})$ has exactly one point.

Lemma 2.2. *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq c, \limsup_{n\to\infty} \|y_n\| \leq c$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = c$ holds for some $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.3. Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$ and $f(t) > 0\, \forall\, t \in (0,\infty)$ and $\|x - Tx\| \geq f(d(x,F(T)))$ for all $x \in C$, where $d(x,F(T))$ denotes the distance from x to $F(T)$.

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. For any $x,y \in H$ and $\alpha \in [0,1]$, it is well-known that

$$\langle x,y\rangle = \frac{1}{2}\left[\|x\|^2 + \|y\|^2 - \|x-y\|^2\right], \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.2}$$

Definition 2.4. Let $A : H \to H$ be an operator. The operator A is called

(i) L-Lipschitz continuous if

$$\|Ax - Ay\| \leq L\|x-y\|, \tag{2.3}$$

where $L > 0$ and $x,y \in H$.

(ii) The operator A is called monotone if

$$\langle Ax - Ay, x-y\rangle \geq 0, \quad \forall x,y \in H. \tag{2.4}$$

(iii) k-inverse strongly monotone (k-ism) if there exists $k > 0$, such that

$$\langle Ax - Ay, x-y\rangle \geq k\|Ax - Ay\|^2, \quad \forall\, x,y \in H, \tag{2.5}$$

(iv) v-strongly monotone (v-sm) if there exists $v > 0$, such that

$$\langle Ax - Ay, x-y\rangle \geq v\|x-y\|^2, \quad \forall\, x,y \in H, \tag{2.6}$$

(v) relaxed (α,k)-cocoercive if there exist $\alpha, k > 0$, such that

$$\langle Ax - Ay, x-y\rangle \geq -\alpha\|Ax - Ay\|^2 + k\|x-y\|^2, \quad \forall\, x,y \in H, \tag{2.7}$$

It is well-known that for any nonexpansive mapping A, the set of the fixed point is closed and convex, in particular, A satisfy the following inequality

(2.8)
$$\langle (x-Ax)-(y-Ay), Ay-Ax\rangle \leq \frac{1}{2}\|(Ax-x)-(Ay-y)\|^2, \quad \forall\, x,y\in H.$$

In addition, for all $x\in H$ and $x^*\in F(A)$, we have

$$\langle x-Ax, x^*-Ax\rangle \leq \frac{1}{2}\|Ax-x\|^2, \quad \forall\, x,y\in H. \tag{2.9}$$

For any $u\in H$, there exists a unique point $P_Cu\in C$ such that

$$\|u-P_C\|\leq \|u-y\|, \quad \forall y\in C.$$

P_C is called the metric projection of H into C. It is well-known that P_C is a nonexpansive mapping of H onto C and that P_C satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2,$$

for all $x,y\in H$. Furthermore, P_Cx is characterized by the properties $P_Cx\in C$,

$$\langle x-P_Cx, P_Cx-y\rangle \geq 0,$$

for all $y\in C$ and

$$\|x-y\|^2 \geq \|x-P_Cx\|^2+\|y-P_Cx\|^2,$$

for all $x\in H$ and $y\in C$.

Lemma 2.5. *Let C be a closed convex subset of a real Hilbert space H and A be a relaxed (α,k)-cocoercive and L-Lipschitzian mapping of C into H such that $\Omega(C,A)\neq\emptyset$. Suppose that*

$$0<\eta<\frac{2(k-\alpha L^2)}{L^2}, \qquad \alpha L^2<k, \tag{2.10}$$

then the operator $G:=P_C(I-\eta A)$ is a contraction mapping.

Proof. Now, observe

$$\begin{aligned}
\|Gx-Gy\| &= \|P_C(x-\eta Ax)-P_C(y-\eta Ay)\|^2\\
&\leq \|(x-y)-(\eta Ax-\eta Ay)\|^2\\
&= \|x-y\|^2-2\eta\langle Ax-Ay, x-y\rangle+\eta^2\|Ax-Ay\|^2\\
&\leq \|x-y\|^2+2\eta\alpha\|Ax-Ay\|^2-2\eta k\|x-y\|^2+\eta^2\|Ax-Ay\|^2\\
&\leq (1-2\eta k)\|x-y\|^2+(2\alpha\eta+\eta^2)L^2\|x-y\|^2\\
&= \left[1-2\eta k+2\alpha\eta L^2+\eta^2L^2\right]\|x-y\|^2,
\end{aligned}$$

which implies that

$$\|TP_C(x-\eta Ax)-TP_C(y-\eta Ay)\|\leq \sqrt{[1-2\eta k+2\alpha\eta L^2+\eta^2L^2]}\|x-y\|$$

$$= \delta\|x - y\|,$$

where

$$\delta = \sqrt{[1 - 2\eta k + 2\alpha\eta L^2 + \eta^2 L^2]} \in [0, 1). \tag{2.11}$$

□

3. Modified Generalized α-Nonexpansive Mappings

In this section, we introduce the notion of modified generalized α-nonexpansive mappings and establish some basic properties for this class of mappings.

