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Firooz Pashaie

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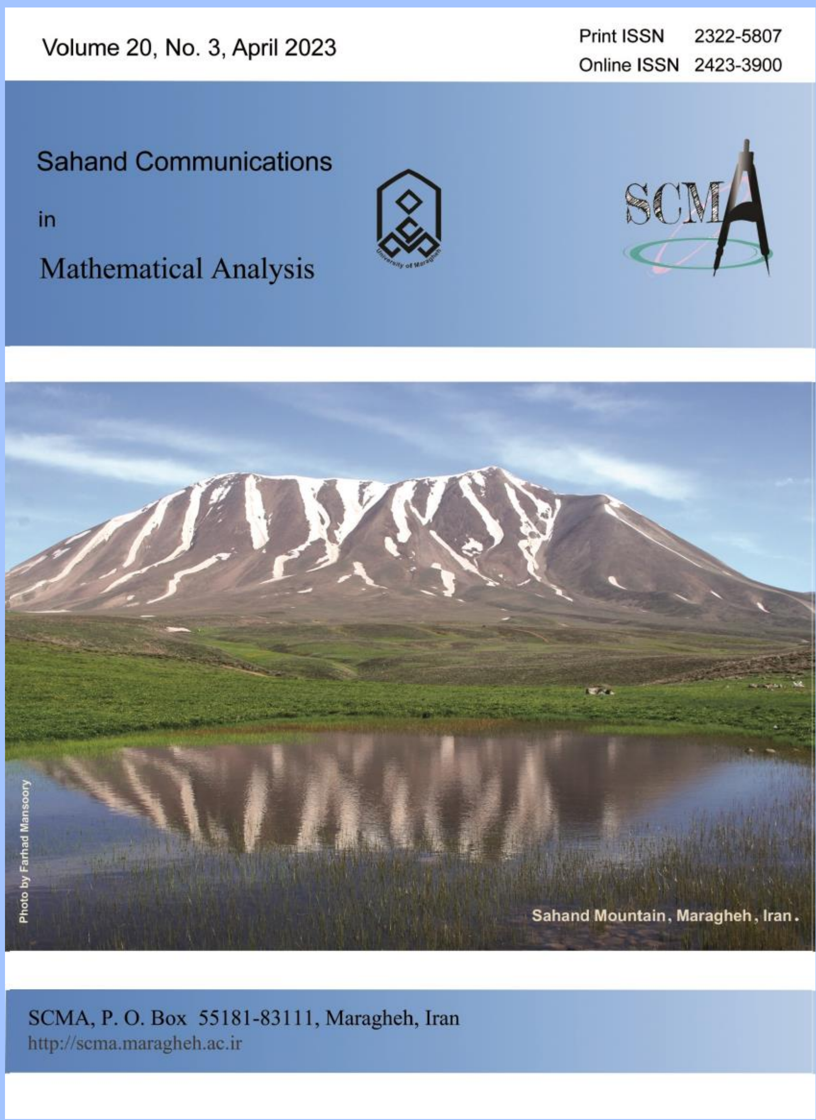

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On Bi-Conservative Hypersurfaces in the Lorentz-Minkowski 4-space $\mathbb{E}_1^4$

Firooz Pashaie

Abstract. In the 1920s, D. Hilbert has showed that the tensor of stress-energy, related to a given functional Λ, is a conservative symmetric bicovariant tensor Θ at the critical points of Λ, which means that $\mathrm{div}\Theta = 0$. As a routine extension, the bi-conservative condition (i.e. $\mathrm{div}\Theta_2 = 0$) on the tensor of stress-bienergy Θ_2 is introduced by G. Y. Jiang (in 1987). This subject has been followed by many mathematicians. In this paper, we study an extended version of bi-conservativity condition on the Lorentz hypersurfaces of the Einstein space. A Lorentz hypersurface M_1^3 isometrically immersed into the Einstein space is called $\mathcal{C}$-bi-conservative if it satisfies the condition $\mathrm{N}_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2$, where N_2 is the second Newton transformation, H_2 is the 2nd mean curvature function on M_1^3 and ∇ is the gradient tensor. We show that the $\mathcal{C}$-bi-conservative Lorentz hypersurfaces of Einstein space have constant second mean curvature.

