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Atefe Razghandi and Ali Akbar Arefijamaal

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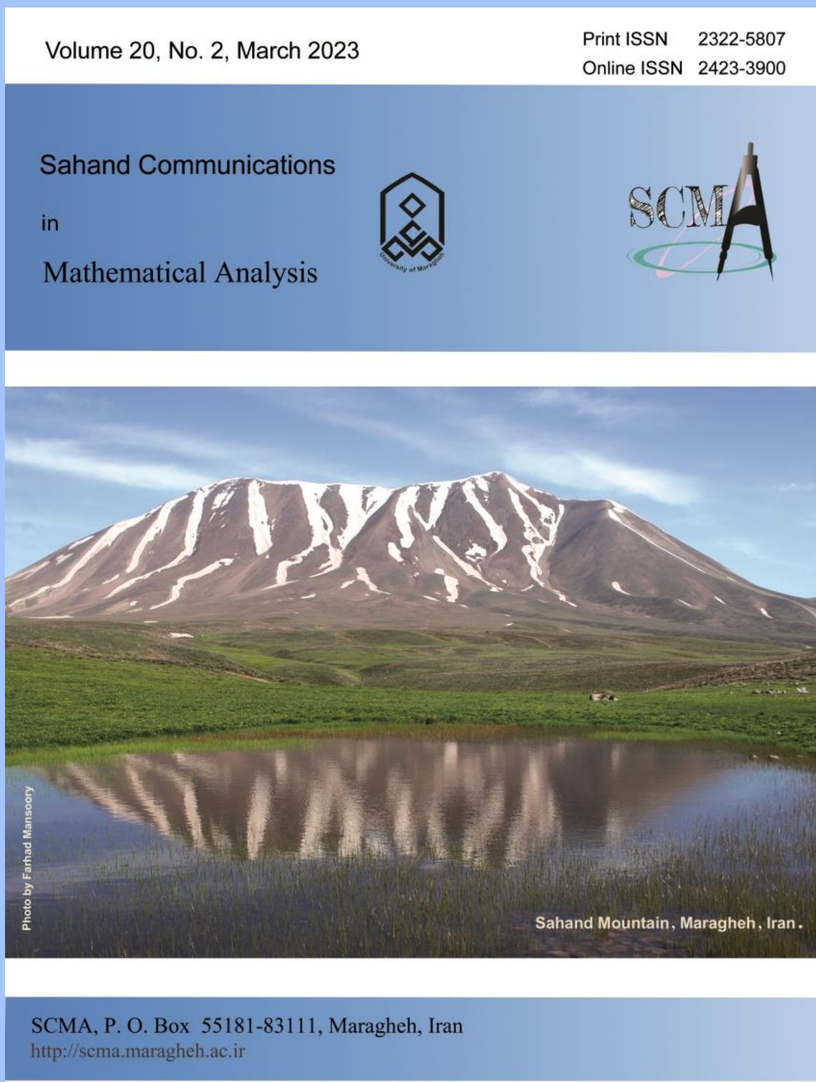

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Duality of Continuous K-g-Frames

Atefe Razghandi[1] and Ali Akbar Arefijamaal[2*]

Abstract. K-g-frames, as an extension of g-frames and K-frames are one of the active fields in frame theory. In this paper, we consider continuous K-g-frames which are a generalization of discrete K-g-frames. We give the necessary and sufficient conditions to characterize their duals. For example, the canonical dual of a given K-g-frame is described by both its frame operator and its alternate duals.

1. Introduction

Frame, appeared first in the early 1950s, for studying some problems in the nonharmonic Fourier series [9]. The notion of an ordinary frame developed to a family indexed by a measurable space [2] and continued by many researchers [3, 15, 17]. The concept of continuous k-g-frame has been introduced in [3]. Also, some properties of continuous k-g-frame have been studied in [4, 12]. Discrete and continuous frames have many applications in many fields. There are several related references on this topic, see [6, 8, 13].