We recall the following. Let C be a nonempty subset of a Banach space X and $T : C \to C$ a self-mapping. A point $x \in X$ is said to be a fixed point of T if $Tx = x$. A mapping $T : C \to C$ is said to be

(i) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;
(ii) mean nonexpansive if there exists $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that $\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$, for all $x, y \in C$;
(iii) Suzuki mean nonexpansive, if there exists $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that $\frac{1}{2}\|Tx - x\| \leq \|x - y\|$ then $\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$, for all $x, y \in C$;
(iv) satisfy condition (C), if $\frac{1}{2}\|Tx - x\| \leq \|x - y\|$ then $\|Tx - Ty\| \leq \|x - y\|$; for all $x, y \in C$;
(v) satisfy condition (C_λ), if $\lambda\|Tx - x\| \leq \|x - y\|$ then $\|Tx - Ty\| \leq \|x - y\|$, $\lambda \in [0, 1)$ for all $x, y \in C$;
(vi) generalized mean nonexpansive mapping if there exists $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$, $\lambda\|Tx - x\| \leq \|x - y\|$ then $\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$;
(vii) α-nonexpansive mapping if there exists $\alpha < 1$ such that for all $x, y \in C$, $\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$;
(viii) generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1]$ such that $\frac{1}{2}\|Tx - x\| \leq \|x - y\|$ then $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|$, for all $x, y \in C$;
(ix) quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T.

Definition 3.1. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ will be called a modified generalized α-nonexpansive mapping if there exist $\alpha, \beta \in [0, 1)$, such that for all $x, y \in C$, $(1 - \beta)\|Tx - x\| \leq \|x - y\|$ then

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|. \tag{3.1}$$

Remark 3.2. It is easy to see that if

(i) $\beta = \frac{1}{2}$, we obtain generalized (α)-nonexpansive mapping.
(ii) $\beta = \frac{1}{2}$ and $\alpha = 0$, we obtain mapping satisfying condition (C).

(iii) every nonexpansive mapping is a modified generalized nonexpansive mapping.

Proposition 3.3. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a modified generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Proof. The proof follows a similar approach as that of generalized α-nonexpansive mapping in [33], thus, we omit it. □

Theorem 3.4. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a modified generalized α-nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Proof. The proof follows a similar approach as that of generalized α-nonexpansive mapping in [33], thus, we omit it. □

Lemma 3.5. *Let C be a nonempty subset of a Banach space X. Suppose that $T : C \to C$ is a generalized α-nonexpansive mapping on C. Then for all $x, y \in C$, we have*

(i) $\|T^2x - Tx\| < \|Tx - x\|$;
(ii) *either* $(1-\beta)\|x-Tx\| \leq \|x-y\|$ *or* $(1-\beta)\|Tx-T^2x\| \leq \|Tx-y\|$;
(iii) *either* $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|$ *or* $\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \alpha\|Ty - Tx\| + (1 - 2\alpha)\|Tx - y\|$.

Proof. The proof follows a similar approach as that of generalized α-nonexpansive mapping in [33], thus, we omit it. □

Lemma 3.6. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ a modified generalized α-nonexpansive mapping. Then for all $x, y \in C$,*

$$\|x - Ty\| \leq \frac{(3 + \alpha)}{(1 - \alpha)}\|x - Tx\| + \|x - y\|.$$

Proof. The proof follows a similar approach as that of generalized α-nonexpansive mapping in [33], thus, we omit it. □

Theorem 3.7. *Let C be a nonempty compact subset Banach space X and $T : C \to C$ is a modified generalized α-nonexpansive mapping. Then T has a fixed point in C if and only if T admits an a.f.p.s.*

Proof. The proof follows a similar approach as that of generalized α-nonexpansive mapping in [33], thus, we omit it. □

4. Convergence Results

In this section, we establish some convergence results for modified generalized α-nonexpansive mapping via a new three steps iterative algorithm in the framework of uniformly convex Banach space. We define our iterative process as follows:

For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = \alpha_n z_n + (1-\alpha_n)Sz_n, \\ x_{n+1} = \beta_n Sz_n + (1-\beta_n)Ty_n, \qquad n \geq 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

Lemma 4.1. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T, S : C \to C$ be modified generalized α-nonexpansive mapping with $F(T) \cap F(S) \neq \emptyset$. Suppose the sequence $\{x_n\}$ is generated by (4.1), then*

(i) $\{x_n\}$ *is bounded;*
(ii) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for all $x^* \in F(T) \cap F(S)$.*

Proof. Let $x^* \in F(T) \cap F(S)$, using (4.1) and Proposition 3.3, we obtain

$$\begin{aligned} \|z_n - x^*\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{4.2}$$

Also, from (4.1), (4.2) and Proposition 3.3, we obtain

$$\begin{aligned} \|y_n - x^*\| &= \|\alpha_n z_n + (1-\alpha_n)Sz_n - x^*\| \\ &\leq \alpha_n\|z_n - x^*\| + (1-\alpha_n)\|Sz_n - x^*\| \\ &\leq \alpha_n\|z_n - x^*\| + (1-\alpha_n)\|z_n - x^*\| \\ &= \|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{4.3}$$

Lastly, using (4.1), (4.3) and Proposition 3.3, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta_n Sz_n + (1-\beta_n)Ty_n - x^*\| \\ &\leq \beta_n\|Sz_n - x^*\| + (1-\beta_n)\|Ty_n - x^*\| \\ &\leq \beta_n\|z_n - x^*\| + (1-\beta_n)\|y_n - x^*\| \\ &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{4.4}$$