1. Introduction

In differential geometry, the study of bi-conservative immersions is an interesting research topic. The subject of bi-conservative immersions has been started by Eells and Sampson [4], which discusses on the critical points of the bienergy functional obtained form the tension field and the zero set of the tangent component of bitension field. Before that time, David Hilbert had proved that the tensor of stress-energy, obtained from a functional Λ, is a conservative symmetric bicovariant tensor Θ at the critical points of Λ ([9]). For the bienergy functional Λ_2, G.Y. Jiang ([10]) has defined the stress-bienergy tensor Θ_2 and proved

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that it satisfies div$\Theta_2 = -\langle \tau^2(\phi), d\phi\rangle$ (see also [2]). In 1995, Hasanis and Vlachos have classified the bi-conservative hypersurfaces (namely H-hypersurfaces) of 3 and 4 dimensional Euclidean spaces ([8]). Some bi-conservative hypersurfaces in Euclidean k-space have been classified in [19]. Recently, in semi-Riemannian context, some (spacelike) bi-conservative hypersurfaces in $\mathbb{E}^5_2$ having diagonal shape operator was studied in [21]. Some other results on biconservative hypersurfaces in semi-Euclidean spaces may be found in [7, 21]. Also, the biconservative hypersurfaces in some Riemannian manifolds have been studied in [5, 6, 20].

Let $\mathbf{x} : M^3_1 \to \mathbb{E}^4_1$ be a Lorentzian (i.e. timelike) hypersurface in the Loentz-Minkowski 4-space $\mathbb{E}^4_1$. We apply an extension of the Laplace operator Δ on M^3_1. By definition, the Laplace operator is given by $\Delta(f) = tr(\nabla^2 f)$ for any $f \in C^\infty(M^3_1)$, where $\nabla^2 f$ is the hessian of f. In fact, Δ stands for the linearized operator of the first variation of the ordinary mean curvature of M^3_1. We use the Cheng-Yau operator $\mathcal{C}$ that is an extension of Δ by definition $\mathcal{C}(f) = tr(\mathrm{N}_1 \circ \nabla^2 f)$, where $\mathrm{N}_1 = nHI - S$ denotes the first Newton transformation associated to the second fundamental from of M^3_1 (see, for instance, [1, 11, 15, 16, 18]). Recently, in [13], we have studied the $\mathcal{C}$-biharmonic spacelike hypersurfaces in $\mathbb{E}^4_1$ satisfying the condition $\mathcal{C}^2\mathbf{x} = 0$. Here, we discuss on $\mathcal{C}$-bi-conservative timelike hypersurfaces with non-diagonal second fundamental form in $\mathbb{E}^4_1$. The hypersurface $\mathbf{x} : M^3_1 \to \mathbb{E}^4_1$ is said to be $\mathcal{C}$-bi-conservative if the tangent component of vector field $\mathcal{C}^2\mathbf{x}$ is null. Equivalently, M^3_1 in $\mathbb{E}^4_1$ is $\mathcal{C}$-bi-conservative if it satisfies the condition $\mathrm{N}_2(\nabla H_2) = \frac{9}{2}H_2\nabla H_2$, where N_2 is the 2nd Newton transformation, H_2 is the 2nd mean curvature function on M^3_1 and ∇ is the gradient tensor. We show that the $\mathcal{C}$-bi-conservative Lorentz hypersurfaces of Einstein space have constant second mean curvature.

2. Preliminaries

We recall some notations and formulae from [11, 12, 14–17]. The pseudo-Euclidean 4-space, $\mathbb{E}^4_1$, is the Euclidean 4-space $\mathbb{R}^4$ endowed with the product $\langle \mathbf{v}, \mathbf{w}\rangle = -v_1w_1 + \sum_{i=2}^{4} v_iw_i$, where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^4$.

For a Lorentzian vector space V^3_1, a basis $\mathcal{B} := \{e_1, e_2, e_3\}$ is said to be orthonormal if it satisfies $\langle e_i, e_j\rangle = \epsilon_i\delta^j_i$ for $i, j = 1, 2, 3$, where $\epsilon_1 = -1$ and $\epsilon_2 = \epsilon_3 = 1$. As usual, δ^j_i stands for the Kronecker delta function. $\mathcal{B}$ is called pseudo-orthonormal if it satisfies $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0$, $\langle e_1, e_2\rangle = -1$ and $\langle e_i, e_3\rangle = \delta^3_i$, for $i = 1, 2, 3$.

The shape operator of a Lorentzian hypersurface $\mathbf{x} : M^3_1 \to \mathbb{E}^4_1$, as a self-adjoint linear map on the tangent space of M^3_1, can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. Where, in cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^3, the matrix representation of the induced metric on M_1^3 is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} \kappa & \lambda & 0 \\ -\lambda & \kappa & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For cases II and III, using a pseudo-orthonormal basis of the tangent space of M_1^3, the induced metric on M_1^3 has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad B_3 = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{pmatrix},$$

respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 2.1. In two cases II and III, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, e_3\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3\}$ where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrix representations $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) as

$$\tilde{B_2} = \begin{pmatrix} \kappa + \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \kappa - \frac{1}{2} & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\sqrt{2}/2 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{pmatrix}$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

Notation: According to four possible matrix representations of the shape operator of M_1^3, we define its principal curvatures, denoted by unified notations κ_i for $i = 1, 2, 3$, as follow.