Being an extension of frames, the concept of g-frames was proposed by Sun [14] and popularized from then on. The concept of K-frames was proposed by Găvruţa [10] which allows the reconstruction of elements from the range of a linear and bounded operator K in a Hilbert space. Though K-frames are similar to ordinary frames there are many differences between them as shown in [5, 11, 16]. By using Găvruţa's idea for g-frames, Xiao et al. [18] proposed the notion of K-g-frames, which it

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* Corresponding author.

was attracted many researchers in recent years; see [3, 7, 15, 19]. However, many problems of continuous K-g-frames such as how to find the canonical and alternate duals, have not been studied. This motivated us to consider the characterization of K-g-duals of continuous K-g-frames.

2. Preliminaries

Throughout this paper, H, $\{H_\omega\}_{\omega\in\Omega}$ will be a separable Hilbert space and a family of Hilbert spaces, respectively and (Ω,μ) a measure space. Also, K is a bounded linear operator on H and $B(H_1,H_2)$ is the set of all bounded linear operators from H_1 to H_2. We denote the range and the null of $K\in B(H_1,H_2)$ by $\mathcal{R}(K)$ and $\mathcal{N}(K)$, respectively and $\pi_{\mathcal{R}(K)}$ is the orthogonal projection on $\mathcal{R}(K)$.

Let $K\in B(H_1,H_2)$ have a closed range. There exists a unique bounded linear operator $K^\dagger\in B(H_2,H_1)$, so-called the pseudo inverse of K [8], satisfying

$$\mathcal{N}\left(K^\dagger\right)=\left(\mathcal{R}(K)\right)^\perp,\ \mathcal{R}\left(K^\dagger\right)=\left(\mathcal{N}(K)\right)^\perp,\ KK^\dagger f=f,\quad f\in\mathcal{R}(K).$$

First, we review some definitions about g- frames. Let

$$\prod_{\omega\in\Omega}H_\omega=\left\{f:\Omega\to\bigcup_{\omega\in\Omega}H_\omega;\quad f(\omega)\in H_\omega\right\},$$

be the Cartesian product of H_ω's, we say that the mapping $F\in\prod_{\omega\in\Omega}H_\omega$ is strongly measurable if F as a mapping of Ω to $(\oplus_{\omega\in\Omega}H_\omega,\mu)_{L^2}$ is measurable. Also

$$(\oplus_{\omega\in\Omega}H_\omega,\mu)_{L^2}=\left\{F\in\prod_{\omega\in\Omega}H_\omega : F\text{ is strongly measurable},\int_\Omega\|F(\omega)\|^2d\mu(\omega)<\infty\right\},$$

with the inner product given by

$$\langle F,G\rangle=\int_\Omega\langle F(\omega),G(\omega)\rangle\,d\mu(\omega),$$

is a Hilbert space [1]. We denote its norm by $\|F\|_2$. The continuous version of k-g-frames have been introduced in [3] in the following way.

Definition 2.1 ([3]). Let $K\in B(H)$ and (Ω,μ) be a measure space with a positive measure μ. A family $\Lambda:=\{\Lambda_\omega\in B(H,H_\omega)\ :\ \omega\in\Omega\}$ is called a continuous K-g-frame or (c-k-g-frame) for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if

(i) $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable, for each $f\in H$;

(ii) there exist constants $0 < C \leq D < \infty$ such that

$$C\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq D\|f\|^2, \quad (f \in H).$$

The constants C and D are called lower and upper c-k-g-frame bounds, respectively. If $K^* = I$, then we obtain a continuous g-frame.