This shows that $\{\|x_n - x^*\|\}$ is bounded and non-increasing for all $x^* \in F(T) \cap F(S)$. Thus, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. $\square$

Lemma 4.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T, S : C \to C$ be modified generalized α-nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by (4.1). Then, $F(T) \cap F(S) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Sx_n - x_n\| = 0$.*

Proof. Suppose that $F(T) \cap F(S) \neq \emptyset$, thus, there exists $x^* \in F(T) \cap F(S)$, then by Lemma 4.1, we obtain that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T) \cap F(S)$. Suppose that

$$\lim_{n\to\infty} \|x_n - x^*\| = c. \tag{4.5}$$

From (4.2) and (4.3), we obtain $\|z_n - x^*\| \leq \|x_n - x^*\|$ and $\|y_n - x^*\| \leq \|x_n - x^*\|$. Taking limsup of both sides, we have

$$\limsup_{n\to\infty} \|z_n - x^*\| \leq c, \tag{4.6}$$

and

$$\limsup_{n\to\infty} \|y_n - x^*\| \leq c. \tag{4.7}$$

From (4.4), we have

$$\|x_{n+1} - x^*\| \leq \beta_n \|z_n - x^*\| + (1 - \beta_n)\|x_n - x^*\|.$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$c \leq \beta_n \liminf_{n\to\infty} \|z_n - x^*\| + (1 - \beta_n)c,$$

$$c \leq \liminf_{n\to\infty} \|z_n - x^*\|. \tag{4.8}$$

From (4.6) and (4.8), we obtain

$$\lim_{n\to\infty} \|z_n - x^*\| = c. \tag{4.9}$$

From (4.4), we have

$$\|x_{n+1} - x^*\| \leq \beta_n \|x_n - x^*\| + (1 - \beta_n)\|y_n - x^*\|.$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$c \leq \beta_n c + (1 - \beta_n)\|y_n - x^*\|,$$

$$c \leq \liminf_{n\to\infty} \|y_n - x^*\|. \tag{4.10}$$

From (4.7) and (4.10), we obtain

$$\lim_{n\to\infty} \|y_n - x^*\| = c. \tag{4.11}$$

Furthermore, using Proposition (3.3), we have

$$\|Tx_n - x^*\| \leq \|x_n - x^*\|, \tag{4.12}$$

$$\|Sx_n - x^*\| \leq \|x_n - x^*\|, \tag{4.13}$$

taking $\limsup\limits_{n\to\infty}$ of both sides, we have

$$\limsup_{n\to\infty} \|Tx_n - x^*\| = c, \tag{4.14}$$

and

$$\limsup_{n\to\infty} \|Sx_n - x^*\| = c, \tag{4.15}$$

Using, (4.5), (4.9), (4.14) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.16}$$

Also, using (4.5), (4.11), (4.15) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|z_n - Sx_n\| = 0. \tag{4.17}$$

In addition, we have

$$\begin{aligned} \|z_n - x_n\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - x_n\| \\ &= \|x_n - \gamma_n x_n + \gamma_n Tx_n - x_n\| \\ &\leq \gamma_n \|x_n - Tx_n\|, \end{aligned} \tag{4.18}$$

taking limit of both sides and using (4.16), we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{4.19}$$

Furthermore, we have

$$\|Sx_n - x_n\| \leq \|Sx_n - z_n\| + \|z_n - x_n\|, \tag{4.20}$$

taking limit of both sides and using (4.19) and (4.17), we obtain

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{4.21}$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty} \|x_n - Tx_n\|$ $= 0 = \lim\limits_{n\to\infty} \|x_n - Sx_n\|$. Let $x^* \in A(C, \{x_n\})$, we have

(4.22)

$$\begin{aligned} r(Tx^*, \{x_n\}) &= \limsup_{n\to\infty} \|Tx^* - x_n\| \\ &\leq \left[\frac{3+\alpha}{1-\alpha}\right] \limsup_{n\to\infty} \|x_n - Tx_n\| + \limsup_{n\to\infty} \|x_n - x^*\| \\ &= \limsup_{n\to\infty} \|x_n - x^*\| \\ &= r(x^*, \{x_n\}). \end{aligned}$$

This implies that $Tx^* \in A(C, \{x_n\})$. Since X is uniformly convex, we obtain that $A(C, \{x_n\})$ has exactly one element, thus, we obtain

$$x^* = Tx^*.$$

Similarly, we obtain

$$x^* = Sx^*,$$

we have $x^* \in F(T) \cap F(S)$, hence, $F(T) \cap F(S) \neq \emptyset$. □

Theorem 4.3. *Let X be a uniformly convex Banach space which satisfies the Opial's condition and C a nonempty closed convex subset of X. Let $T, S : C \to C$ be modified generalized α-nonexpansive mapping with $F(T) \cap F(S) \neq \emptyset$ and sequence $\{x_n\}$ generated by (4.1). Then, the sequence $\{x_n\}$ converges weakly to a common fixed point of T and S.*