In case I, we put $\kappa_i := \lambda_i$, for $i = 1, 2, 3$, where λ_i's are the eigenvalues of B_1.

In cases II, where the matrix representation of S is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_3 := \lambda$.

In case III, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$.

Finally, in the case IV, where the shape operator has matrix representation $\tilde{B}_4$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_3 := \eta$.

The characteristic polynomial of S on M_1^3 is of the form

$$Q(t) = \sum_{i=1}^{3}(t - \kappa_i) = \sum_{j=0}^{3}(-1)^j s_j t^{3-j},$$

where, $s_0 := 1$, $s_1 = \sum_{j=1}^{3} \kappa_j$, $s_2 := \sum_{1\leq i_1<i_2\leq 3} \kappa_{i_1}\kappa_{i_2}$ and $s_3 := \kappa_1\kappa_2\kappa_3$.

For $j = 1, 2, 3$, the j-th mean curvature H_j of M is defined by $H_j = \frac{1}{\binom{3}{j}} s_j$. When H_j is identically null, M_1^n is said to be $(j-1)$-minimal. Remember that, a timeike hypersurface $x : M_1^3 \to \mathbb{E}_1^4$ with diagonalizable shape operator is said to be isoparametric if all of its principal curvatures are constant. But, a timelike hypersurface $x : M_1^3 \to \mathbb{E}_1^4$ with non-diagonalizable shape operator is called isoparametric if the minimal polynomial its shape operator has constant coefficients.

Remark 2.2. Here we remind Theorem 4.10 from [12], which assures us that there is no isoparametric timelike hypersurface of $\mathbb{E}_1^4$ with complex principal curvatures.

The jth Newton transformation on M_1^3, $\mathrm{N}_j : \chi(M_1^3) \to \chi(M_1^3)$, is defined (inductively) by

$$\mathrm{N}_0 = \mathrm{I}, \qquad \mathrm{N}_j = s_j\mathrm{I} - S \circ \mathrm{N}_{j-1}, \quad (j = 1, 2, 3), \tag{2.1}$$

where, I is the identity map.

Using its explicit formula, $\mathrm{N}_j = \sum_{i=0}^{j}(-1)^i s_{j-i} S^i$ (where $S^0 = \mathrm{I}$), it can be seen that, N_j is self-adjoint and commutative with S (see [1, 15]).

Now, we define a notation as

$$\mu_{j;k} = \sum_{l=0}^{k}(-1)^l \binom{n}{k-1} H_{k-l}\kappa_j^l, \quad (1 \leq j \leq 3,\ 1 \leq k < 3) \tag{2.2}$$

Corresponding to the four possible forms $\tilde{B}_i$ (for $1 \leq i \leq 4$) of S, the Newton transformation N_j has different representations. In the case I, where $S = \tilde{B}_1$, we have $\mathrm{N}_j = diag[\mu_{1;j}, \mu_{2;j}, \mu_{3;j}]$, for $j = 1, 2$.

When $S = B_2$ (in the case II), we have

$$\mathrm{N}_1 = \begin{pmatrix} \kappa + \lambda - \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \kappa + \lambda + \frac{1}{2} & 0 \\ 0 & 0 & 2\kappa \end{pmatrix},$$

$$\mathrm{N}_2 = \begin{pmatrix} (\kappa - \frac{1}{2})\lambda & -\frac{1}{2}\lambda & 0 \\ \frac{1}{2}\lambda & (\kappa + \frac{1}{2})\lambda & 0 \\ 0 & 0 & \kappa^2 \end{pmatrix}.$$

In the case III, we have $S = B_3$, and

$$\mathrm{N}_1 = \begin{pmatrix} 2\kappa & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\kappa & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa \end{pmatrix}, \qquad \mathrm{N}_2 = \begin{pmatrix} \kappa^2 - \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}\kappa \\ \frac{1}{2} & \kappa^2 + \frac{1}{2} & \frac{\sqrt{2}}{2}\kappa \\ \frac{\sqrt{2}}{2}\kappa & \frac{\sqrt{2}}{2}\kappa & \kappa^2 \end{pmatrix}.$$