If the right-hand inequality in (ii) holds, the family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a continuous g-Bessel family with Bessel constant D and it called C-tight K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if

$$C\|K^*f\|^2 = \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega), \quad (f \in H).$$

The C-tight K-g-frame is said to be Parseval c-k-g-frame if $C = 1$. In this case, it is easy to check that

$$KK^*f = \int_\Omega \Lambda_\omega^* \Lambda_\omega f d\mu(\omega), \quad (f \in H).$$

For a c-k-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$, the operator $T_\Lambda : (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2} \to H$ weakly defined by

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F, g\rangle \, d\mu(\omega), \quad (F \in (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, \; g \in H),$$

is bounded operator [3].

Definition 2.2 ([3])**.** Suppose that Λ_ω is a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bounds A, B. We define $S : H \to H$ by

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega g\rangle \, d\mu(\omega), \quad f, g \in H,$$

and we call it the c-K-g-frame operator.

Notice that the c-k-g-frame operator S_Λ is not invertible, in general. However, it is self-adjoint and if K has a close range then S_Λ is invertible on $\mathcal{R}(K)$ [16]. In addition,

$$\begin{aligned} D_\Lambda^{-1}\|f\|^2 &\leq \left\langle \left(S_\Lambda|_{\mathcal{R}(K)}\right)^{-1} f, f \right\rangle \\ &\leq C_\Lambda^{-1}\|K^\dagger\|^2\|f\|^2, \quad (f \in H). \end{aligned}$$

In the next, we write S_Λ^{-1} instead of $\left(S_\Lambda \,|_{\mathcal{R}(K)}\right)^{-1}$. It is not self-adjoint, however $S_\Lambda^{-1}S_\Lambda$ is the identity on $\mathcal{R}(K)$, in particular, $S_\Lambda^{-1}S_\Lambda = \left(S_\Lambda^{-1}S_\Lambda\right)^*$. Suppose $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$. A continuous g-Bessel mapping $\{\Gamma_\omega\}_{\omega\in\Omega}$ for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ is said to be a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ if $T_\Lambda T_\Gamma^* = K$ or equivalently

$$\langle Kf, g\rangle = \int_\Omega \langle \Lambda_\omega^* \Gamma_\omega f, g\rangle \, d\mu(\omega), \quad (f, g \in H).$$

A K-g-dual $\widetilde{\Lambda} = \{\widetilde{\Lambda}_\omega\}_{\omega\in\Omega}$ of $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called the canonical dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$, if $\|T_{\widetilde{\Lambda}}\| \leq \|T_\Gamma\|$ for every K-g-dual $\{\Gamma_\omega\}_{\omega\in\Omega}$ of $\{\Lambda_\omega\}_{\omega\in\Omega}$ [15]. Duals of c-k-g-frames have been introduced and characterized by atomic cg-systems [12].

In the next section, we consider the duality of c-k-g-frames and give the necessary and sufficient conditions for a K-g-dual to be the canonical K-g-dual.

3. Main Results

In this section, we characterize the (canonical) duals of a c-k-g-frame. This leads to obtaining more duals from a given dual. At the end, we give the relationship between K-g-duals and the canonical K-g-dual. Let $K \in B(H)$, and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g frame. Then $\left\{\Lambda_\omega \left(S_\Lambda^{-1}\right)^* K\right\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ [4, Theorem 3.3].

As a consequence, if $K \in B(H)$ and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-k-g-frame, then $\left\{\Lambda_\omega \left(S_\Lambda^{-1/2}\right)^* K^*\right\}_{\omega\in\Omega}$ is a Parseval c-k-g-frame. Similar to discrete frames [15] we obtain the following K-g-duals for Parseval frames.

Proposition 3.1. *Let $K \in B(H)$ has a close range and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-k-g-frame. Then*

(i) *If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is Parseval, then $\{\Lambda_\omega \left(K^\dagger\right)^*\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$.*

(ii) $\left\{\Lambda_\omega \left(S_\Lambda^{-1/2}\right)^* K^* \left(K^\dagger\right)^*\right\}_{\omega\in\Omega}$ *is a K-g-dual of* $\left\{\Lambda_\omega \left(S_\Lambda^{-1/2}\right)^* K^*\right\}_{\omega\in\Omega}$.