Proof. It has been established in Lemma 4.1 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and $\{x_n\}$ is bounded. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$ and $F(S)$. Let x and y be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Theorem 4.2, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Since $I - T$ and $I - S$ are demiclosed at zero, therefore, we have $x, y \in F(T) \cap F(S)$. Since, $x, y \in F(T) \cap F(S)$, thus, from Lemma 4.1, we have $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exists. Now, suppose that $x \neq y$, then by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x\| &= \lim_{k\to\infty} \|x_{n_k} - x\| \\
&< \lim_{k\to\infty} \|x_{n_k} - y\| \\
&= \lim_{n\to\infty} \|x_n - y\| \\
&= \lim_{j\to\infty} \|x_{n_j} - y\| \\
&< \lim_{j\to\infty} \|x_{n_j} - x\| \\
&= \lim_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

This is a contradiction. So, $x = y$. Hence, $\{x_n\}$ converges weakly to a common fixed point of T and S and this completes the proof. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T, S : C \to C$ be a modified generalized α-nonexpansive mapping and the sequence $\{x_n\}$ defined by (4.1) and $F(T) \cap F(S) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to a common fixed point of $F(T) \cap F(S)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$ where $d(x, F(T) \cap F(S)) = \inf\{\|x - x^*\| : x^* \in F(T) \cap F(S)\}$.*

Proof. Let $\{x_n\}$ converges to x^* a fixed point of T and S.

Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)\cap F(S)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)\cap F(S)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)\cap F(S)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$. It follows from Lemma 4.1 $\lim_{n\to\infty} \|x_n - x^*\|$ exists and $\lim_{n\to\infty} d(x_n, F(T)\cap F(S))$ exists for all $x^* \in F(T) \cap F(S)$. Based on our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T) \cap F(S)) = 0$. Suppose $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$ and $\{u_k\}$ is a sequence of $F(T) \cap F(S)$ such that for all $n \in \mathbb{N}$,

$$\|x_{n_k} - u_k\| < \frac{1}{2^k},$$

it follows from (4.4) that $\|x_{n+1} - u_k\| \leq \|x_n - u_k\| < \frac{1}{2^k}$, hence

$$\begin{aligned}\|u_{k+1} - u_k\| &\leq \|u_{k+1} - x_{n+1}\| + \|x_{n+1} - u_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Thus, $\{u_k\}$ is a Cauchy sequence in $F(T) \cap F(S)$. Also, by Theorem 3.4, $F(T) \cap F(S)$ is closed. Thus $\{u_k\}$ is a convergent sequence in $F(T) \cap F(S)$. Now, suppose that $\{u_k\}$ converges to $p \in F(T) \cap F(S)$. Since

$$\|x_{n_k} - p\| \leq \|x_{n_k} - u_k\| + \|u_k - p\| \to 0 \text{ as } k \to \infty,$$

we obtain $\lim_{k\to\infty} \|x_{n_k} - p\| = 0$ and so $\{x_{n_k}\}$ converges strongly to $p \in F(T) \cap F(S)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. □

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T, S be modified generalized α-nonexpansive mapping and the sequence $\{x_n\}$ defined by (4.1) such that $F(T)\cap F(S) \neq \emptyset$. Suppose that T and S satisfy condition (I), then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Using Lemma 4.1 and Theorem 4.2, we obtain that $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Using the fact that

$$\begin{aligned}0 &\leq \lim_{n\to\infty} f(d(x_n, F(T) \cap F(S)) \\ &\leq \lim_{n\to\infty} \|x_n - Tx_n\| \\ &= 0, \quad \forall x \in C,\end{aligned}$$

and

$$\begin{aligned}0 &\leq \lim_{n\to\infty} f(d(x_n, F(T)\cap F(S)) \\ &\leq \lim_{n\to\infty} \|x_n - Sx_n\| \\ &= 0, \quad \forall x \in C,\end{aligned}$$

where $\lim_{n\to\infty} f(d(x_n, F(T)\cap F(S))) = 0$, since, f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0,\infty)$, then $\lim_{n\to\infty} d(x_n, F(T)\cap F(S)) = 0$. Thus using Theorem 4.4, $\{x_n\}$ converges strongly to $p \in F(T)$. □

5. Applications and Numerical Example

5.1. **Application to Constrained Optimization and Variational Inequality Problem.** Let $f : C \to \mathbb{R}$ be a convex mapping where C is a closed and convex subset of a Hilbert space H. Considering the convex minimization problem

$$\min_{x\in C} f(x). \tag{5.1}$$

Let $P_C : H \to C$ be a projection map and f be Frêchet differentiable. Denote the gradient of f by ∇f. It is well-known that x^* solves (5.1) if and only if the following variational inequality holds:

$$x^* \in C, \langle \nabla f x^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \tag{5.2}$$

where $x^* \in \Omega(C, A)$. In addition x^* solves (5.1) if and only if $x^* = P_C(x^* - \eta\nabla f(x^*))$, where $\eta > 0$. In order to solve (5.1) the gradient projection algorithm (GPA) is usually used and it is defined as

$$x_{n+1} = P_C(x_n - \eta\nabla f(x_n)),$$

where $x_0 \in C$ and η is the step size. Now, suppose that $T = S = P_C(I - \eta\nabla f)$ and A is as the gradient of a convex function f in the iterative process (4.1), then we get the following iterative process which converges to a solution of a convex minimization problem (5.1),

$$\begin{cases} x_0 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n P_C(I-\eta\nabla f)x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n P_C(I-\eta\nabla f)z_n, \\ x_{n+1} = \beta_n P_C(I-\eta\nabla f)z_n + (1-\beta_n)P_C(I-\eta\nabla f)y_n, \qquad n \geq 1, \end{cases} \tag{5.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\eta > 0$.