In the case IV, $S = B_4$,

$$\mathrm{N}_1 = \begin{pmatrix} \kappa + \eta & -\lambda & 0 \\ \lambda & \kappa + \eta & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad \mathrm{N}_2 = \begin{pmatrix} \kappa\eta & -\lambda\eta & 0 \\ \lambda\eta & \kappa\eta & 0 \\ 0 & 0 & \kappa^2 + \lambda^2 \end{pmatrix}.$$

Fortunately, in all cases we have the following important identities for $j = 1, 2$, similar to those in [1, 15].

$$\begin{aligned} &\text{(i) } \mathrm{tr}(\mathrm{N}_1) = 6\mathrm{H}_1, \\ &\text{(ii) } \mathrm{tr}(\mathrm{N}_2) = 3\mathrm{H}_2, \\ &\text{(iii) } \mathrm{tr}(\mathrm{N}_1 \circ \mathrm{S}) = 6\mathrm{H}_2, \\ &\text{(iv) } \mathrm{tr}(\mathrm{N}_2 \circ \mathrm{S}) = 3\mathrm{H}_3, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\text{(i) } \mathrm{trS}^2 = 9\mathrm{H}_1^2 - 6\mathrm{H}_2, \\ &\text{(ii) } \mathrm{tr}(\mathrm{N}_1 \circ \mathrm{S}^2) = 9\mathrm{H}_1\mathrm{H}_2 - 3\mathrm{H}_3, \\ &\text{(iii) } \mathrm{tr}(\mathrm{N}_2 \circ \mathrm{S}^2) = 3\mathrm{H}_1\mathrm{H}_2. \end{aligned} \tag{2.4}$$

The linearized operator of the $(j+1)$th mean curvature of M, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(\mathrm{N}_j \circ \nabla^2 f)$, where, $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for every $X, Y \in \chi(M)$.

Associated to the orthonormal frame $\{e_1, e_2, e_3\}$ of tangent space on a local coordinate system in the hypersurface $x : M_1^3 \to \mathbb{E}_1^4$, the Cheng-Yau operator $\mathcal{C} = L_1$ has an explicit expression as

$$\mathcal{C}(f) = \sum_{i=1}^{3} \epsilon_i \mu_{i,1} \left(e_i e_i f - \nabla_{e_i} e_i f\right). \tag{2.5}$$

For a Lorentzian hypersurface $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{E}_1^4$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$ where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [1, 15].

$$\nabla\langle \mathbf{x}, \mathbf{a}\rangle = \mathbf{a}^T, \qquad \nabla\langle \mathbf{n}, \mathbf{a}\rangle = -S\mathbf{a}^T. \tag{2.6}$$

$$\mathcal{C}\mathbf{x} = c_1 H_2 \mathbf{n}, \qquad \mathcal{C}\mathbf{n} = -3\nabla(H_2) - 3[3H_1H_2 - H_3]\mathbf{n}, \tag{2.7}$$

(2.8) $\mathcal{C}^2x = -6[9H_2\nabla H_2 - 2\mathrm{N}_2\nabla H_2] - 6\left[9H_1H_2^2 + 3H_2H_3 - \mathcal{C}H_2\right]\mathbf{n}.$

Assume that a hypersurface $x : M_1^3 \to \mathbb{E}_1^4$ satisfies the condition $\mathcal{C}^2x = 0$, then it is said to be $\mathcal{C}$-bi-harmonic. An $\mathcal{C}$-bi-harmonic hypersurface $\mathbf{x} : M_1^3 \to \mathbb{E}_1^4$ is said to be proper-$\mathcal{C}$-bi-harmonic, if it satisfies the condition $\mathcal{C}\mathbf{x} \neq 0$.

By equalities (2.7) and (2.8), from the condition $\mathcal{C}(H_2\mathbf{n}) = 0$ we obtain simpler conditions on M_1^3 to be a $\mathcal{C}$-bi-harmonic hypersurface in $\mathbb{E}_1^4$, as:

$$\text{(i) } \mathcal{C}H_2 = 3\left(3H_1H_2^2 - H_2H_3\right), \qquad \text{(ii) } \mathrm{N}_2\nabla H_2 = \frac{9}{2}H_2\nabla H_2. \tag{2.9}$$

A timelike hypersurface $x : M_1^3 \to \mathbb{E}_1^4$ is said to be $\mathcal{C}$-bi-conservative, if its 2th mean curvature satisfies the condition (2.9)(ii).