Using [4, Theorem 3.3], it follows that every c-k-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$ has a K-g-dual. In the next, by using a given K-g-dual, we characterize all K-g-duals of $\{\Lambda_\omega\}_{\omega\in\Omega}$. For this means, we define the orthogonal projection P_α for every $\alpha \in \Omega$ by

$$\begin{aligned} P_\alpha : (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2} &\to H_\alpha \\ \{f_\omega\}_{\omega\in\Omega} &\to f_\alpha. \end{aligned}$$

Theorem 3.2. *Suppose $K \in B(H)$ has a closed range and $\{\Lambda'_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$, then $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ if and only if there exists $\Phi \in B(H, (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2})$ with $T_\Lambda\Phi = 0$ such that*

$$\Gamma_\omega = \Lambda'_\omega + P_\omega\Phi, \quad (\omega \in \Omega).$$

Proof. Assume that $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Define $\Phi : H \to (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}$; $f \mapsto \Phi f$ by

$$(\Phi f)_\omega = \Gamma_\omega f - \Lambda'_\omega f.$$

Then, $\Phi \in B\left(H, (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}\right)$. Indeed,

$$\begin{aligned}\|\Phi f\|^2 &= \int \|(\Phi f)_\omega\|^2 \, d\mu(\omega) \\ &= \int \|\Gamma_\omega f - \Lambda'_\omega f\|^2 d\mu(\omega) \\ &\le \int \left(\|\Gamma_\omega f\|^2 + \|\Lambda'_\omega f\|^2 + 2\|\Gamma_\omega f\|\|\Lambda'_\omega f\|\right) d\mu(\omega) \\ &\le \left(\sqrt{D_\Gamma} + \sqrt{D_{\Lambda'}}\right)^2 \|f\|^2.\end{aligned}$$

In addition,

$$\begin{aligned}T_\Lambda \Phi f &= \int_\Omega \Lambda^*_\omega \left(\Phi f\right)_\omega d\mu\left(\omega\right) \\ &= \int_\Omega \Lambda^*_\omega \left(\Gamma_\omega f - \Lambda'_\omega f\right) d\mu\left(\omega\right) \\ &= Kf - Kf \\ &= 0.\end{aligned}$$

Conversely, let $\Gamma_\omega = \Lambda'_\omega + P_\omega \Phi$ where $\Phi \in B\left(H, (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}\right)$ and $T_\Lambda \Phi = 0$. Then

$$\begin{aligned}\int_\Omega \Lambda^*_\omega \Gamma_\omega f d\mu(\omega) &= \int_\Omega \Lambda^*_\omega \Lambda'_\omega f d\mu(\omega) + \int_\Omega \Lambda^* P_\omega \Phi f d\mu(\omega) \\ &= Kf + \int_\Omega \Lambda^* \left(\Phi f\right)_\omega d\mu(\omega) \\ &= Kf + T_\Lambda \Phi \\ &= Kf.\end{aligned}$$

It completes the proof. □

From the above theorem, it follows that K-g-duals of $\{\Lambda_\omega\}_{\omega\in\Omega}$ can be written as $\Gamma_\omega = \Lambda_\omega \left(S_\Lambda^{-1}\right)^* K + P_\omega \Phi$ where $\Phi \in B\left(H, (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}\right)$ satisfies $T_\Lambda \Phi = 0$.