Theorem 5.1. *Suppose that problem* (5.1) *has a solution. Let $f : C \to \mathbb{R}$ is a convex mapping such that its gradient ∇f is a relaxed (α, k)-cocorcive and L-Lipschitzian mapping of C into H. Let $\{x_n\}$ be sequence defined by* (5.3) *for any $x_0 \in C$, such that condition* (2.10) *and δ is defined as*

(2.11) *hold, then* $\{x_n\}$ *obtained from* (5.3) *converges weakly to* x^* *to the solution of* (5.1).

Proof. Similarly to Theorem 4.4, the prove follows the same approach, by taking $T = S = P_C(I - \eta\nabla f)$, which is a nonexpansive mapping, and every nonexpansive mapping is a modified generalized α-nonexpansive mapping, and $A = \nabla f$. The results follow from Theorem 4.3. □

Theorem 5.2. *Suppose that problem* (5.1) *has a solution. Let* $f : C \to \mathbb{R}$ *is a convex mapping such that its gradient* ∇f *is a relaxed* (α, k)*-cocorcive and L-Lipschitzian mapping of* C *into* H*. Let* $\{x_n\}$ *be sequence defined by* (5.3) *for any* $x_0 \in C$*, such that condition* (2.10) *and* δ *is defined as in* (2.11) *hold, then* $\{x_n\}$ *obtained from* (5.3) *converges strongly to* x^* *to the solution of* (5.1) *if and only if*

$$\liminf_{n\to\infty} d(x_n, \Omega(C, A)) = 0.$$

Proof. Similarly to Theorem 4.4, the prove follows the same approach, by taking $T = S = P_C(I - \eta\nabla f)$, which is a nonexpansive mapping, and every nonexpansive mapping is a modified generalized α-nonexpansive mapping, and $A = \nabla f$. The results follow from Theorem 4.4.

(5.4)

$$\begin{cases} x_0 \in C, \\ z_n = (1-\gamma_n)x_n + \gamma_n P_C(I-\eta T)x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n P_C(I-\eta T)z_n, \\ x_{n+1} = \beta_n P_C(I-\eta T)z_n + (1-\beta_n)P_C(I-\eta T)y_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\eta > 0$. □

Theorem 5.3. *Suppose that problem* (5.1) *has a solution. Let* $P_C : H \to C$ *be a metric projection defined on a nonempty closed convex subset of* C *into* H *and* $T : C \to C$ *be* v*-inverse strongly monotone mapping with* $v > 0$ *is a constant and* $\eta \in (0, 2v)$*. Let* $\{x_n\}$ *be sequence defined by* (5.4) *then* $\{x_n\}$ *obtained from* (5.4) *converges weakly to* x^* *to the solution of* (5.1).

Proof. Since $\eta \in (0, 2v)$, it is well-known that $P_C(I-\eta T)$ is nonexpansive and since every nonexpansive is a modified generalized α-nonexpansive mapping. Thus, our result follows from Theorem 4.3, by taking $T = S = P_C(I - \eta T)$. □

Theorem 5.4. *Suppose that problem* (5.1) *has a solution. Let* $P_C : H \to C$ *is a metric projection defined on a nonempty closed convex subset of* C *into* H *and* $T : C \to C$ *be* v*-inverse strongly monotone mapping with* $v > 0$ *is a constant and* $\eta \in (0, 2v)$*. Let* $\{x_n\}$ *be a sequence defined* *by* (5.4) *then* $\{x_n\}$ *obtained from* (5.4) *converges strongly to* x^* *to the solution of* (5.1) *if and only if* $\liminf_{n\to\infty} d(x_n, \Omega(C, A)) = 0$.

Proof. Since $\eta \in (0, 2v)$, it is well-known that $P_C(I-\eta T)$ is nonexpansive and since every nonexpansive is a modified generalized α-nonexpansive mapping. Thus, our result follows from Theorem 4.4, by taking $T = S = P_C(I - \eta T)$. □

5.2. **Application to Split Feasibility Problem.** Let C and Q be closed convex and nonempty subsets of two real Hilbert spaces H_1 and H_2, respectively and let $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is defined as:

$$x \in C \text{ such that } y = Ax \in Q. \tag{5.5}$$

The concept of SFP was introduced by Censor and Elfving [9] in the framework of finite-dimensional Hilbert spaces. The SFP has found applications in many real-life problems such as image recovery, signal processing, control theory, data compression, computer tomography and so on (see [10, 11] and the references therein). We denote the solution set of (5.5) by Ω. It is well-known that the solution set Ω is closed, convex and nonempty. In 2002, Byrne [8] introduced an iterative method for solving SFP (5.5). He defined the iterative method as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \tag{5.6}$$

for all $n \in \mathbb{N}$, where $0 < \gamma < \frac{2}{\|A\|^2}$, P_C and P_Q are the metric projections onto set C and Q, respectively and $A^* : H_2 \to H_1$ is the adjoint operator of A. The SFP has been studied by many researchers in this area, for details about some methods for solving the SFP (5.5), see [9, 10] and the references therein.