The structure equations of $\mathbb{E}_1^4$ are given by

$$d\omega_i = \sum_{j=1}^{4} \omega_{ij} \wedge \omega_j, \ \omega_{ij} + \omega_{ji} = 0, \tag{2.10}$$

$$d\omega_{ij} = \sum_{l=1}^{4} \omega_{il} \wedge \omega_{lj}. \tag{2.11}$$

With restriction to M, we have $\omega_4 = 0$ and then,

$$0 = d\omega_4 = \sum_{i=1}^{3} \omega_{4,i} \wedge \omega_i. \tag{2.12}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{4,i} = \sum_{j=1}^{3} h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.13}$$

This gives the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_4$. The mean curvature H is given by $H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}$. From (2.10) - (2.13) we obtain the structure equations of M as follow.

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.14}$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.15}$$

for $i, j = 1, 2, 3$, and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$dh_{ij} = \sum_{k=1}^{3} h_{ijk}\omega_k + \sum_{k=1}^{3} h_{kj}\omega_{ik} + \sum_{k=1}^{3} h_{ik}\omega_{jk}.$$

Thus, by exterior differentiation of (2.13), we obtain the Codazzi equation $h_{ijk} = h_{ikj}$.

The next lemma can be proved by a similar proof as in [18].

Lemma 2.3. *Let M_1^3 be a timelike hypersurface in $\mathbb{E}_1^4$ of type I with principal curvatures of constant multiplicities. Then the distribution of the space of principal directions corresponding to the principal curvatures is completely integrable. In addition, if a principal curvature is of multiplicity greater than one, then it will be constant on each integral submanifold of the corresponding distribution.*

Now, we see two examples of non-$\mathcal{C}$-bi-conservative timelike hypersurfaces in $\mathbb{E}_1^4$.

Example 2.4. Let $M_1^3(r)$ be the product $\mathbb{S}_1^2(r) \times \mathbb{E}^1 \subset \mathbb{E}_1^4$ where $r > 0$. It has another representation as

$$M_1^3(r) = \left\{(y_1, \ldots, y_4) \in \mathbb{E}_1^4 | -y_1^2 + y_2^2 + y_3^2 = r^2\right\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$, $\kappa_3 = 0$, and the constant higher order mean curvatures $H_1 = \frac{2}{3}r^{-1}$, $H_2 = \frac{1}{3}r^{-2}$ and $H_3 = 0$. One see that $M_1^3(r)$ is $\mathcal{C}$-bi-conservative.

Example 2.5. Let $\bar{M}_1^3(r)$ be the product $\mathbb{E}_1^2 \times \mathbb{S}^1(r) \subset \mathbb{E}_1^4$ where $r > 0$. It can be represented as

$$\bar{M}_1^3(r) = \{(y_1, \ldots, y_4) \in \mathbb{E}_1^4 | y_3^2 + y_4^2 = r^2\},$$

with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(0, 0, y_3, y_4)$. it has two distinct principal curvatures $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = \frac{1}{r}$, and the constant higher order mean curvatures $H_1 = \frac{1}{3r}$, and $H_k = 0$ for $k = 2, 3$. Clearly, $\bar{M}_1^3(r)$ is $\mathcal{C}$-bi-conservative.

3. Main results

In this section, we give five theorems on the $\mathcal{C}$-bi-conservative connected orientable timelike hypersurface in $\mathbb{E}_1^4$ with constant ordinary mean curvature. Theorems 3.1 and 3.2 are appropriated to the case that the shape operator on hypersurface is diagonalizable. Theorems 3.3, 3.4 and 3.5 are related to the cases that the shape operator on hypersurface is of type II, III and IV, respectively.

Theorem 3.1. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an $\mathcal{C}$-bi-conservative connected orientable timelike hypersurface which has diagonal shape operator, constant ordinary mean curvature and three distinct real principal curvatures. Then, M_1^3 is isoparametric and its second mean curvature is constant.*

Proof. Suppose that, H_2 is non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_1, such that $Se_i = \lambda_i e_i$ and then, $P_2 e_i = \mu_{i,2} e_i$ for $i = 1, 2, 3$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.9)(ii), we get

$$e_i(H_2)\left(\mu_{i,2} - \frac{9}{2}H_2\right) = 0, \tag{3.1}$$

for $i = 1, 2, 3$. Each point of $\mathcal{U}$ has an open neighborhood on which we have $e_i(H_2) \neq 0$ for at least one i. So, without loss of generality, we can assume that $e_1(H_2) \neq 0$ and then we have $\mu_{1,2} = \frac{9}{2}H_2$, (locally) on $\mathcal{U}$, which gives $\lambda_2\lambda_3 = \frac{9}{2}H_2$. Now, we prove three simple claims.