Corollary 3.3. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-k-g-frame. Then K-g-duals of $\{\Lambda_\omega\}_{\omega\in\Omega}$ are precisely the family $P_\omega V^*$, where $V \in B\left((\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}, H\right)$ satisfies the condition $K = T_\Lambda V^*$.*

Theorem 3.4. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-k-g-frame.Then $\widetilde{\Lambda}_\omega = \Lambda_\omega \left(S_\Lambda^{-1}\right)^* K$.*

Proof. Using [4, Theorem 3.3], it follows that ${\Lambda'}_\omega = \Lambda_\omega \left(S_\Lambda^{-1}\right)^* K$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. To complete the proof, it is enough to show that $\|T_{\Lambda'}\| \le \|T_\Gamma\|$ for any K-g-dual $\{\Gamma_\omega\}_{\omega\in\Omega}$ of $\{\Lambda_\omega\}_{\omega\in\Omega}$. By Theorem 3.2

there exists $\Phi \in B\left(H, (\oplus_{\omega\in\Omega} H_\omega, \mu)_{L^2}\right)$ such that $\Gamma_\omega = {\Lambda'}_\omega + P_\omega \Phi$, $\omega \in \Omega$ and $T_\Lambda \Phi = 0$. Thus,

$$\begin{aligned}
\|T_\Gamma^* f\|^2 &= \left\langle T_{\Lambda'}^* f + \Phi f, T_{\Lambda'}^* f + \Phi f \right\rangle \\
&= \left\| T_{\Lambda'}^* f \right\|^2 + \left\langle T_{\Lambda'}^* f, \Phi f \right\rangle + \left\langle \Phi f, T_{\Lambda'}^* f \right\rangle + \|\Phi f\|^2 \\
&= \left\| T_{\Lambda'}^* f \right\|^2 + 2\mathrm{Re}\left\langle f, T_{\Lambda'} \Phi f \right\rangle + \|\Phi f\|^2 \\
&= \left\| T_{\Lambda'}^* f \right\|^2 + 2\mathrm{Re}\left\langle f, K^* S_\Lambda^{-1} T_\Lambda \Phi f \right\rangle + \|\Phi f\|^2 \\
&= \left\| T_{\Lambda'}^* f \right\|^2 + \|\Phi f\|^2.
\end{aligned}$$

Hence, $\|T_{\Lambda'}\| \leq \|T_\Gamma\|$. By the uniqueness of the canonical dual, see Proposition 2.4 of [15], we obtain $\widetilde{\Lambda}_\omega = \Lambda_\omega \left(S_\Lambda^{-1}\right)^* K$. □

As a consequence of Theorem 3.2, by using K-g-duals we make more K-g-duals.

Theorem 3.5. *Let K be a self-adjoint and a closed range operator, also let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a c-k-g-frame. Then $\{\Theta_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ if and only if*

$$\Theta_\omega = \widetilde{\Lambda}_\omega + \Gamma_\omega S_\Lambda - \widetilde{\Lambda}_\omega S_\Lambda, \tag{3.1}$$

on $\mathcal{R}(K)$, which $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

Proof. Let $\{\Theta_\omega\}_{\omega\in\Omega}$ be a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Putting

$$\Gamma_\omega f = \begin{cases} \left(\Theta_\omega - \widetilde{\Lambda}_\omega + \widetilde{\Lambda}_\omega S_\Lambda\right) S_\Lambda^{-1} f, & f \in S_\Lambda \mathcal{R}(K), \\ 0, & \text{otherwise.} \end{cases}$$

In fact, $\Gamma_\omega S_\Lambda = 0$ on $\mathcal{R}(K)^\perp$. Then $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a g-Bessel mapping and clearly (3.1) holds. Moreover, for every $f \in \mathcal{R}(K)$, we have

$$\begin{aligned}
T_\Lambda T_\Gamma^* S_\Lambda f &= T_\Lambda T_\Theta^* f - T_\Lambda T_{\widetilde{\Lambda}}^* f + T_\Lambda T_{\widetilde{\Lambda}}^* S_\Lambda f \\
&= Kf - Kf + KS_\Lambda f \\
&= KS_\Lambda f.
\end{aligned}$$