Lemma 5.5. *[16] Let operator* $T = P_C(I - \gamma A^*(I - P_Q)A)$, *with* $0 < \gamma < \frac{2}{\|A\|^2}$. *Then,* T *is a nonexpansive map.*

Since the solution set of problem (5.5) is not empty, it has been established that any $x \in C$ is a solution of SFP (5.5) if and only if it satisfies the following fixed point equation:

$$P_C(I - \gamma A^*(I - P_Q)A)x = x. \tag{5.7}$$

That is the solution set $\Omega = F(T)$.

Theorem 5.6. *Let* $\{x_n\}$ *be the sequence defined by the iterative method* (4.1) *with* $S = T = P_C(I - \gamma A^*(I - P_Q)A$, *where* A *is defined in the above and* $0 < \gamma < \frac{2}{\|A\|^2}$, *then* $\{x_n\}$ *converges weakly to the solution of SFP* (5.5).

Proof. It is easy to see that all the conditions in Lemma 5.5 are satisfied, thus, $S = T = P_C(I - \gamma A^*(I - P_Q)A$ is a nonexpansive mapping and since every nonexpansive mapping is a modified generalized α-nonexpansive mapping. Thus, the result follows from Theorem 4.3. □

Theorem 5.7. *Let $\{x_n\}$ be the sequence defined by the iterative method* (4.1) *with $S = T = P_C(I - \gamma A^*(I - P_Q)A$, where A is defined in the above and $0 < \gamma < \frac{2}{\|A\|^2}$, then, $\{x_n\}$ converges strongly to the solution of SFP* (5.5)*.*

Proof. It is easy to see that all the conditions in Lemma 5.5 are satisfied, thus, $S = T = P_C(I - \gamma A^*(I - P_Q)A$ is a nonexpansive mapping and since every nonexpansive mapping is a modified generalized α-nonexpansive mapping. Thus, the result follows from Theorem 4.4. □

Theorem 5.8. *Let $\{x_n\}$ be the sequence defined by the iterative method* (4.1) *with $S = T = P_C(I - \gamma A^*(I - P_Q)A$, where A is defined in the above and $0 < \gamma < \frac{2}{\|A\|^2}$, then $\{x_n\}$ converges strongly to the solution of SFP* (5.5) *if and only if $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.*

Proof. It is easy to see that all the conditions in Lemma 5.5 are satisfied, thus, $S = T = P_C(I - \gamma A^*(I - P_Q)A$ is a nonexpansive mapping and since every nonexpansive mapping is a modified generalized α-nonexpansive mapping. Thus, the result follows from Theorem 4.4. □

5.3. **Application to Nonlinear Integral Equation.** In this section, we present an application of our result to nonlinear integral equation of the form:

$$x(t) = g(t) + \gamma \int_a^b M(t,s)h(t,x(s))ds, \tag{5.8}$$

where $\gamma \in (0,1], M : [a,b] \times [a,b] \to \mathbb{R}^+,\ h : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real-valued functions defined on $[a,b]$ with ordered relation $\leq$ in X defined as for $x, y \in X, x \leq y$ if and only if $x(s) \leq y(s)$ for all $s \in [a,b]$. We defined $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ by $\|x - y\| = \sup_{s\in[a,b]} |x(s) - y(s)|$.

Theorem 5.9. *Let $X = C([a,b],\mathbb{R})$ and $T : X \to X$ the operator given by*

$$Tx(t) = g(t) + \gamma \int_a^b M(t,s)h(t,x(s))ds,$$

for all $t, s \in [a,b]$, where $\gamma \in [0,1], M : [a,b]\times[a,b] \to \mathbb{R}^+, h : [a,b]\times\mathbb{R} \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real-valued functions defined on $[a,b]$. Furthermore, suppose the following condition holds

(i) *there exists a continuous mapping* $\upsilon : X \times X \to [0, \infty)$ *such that*

$$|h(s, x(s)) - h(s, y(s))| \leq \upsilon(x, y)|x(s) - y(s)|$$

for all $s \in [a, b]$ *and* $x, y \in X$*;*

(ii) *there exists* $\omega \in [0, 1]$*, such that*

$$\int_a^b M(t, s)\upsilon(x, y) \leq \omega.$$

Then the integral equation (5.8) *has a solution.*

Proof. Without loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s) - x(s)| : s \in [a, b]\} \geq \sup\{|Tx(s) - x(s)| : s \in [a, b]\},$$

which implies that

$$\frac{1}{2}\|Tx - x\| \leq \|Tx - x\| \leq \|y - x\|,$$

where $\lambda \in [0, 1)$. Thus, we have

$$\begin{aligned} |Ty(s) - Tx(s)| &= \left\| g(t) + \gamma \int_a^b M(t, s)h(t, y(s)) - g(t) \right. \\ &\qquad \left. -\gamma \int_a^b M(t, s)h(t, x(s))ds \right\| \\ &\leq \gamma \int_a^b |M(t, s)[h(t, y(s)) - h(t, x(s))]|ds \\ &\leq \gamma \int_a^b M(t, s)\upsilon(x, y)|y(s) - x(s)|ds \\ &\leq \sup_{s \in [a,b]} |y(s) - x(s)|\gamma \int_a^b M(t, s)\mu(x, y)ds \\ &\leq \gamma\omega\|y - x\| \\ &\leq \|y - x\|. \end{aligned}$$

Thus, we have

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \Rightarrow \quad \|Tx - Ty\| \leq \|x - y\|.$$