Claim 1: $e_2(H_2) = e_3(H_2) = 0$.
If $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (3.1) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2}H_2$ or $\mu_{1,2} = \mu_{3,2} = \frac{9}{2}H_2$, which give $\lambda_3(\lambda_2 - \lambda_1) = 0$ or $\lambda_2(\lambda_1 - \lambda_3) = 0$. But, since λ_i's are assumed to be mutually distinct, we get $\lambda_3 = 0$ or $\lambda_2 = 0$, which gives $H_2 = 0$ on $\mathcal{U}$. The result is in contradiction with the definition of $\mathcal{U}$.

Claim 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$.
Since H is assumed to be constant on M, we have

$$e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2).$$

On the other hand, from two recent results $e_2(H_2) = 0$ and $\lambda_2\lambda_3 = \frac{9}{2}H_2$ we get

$$e_2(\lambda_1\lambda_3) + e_2(\lambda_1\lambda_2) = e_2\left(3H_2 - \frac{9}{2}H_2\right) = 0,$$

which gives $\lambda_1 e_2(\lambda_2+\lambda_3)+(\lambda_2+\lambda_3)e_2\lambda_1=0$, and then we have

$$\lambda_1 e_2(3H-\lambda_1)+(\lambda_2+\lambda_3)e_2\lambda_1=(-\lambda_1+\lambda_2+\lambda_3)e_2\lambda_1=0.$$

Therefore, assuming $e_2(\lambda_1)\neq 0$, we get $\lambda_1=\lambda_2+\lambda_3$ which gives contradiction

$$e_2(\lambda_1)=e_2(\lambda_2+\lambda_3)=e_2(3H-\lambda_1)=-e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1)=0$.

Similarly, one can show $e_3(\lambda_1)=0$. So, Claim 2 is proved.

Claim 3: $e_2(\lambda_3)=e_3(\lambda_2)=0$.

Using the notations

$$\nabla_{e_i}e_j=\sum_{k=1}^{3}\omega_{ij}^k e_k, \quad (i,j=1,2,3), \tag{3.2}$$

and the compatibility condition $\nabla_{e_k}\langle e_i,e_j\rangle=0$, we have

$$\omega_{ki}^i=0, \qquad \omega_{ki}^j+\omega_{kj}^i=0, \quad (i,j,k=1,2,3) \tag{3.3}$$

and applying the Codazzi equation (see [14], page 115, Corollary 34(2))

$$(\nabla_V S)W=\nabla_W S)V, \quad (\forall V,W\in\chi(M)) \tag{3.4}$$

on the basis $\{e_1,e_2,e_3\}$, we get for distinct $i,j,k=1,2,3$

$$(a)\ e_i(\lambda_j)=(\lambda_i-\lambda_j)\omega_{ji}^j, \qquad (b)\ (\lambda_i-\lambda_j)\omega_{ki}^j=(\lambda_k-\lambda_j)\omega_{ik}^j. \tag{3.5}$$

Also, by a straightforward computation of components of the identity $\nabla_{e_i}S)e_j-\nabla_{e_j}S)e_i\equiv 0$ for distinct $i,j=1,2,3$, we get $[e_2,\ e_3](H_2)=0$, $\omega_{12}^1=\omega_{13}^1=\omega_{13}^2=\omega_{21}^3=\omega_{32}^1=0$ and

$$\omega_{21}^2=\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}, \qquad \omega_{31}^3=\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}, \tag{3.6}$$
$$\omega_{23}^2=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}, \qquad \omega_{32}^3=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}.$$

Therefore, the covariant derivatives $\nabla_{e_i}e_j$ simplify to $\nabla_{e_1}e_k=0$ for $k=1,2,3$, and

$$\nabla_{e_2}e_1=\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}e_2, \qquad \nabla_{e_3}e_1=\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}e_3, \tag{3.7}$$
$$\nabla_{e_2}e_2=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_1, \qquad \nabla_{e_3}e_2=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}e_3,$$
$$\nabla_{e_2}e_3=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}e_2, \qquad \nabla_{e_3}e_3=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_1+\frac{e_2(\lambda_3)}{\lambda_3-\lambda_2}e_2.$$

Now, the Gauss equation for $\langle R(e_2,e_3)e_1,\ e_2\rangle$ and $\langle R(e_2,e_3)e_1,\ e_3\rangle$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{3.8}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right). \tag{3.9}$$

We also have the Gauss equation for $\langle R(e_1,e_2)e_1,\ e_2\rangle$ and $\langle R(e_3,e_1)e_1,\ e_3\rangle$, which give the following relations