Also, if $f \in \mathcal{R}(K)^\perp$, then

$$\begin{aligned}
KS_\Lambda f &= (KS_\Lambda)^* f \\
&= S_\Lambda K^* f \\
&= 0 \\
&= T_\Lambda T_\Gamma^* S_\Lambda f.
\end{aligned}$$

Thus, $T_\Lambda T_\Gamma^* = K$. i.e. $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Conversely, if $\Theta_\omega = \widetilde{\Lambda}_\omega + \Gamma_\omega S_\Lambda - \widetilde{\Lambda}_\omega S_\Lambda$, for some K-g-dual $\{\Gamma_\omega\}_{\omega\in\Omega}$ of $\{\Lambda_\omega\}_{\omega\in\Omega}$, then

$$\begin{aligned} T_\Lambda T_\Theta^* &= T_\Lambda T_{\widetilde{\Lambda}}^* + T_\Lambda T_\Gamma^* S_\Lambda - T_\Lambda T_{\widetilde{\Lambda}}^* S_\Lambda \\ &= K, \end{aligned}$$

therefore, $\{\Theta_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. □

In the following, we describe the canonical dual of Parseval c-k-g-frames when K has a closed range.

Proposition 3.6. *Let K have a closed range and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval c-k-g-frame. Then $\widetilde{\Lambda}_\omega = \Lambda_\omega \left(K^\dagger\right)^*$, for every $\omega \in \Omega$.*

Proof. For every $f \in \mathcal{R}(K^*)$, we have $f = K^*(K^*)^\dagger f = K^*(K^\dagger)^* f$. Thus,

$$Kf = KK^*(K^*)^\dagger f, \quad (f \in \mathcal{R}(K^*))\,. \tag{3.2}$$

Since $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a Parseval c-k-g-frame, it follows that $KK^* = S_\Lambda$. Hence, by (3.2) for every $f \in \mathcal{R}(K^*)$, we have

$$\begin{aligned} \Lambda_\omega \left(S_\Lambda^{-1}\right)^* Kf &= \Lambda_\omega \left(S_\Lambda^{-1}\right)^* KK^* \left(K^*\right)^\dagger f \\ &= \Lambda_\omega \left(S_\Lambda^{-1}\right)^* S_\Lambda \left(K^*\right)^\dagger f \\ &= \Lambda_\omega \left(K^\dagger\right)^* f. \end{aligned}$$

Also, if $f \in \mathcal{R}(K^*)^\perp = \mathcal{N}(K)$ then $\left(S_\Lambda^{-1}\right)^* Kf = 0$. On the other hands, if $f \in \mathcal{R}(K^*)^\perp$, then the identity $\mathcal{R}(K^*)^\perp = \mathcal{N}\left(K^\dagger\right)^*$ shows that $\left(K^\dagger\right)^* f = 0$. Therefore,

$$\Lambda_\omega \left(S_\Lambda^{-1}\right)^* Kf = \Lambda_\omega \left(K^\dagger\right)^* f, \quad (f \in H)\,. \qquad \square$$

Several characterizations of the canonical dual are given in the following theorem.

Theorem 3.7. *Let $\{\Gamma_\omega\}_{\omega\in\Omega}$ be a K-g-dual of c-k-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$. Then the following are equivalent:*

(i) $\{\Gamma_\omega\}_{\omega\in\Omega}$ *is the canonical dual of* $\{\Lambda_\omega\}_{\omega\in\Omega}$.
(ii) $S_\Gamma = S_{\widetilde{\Lambda}}$.
(iii) $S_\Gamma = T_\Gamma T_\Theta^*$, *for every K-g-dual* $\{\Theta_\omega\}_{\omega\in\Omega}$ *of* $\{\Lambda_\omega\}_{\omega\in\Omega}$.
(iv) $\left\langle \widetilde{\Lambda}_\omega^* \Lambda_\omega \left(K^\dagger\right)^* f, \Gamma_\xi f \right\rangle = \left\langle {\Gamma_\omega}^* \Lambda_\omega \left(K^\dagger\right)^* f, \widetilde{\Lambda}_\xi f \right\rangle$, *for every* $f \in H$, $\omega, \xi \in \Omega$ *and* $\Gamma_\omega = 0$ *on* $\mathcal{R}\left(K^*\right)^\perp$.