Clearly, T satisfies condition (C) and so, T is a modified generalized α-nonexpansive mapping and all the conditions in Theorem 3.7 are satisfied, as such T has a fixed point, that is the integral equation (5.8) has a solution. □

5.4. **Numerical Example.**

Example 5.10. Let $H = L^2([0,1])$, H is a Hilbert space with the induced inner product

$$\begin{aligned}\|x(t)\|_2 &= \sqrt{\langle x(t), x(t)\rangle} \\ &= \left(\int_0^1 x^2(t)dt\right)^{\frac{1}{2}}, \quad \forall\, x \in L^2([0,1]).\end{aligned}$$

It is well-known that the set $C = \{x \in L^2([0,1]) : \|x(t)\|_2 \leq 1\}$ is closed and convex subset of H. We define $f : C \to H$ as $f(x) = \|x(t)\|_2^2$, f is a convex function and $x(0) = 0$ a unique minimum of f. In addition, f is Frêchet differentiable at x and its gradient $\nabla f : C \to H$ is defined as $\nabla f(x) = 2x$. Now, observe

$$\begin{aligned}\|\nabla f(x(t)) - \nabla f(y(t))\|_2 &= \left(\int_0^1 (2x(t) - 2y(t))^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 (2(x(t) - y(t)))^2 dt\right)^{\frac{1}{2}} \\ &= 2\left(\int_0^1 (x(t) - y(t))^2 dt\right)^{\frac{1}{2}} \\ &= 2\|x(t) - y(t)\|_2,\end{aligned}$$

clearly, $\nabla f(x)$ is 2-Lipschitzian mapping. In addition, we have

$$\begin{aligned}&\langle \nabla f(x(t)) - \nabla f(y(t)), x(t) - y(t)\rangle \\ &= \int_0^1 (2x(t) - 2y(t)(x(t) - y(t)))dt \\ &= 2\int_0^1 (x(t) - y(t))^2 dt \\ &= (3-1)\int_0^1 (x(t) - y(t))^2 dt \\ &= -\int_0^1 (x(t) - y(t))^2 dt + 3\int_0^1 (x(t) - y(t))^2 dt \\ &= -\|x(t) - y(t)\|^2 + 3\|x(t) - y(t)\|^2 \\ &= -\frac{1}{4}\|2x(t) - 2y(t)\|^2 + 3\|x(t) - y(t)\|^2.\end{aligned}$$

Clearly, $\nabla f(x)$ is a relaxed $(\frac{1}{4}, 3)$-cocorcive. we have $L = 2, \alpha = \frac{1}{4}$ and $k = 3$, then condition (2.10) takes the form

$$0 < \eta < 1 < 3. \tag{5.9}$$

Let us choose $\eta = \frac{1}{6}, \gamma_n = \frac{2}{3n+7}, \beta_n = \frac{5}{19n+2}$ and $\alpha_n = \frac{1}{2n+15}$, then iterative scheme (5.3) becomes

$$\begin{cases} x_0 \in C, \\ z_n = \left(1 - \frac{2}{3n+7}\right) x_n + \frac{2}{3n+7} P_C\left(\frac{2}{3} x_n\right), \\ y_n = \left(1 - \frac{1}{2n+15}\right) z_n + \frac{1}{2n+15} P_C\left(\frac{2}{3} z_n\right), \\ x_{n+1} = \frac{5}{19n+2} P_C\left(\frac{2}{3} z_n\right) + \left(1 - \frac{5}{19n+2}\right) P_C\left(\frac{2}{3} y_n\right), \qquad n \geq 1, \end{cases}$$

where

$$\begin{cases} x(t), & \text{if } x(t) \in C, \\ \frac{x(t)}{\|x(t)\|}, & \text{if } x(t) \notin C. \end{cases}$$

We plot the graph of error against number of iterations with tolerance level($||x_{n+1} - x_n|| = 10 \times e^{-5}$) and varying values of x_0. We consider the following cases.

Case 1: Take $x_0 = 2t^{\frac{1}{3}} + 3t^2 + 5t^3$,

Case 2: Take $x_0 = t^4 + 3t^3 - 5t + 7$,

Case 3: Take $x_0 = \frac{t^2}{2} + 7$,

Case 4: Take $x_0 = \cos t + \sin t$.

Example 5.11. Let $H = \ell_2(\mathbb{R})$, where

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2, \ldots, x_i \ldots), \quad x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\},$$

with inner product $\langle ., . \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ and the norm $\|.\| : \ell_2 \to \mathbb{R}$ by $\|x\| := \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$, where $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$. Let $T : \ell_2 \to \ell_2$ is defined by $Tx = \left(\frac{x_1}{3}, \frac{x_2}{3}, \ldots, \frac{x_i}{3}, \ldots\right)$, for all $x = \{x_i\}_{i=1}^{\infty} \in \ell_2$. Furthermore, let $C := \{x \in \ell_2 : \|x\| \leq 1\}$ be the unit ball. Then, we define the metric projection P_C as:

$$P_C(x) = \begin{cases} \frac{x}{x_{\ell_2}}, & \text{if } x_{\ell_2} > 1, \\ x, & \text{if } x_{\ell_2} \leq 1. \end{cases}$$

With respect to Algorithm 1.3, Algorithm 1.6 and Algorithm 4.1 ($S = T$), we randomly choose $x_0 \in H$. We choose $\beta_n = \frac{2n}{5n+3}, \alpha_n = \frac{3n}{8n+2}$ and $\gamma_n = \frac{8n+5}{20n+4}$ We consider the following cases for our numerical experiment.