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)+\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)^2=\lambda_1\lambda_2, \tag{3.10}$$
$$e_1\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)+\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)^2=\lambda_1\lambda_3.$$

Finally, we obtain from the Gauss equation for $\langle R(e_3,e_1)e_2,e_3\rangle$ that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\right)=\frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)}. \tag{3.11}$$

On the other hand, we consider the condition (2.9)(ii). It follows from Claim I that

(3.12)

$$-\mu_{1,1}e_1e_1(H_2)$$
$$+\left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)e_1(H_2)-9H_2^2\left(H-\frac{3}{2}\lambda_1\right)=0.$$

By differentiating (3.12) along on e_2 respectively e_3, and using (3.8), (3.9) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{3.13}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}-\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{3.14}$$

Using (3.7), we find that

$$[e_1,\ e_2]=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_2. \tag{3.15}$$

Applying both sides of the equality (3.15) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (3.13), (3.10), and (3.11), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.16}$$

(3.16) shows that $e_2(\lambda_3) = 0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1}. \tag{3.17}$$

From equation (3.17), by differentiating on its both sides along e_1 and applying (3.10), we get $\lambda_2 = \lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3) = 0$.

Analogously, using (3.7), we find that $[e_1,\ e_3] = \frac{e_1(\lambda_3)}{\lambda_3-\lambda_1} e_3$. By a similar manner, we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} + \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) = 0, \tag{3.18}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3) = e_3(\lambda_2) = 0$ which, by applying the Gauss equation for $\langle R(e_2, e_3)e_1, e_3\rangle$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} = \lambda_2\lambda_3. \tag{3.19}$$

Finally, using (3.10), differentiating (3.19) along e_1 gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} + \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) = 0, \tag{3.20}$$

which implies $\lambda_2\lambda_3 = 0$ (since we have seen above that $\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1} + \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right) \neq 0$). Therefore, we obtain $H_2 = 0$ on $\mathcal{U}$, which is a contradiction. Hence H_2 is constant on M_1^3. Finally, we get that M_1^3 is isoparametric. $\square$

Theorem 3.2. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be a $\mathcal{C}$-bi-conservative Lorentzian hypersurfaces of $\mathbb{E}_1^4$ with diagonalizable shape operator (i.e of type I) and constant ordinary mean curvature. If M_1^3 has exactly two distinct principal curvatures, then it is isoparametric and its second mean curvature is constant.*

Proof. By assumption, M_1^3 has two distinct principal curvatures λ and μ of multiplicities 2 and 1, respectively. Defining the open subset $\mathcal{U}$ of M as $\mathcal{U} := \{p \in M_1^3 : \nabla H_2^2(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \varnothing$, we consider $\{e_1, e_2, e_3\}$ as a local orthonormal frame of principal directions of S on $\mathcal{U}$ such that $Se_i = \lambda_i e_i$ for $i = 1, 2, 3$. By assumption, we have

$$\lambda_1 = \lambda_2 = \lambda, \qquad \lambda_3 = \mu.$$

Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \lambda\mu, \qquad \mu_{3,2} = \lambda^2, \qquad 3H = 2\lambda + \mu, \qquad 3H_2 = \lambda^2 + 2\lambda\mu. \tag{3.21}$$

By condition (2.9)(ii), we have

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{3.22}$$

Then, using the polar decomposition

$$\nabla H_2 = \sum_{i=1}^{3} \epsilon_i \langle \nabla H_2, e_i \rangle e_i, \tag{3.23}$$

we see that (3.22) is equivalent to

$$\epsilon_i \langle \nabla H_2, e_i \rangle \left(\mu_{i,2} - \frac{9}{2} H_2 \right) = 0$$

on $\mathcal{U}$ for $i = 1, 2, 3$. Hence, for every i such that $\langle \nabla H_2, e_i \rangle \neq 0$ on $\mathcal{U}$ we get

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{3.24}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathcal{U}$, which gives one or both of the following states.

Case 1. $\langle \nabla H_2, e_i \rangle \neq 0$, for $i = 1$ or $i = 2$. By equalities (3.21) and (3.24), we obtain

$$\lambda\mu = \frac{9}{2} \left(\frac{2}{3} \lambda\mu + \frac{1}{3} \lambda^2 \right),$$

which gives

$$\lambda \left(6H - \frac{5}{2} \lambda \right) = 0. \tag{3.25}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\lambda = \frac{12}{5} H$, $\mu = -\frac{9}{5} H$ and $H_2 = -\frac{72}{25} H^2$.