Proof. $(i) \Leftrightarrow (ii)$ By using Theorem 3.2 there exists $\Phi \in B\left(H, \left(\oplus_{\omega\in\Omega} H_\omega, \mu\right)_{L^2}\right)$ such that $\Gamma_\omega = \widetilde{\Lambda}_\omega + P_\omega \Phi$, for $\omega \in \Omega$ and $T_\Lambda \Phi = 0$. Hence, for any $f \in H$, we have

$$\begin{aligned}\langle S_\Gamma f, f\rangle &= \langle T_\Gamma^* f, T_\Gamma^* f\rangle \\ &= \left\langle T_{\widetilde{\Lambda}}^* f + \Phi f, T_{\widetilde{\Lambda}}^* f + \Phi f\right\rangle \\ &= \left\|T_{\widetilde{\Lambda}}^* f\right\|^2 + \left\langle T_{\widetilde{\Lambda}}^* f, \Phi f\right\rangle + \left\langle \Phi f, T_{\widetilde{\Lambda}}^* f\right\rangle + \|\Phi f\|^2 \\ &= \left\|T_{\widetilde{\Lambda}}^* f\right\|^2 + 2\mathrm{Re}\left\langle f, T_{\widetilde{\Lambda}} \Phi f\right\rangle + \|\Phi f\|^2 \\ &= \left\|T_{\widetilde{\Lambda}}^* f\right\|^2 + 2\mathrm{Re}\left\langle f, K^* S_\Lambda^{-1} T_\Lambda \Phi f\right\rangle + \|\Phi f\|^2 \\ &= \left\langle S_{\widetilde{\Lambda}} f, f\right\rangle + \|\Phi f\|^2.\end{aligned}$$

Therefore, $S_\Gamma = S_{\widetilde{\Lambda}}$ if and only if $\{\Gamma_\omega\}_{\omega\in\Omega}$ is the canonical dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

$(i) \Leftrightarrow (iii)$ Let $\{\Gamma_\omega\}_{\omega\in\Omega}$ be the canonical dual and $\{\Theta_\omega\}_{\omega\in\Omega}$ is a K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$. Then

$$\begin{aligned}S_\Gamma &= T_{\widetilde{\Lambda}} T_{\widetilde{\Lambda}}^* \\ &= K^* \left(S_\Lambda^{-1}\right)^* T_\Lambda T_{\widetilde{\Lambda}}^* \\ &= K^* \left(S_\Lambda^{-1}\right)^* K \\ &= K^* \left(S_\Lambda^{-1}\right)^* T_\Lambda T_\Theta^* \\ &= T_\Gamma T_\Theta^*.\end{aligned}$$

Conversely, if $S_\Gamma = T_\Gamma T_\Theta^*$, for a K-g-dual $\{\Theta_\omega\}_{\omega\in\Omega}$ of $\{\Lambda_\omega\}_{\omega\in\Omega}$, then we have

$$\begin{aligned}\|T_\Gamma\|^2 &= \|T_\Gamma T_\Gamma^*\|^2 \\ &= \|T_\Gamma T_\Theta^*\|^2 \\ &\leq \|T_\Gamma\|\|T_\Theta\|.\end{aligned}$$

Hence $\|T_\Gamma\| \leq \|T_\Theta\|$, so $\{\Gamma_\omega\}_{\omega\in\Omega}$ is the canonical K-g-dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$.