Case 1: Take $x_0 = (3.0218, -4.1491, 0, \ldots, 0, \ldots)^T$.

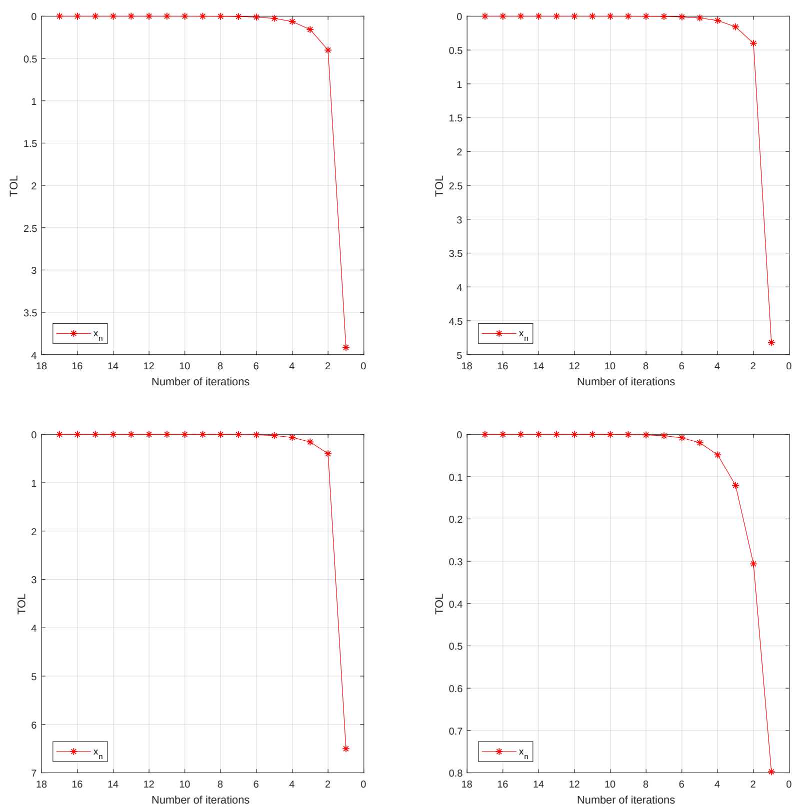

FIGURE 1. Example 5.10, Top Left: Case 1; Top Right: Case 2; Bottom Left: Case 3; Bottom Right: Case 4.

Case 2: Take $x_0 = (2.5141, -1.6457, 0, \ldots, 0, \ldots)^T$.

Case 3: Take $x_0 = (1.2341, -5.0554, 0, \ldots, 0, \ldots)^T$.

Case 4: Take $x_0 = \left(-\sqrt{3}, \sqrt{\frac{1}{17}}, 0, \ldots, 0, \ldots\right)^T$.

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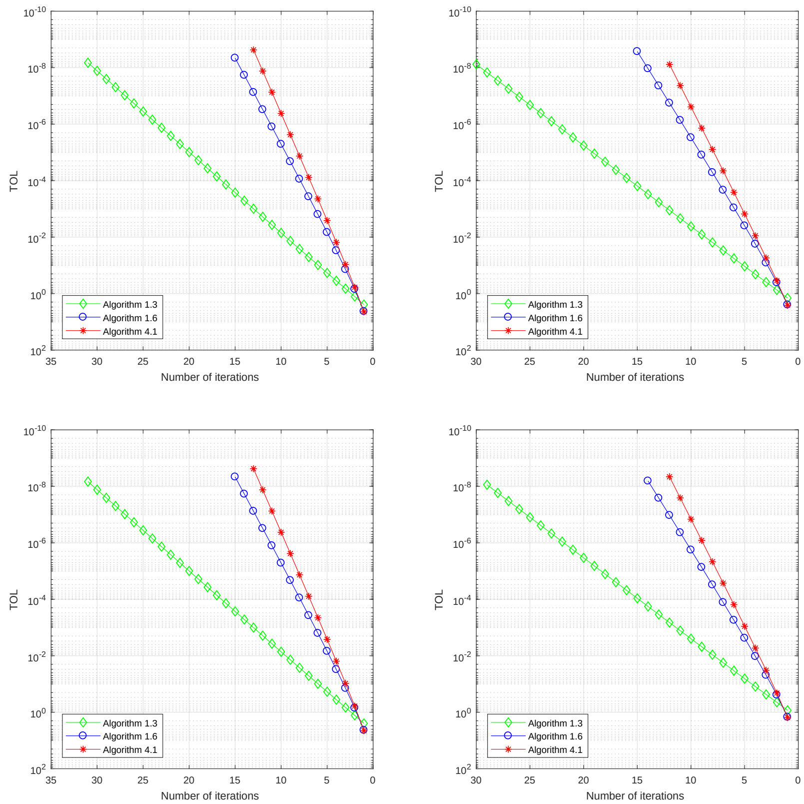

FIGURE 2. Example 5.11, Case 1 (top left); Case 2 (top right); Case 3 (bottom left); Case 4 (bottom right)

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