$(i) \Leftrightarrow (iv)$ If $\{\Gamma_\omega\}_{\omega\in\Omega}$ is the canonical dual of $\{\Lambda_\omega\}_{\omega\in\Omega}$ then trivially (4) holds. Conversely, if $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a K-g-dual and $\Gamma_\omega = 0$ on $\mathcal{R}\left(K^*\right)^\perp$, then for every $f \in \mathcal{R}\left(K^*\right)$, we obtain

$$\begin{aligned}\langle f, \Gamma_\xi f\rangle &= \left\langle K^*\left(K^*\right)^\dagger f, \Gamma_\xi f\right\rangle \\ &= \left\langle \int \widetilde{\Lambda}_\omega^* \Lambda_\omega \left(K^*\right)^\dagger f d\mu(\omega), \Gamma_\xi f\right\rangle\end{aligned}$$

$$= \left\langle \int \Gamma_\omega{}^* \Lambda_\omega \left(K^*\right)^\dagger f d\mu\left(\omega\right), \widetilde{\Lambda}_\xi f \right\rangle$$
$$= \left\langle f, \widetilde{\Lambda}_\xi f \right\rangle.$$

On the other hand, $\Gamma_\omega = \widetilde{\Lambda}_\omega = 0$ on $\mathcal{R}\left(K^*\right)^\perp$ by the assumption. So $\Gamma_\omega = \widetilde{\Lambda}_\omega$, for every $\omega \in \Omega$. □

We end this section by a stability result about K-g-duals.

Proposition 3.8. *Let $\{\Gamma_\omega\}_{\omega\in\Omega}$ be a K-g-dual of c-k-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$. If a g-Bessel family $\Theta \in B\left(H, \left(\oplus_{\omega\in\Omega} H_\omega, \mu\right)_{L^2}\right)$ satisfies in the condition $\|T_\Gamma^* - T_\Theta^*\| < \left\|T_\Lambda\|^{-1}\right\| K^\dagger\|^{-1}$, then $T_\Lambda T_\Theta^* K^\dagger$ is invertible on $\mathcal{R}\left(K\right)$ and $\left\{\Lambda_\omega \left(\left(T_\Lambda T_\Theta^* K^\dagger \mid_{\mathcal{R}(K)}\right)^{-1}\right)^* K\right\}_{\omega\in\Omega}$ is a K-g-dual of $\Theta K^\dagger$.*

Proof. Putting $L = T_\Lambda T_\Theta^* K^\dagger \mid_{\mathcal{R}(K)}$. Then for any $f \in \mathcal{R}\left(K\right)$, we have

$$\begin{aligned}\|f - Lf\| &= \|T_\Lambda T_\Gamma^* K^\dagger f - T_\Lambda T_\Theta^* K^\dagger f\| \\ &\leq \|T_\Lambda\| \, \|T_\Gamma^* - T_\Theta^*\| \, \|K^\dagger\| \|f\| \\ &:= \beta \|f\|.\end{aligned}$$

Therefore $\left(1 - \beta\right) \|f\| \leq \|Lf\|$, for any $f \in \mathcal{R}\left(K\right)$. Hence $L : \mathcal{R}\left(K\right) \to L\left(\mathcal{R}\left(K\right)\right)$ is an invertible operator and for any $f \in H$, we have

$$\begin{aligned}T_{\Theta K^\dagger} T_\Lambda^* \Lambda \left(\left(L \mid_{\mathcal{R}(K)}\right)^{-1}\right)^* Kf &= \left(K^\dagger\right)^* T_\Theta T_\Lambda^* \left(\left(L \mid_{\mathcal{R}(K)}\right)^{-1}\right)^* Kf \\ &= \left(\left(L \mid_{\mathcal{R}(K)}\right)^{-1} L\right)^* Kf \\ &= Kf.\end{aligned}$$

It completes the proof. □

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[1] Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

Email address: a.razghandi@hsu.ac.ir, ateferazghandi@yahoo.com

[2] Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

Email address: arefijamaal@hsu.ac.ir, arefijamaal@gmail.com