Case 2. $\langle \nabla H_2, e_3 \rangle \neq 0$. By equalities (3.21) and (3.24), we obtain

$$\lambda^2 = \frac{9}{2} \left(\frac{2}{3} \lambda\mu + \frac{1}{3} \lambda^2 \right),$$

which gives

$$\lambda \left(9H - \frac{11}{2} \lambda \right) = 0. \tag{3.26}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\lambda = \frac{18}{11} H$, $\mu = -\frac{3}{11} H$ and $H_2 = \frac{216}{121} H^2$. Therefore, H_2 is constant. Finally, we get that M_1^3 is isoparametric. □

Theorem 3.3. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an $\mathcal{C}$-bi-conservative connected orientable timelike hypersurface with shape operator of type II in $\mathbb{E}_1^4$. If M_1^3 has constant ordinary mean curvature, then its 2nd mean curvature has to be constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_2$, such that $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda e_3$ and then, we have $P_2 e_1 = (\kappa - \frac{1}{2})\lambda e_1 + \frac{1}{2}\lambda e_2$, $P_2 e_2 = -\frac{1}{2}\lambda e_1 + (\kappa + \frac{1}{2})\lambda e_2$ and $P_2 e_3 = \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.9)(ii) we get

$$
\begin{aligned}
&\text{(i) } \epsilon_1 \mathrm{e}_1(\mathrm{H}_2)\left[(\kappa - \frac{1}{2})\lambda - \frac{9}{2}\mathrm{H}_2\right] = \epsilon_2 \mathrm{e}_2(\mathrm{H}_2)\frac{\lambda}{2} \\
&\text{(ii) } \epsilon_2 \mathrm{e}_2(\mathrm{H}_2)\left[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}\mathrm{H}_2\right] = -\epsilon_1 \mathrm{e}_1(\mathrm{H}_2)\frac{\lambda}{2} \\
&\text{(iii) } \epsilon_3 \mathrm{e}_3(\mathrm{H}_2)\left(\kappa^2 - \frac{9}{2}\mathrm{H}_2\right) = 0.
\end{aligned}
\tag{3.27}
$$

Now, we prove some simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (3.27)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$
\begin{aligned}
&\text{(i) } \left(\kappa - \frac{1}{2}\right)\lambda - \frac{9}{2}\mathrm{H}_2 = \frac{\epsilon_2 \mathrm{e}_2(\mathrm{H}_2)}{\epsilon_1 \mathrm{e}_1(\mathrm{H}_2)}\frac{\lambda}{2} \\
&\text{(ii) } \frac{\epsilon_2 \mathrm{e}_2(\mathrm{H}_2)}{\epsilon_1 \mathrm{e}_1(\mathrm{H}_2)}\left[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}\mathrm{H}_2\right] = -\frac{\lambda}{2},
\end{aligned}
\tag{3.28}
$$

which, by substituting (i) in (ii), gives $\frac{\lambda}{2}(1+u)^2 = 0$, where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. Then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then we get $H_2 = 0$ from (3.28)(i). Also, by assumption $\lambda \neq 0$ we get $u = -1$ which gives $\kappa\lambda = \frac{9}{2}H_2$, then $\kappa(3\kappa + 4\lambda) = 0$ and finally $\kappa = -\frac{4}{3}\lambda$ (since $\kappa = 0$ gives $H_2 = 0$ again). Hence, we have $H_2 = \frac{2}{9}\kappa\lambda = -\frac{8}{27}\lambda^2$ and $H_1 = -\frac{5}{9}\lambda$, and since H_1 is assumed to be constant, H_2 has to be constant and we have $e_1(H_2) = 0$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then by (3.27)(iii) we get $\kappa^2 = \frac{9}{2}H_2$, then $\kappa(\kappa + 6\lambda) = 0$, which gives $\kappa = 0$ or $\kappa = -6\lambda$. If $\kappa = 0$, then $H_2 = 0$, and if $\kappa = -6\lambda$ then since $H_1 = -\frac{11}{3}\lambda$ is assumed to be constant, we get that H_2 is constant and then $e_3(H_2) = 0$. Which is a contradiction, so we have $e_3(H_2) = 0$. ∎

Theorem 3.4. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an $\mathcal{C}$-bi-conservative connected orientable timelike hypersurface with shape operator of type III in $\mathbb{E}_1^4$. Then M_1^3 has constant second mean curvature.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U}=\{p\in M:\nabla H_2^2(p)\neq 0\}$, we try to show $\mathcal{U}=\emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1,e_2,e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_3$, such that $Se_1=\kappa e_1+\frac{\sqrt{2}}{2}e_3$, $Se_2=\kappa e_2-\frac{\sqrt{2}}{2}e_3$, $Se_3=-\frac{\sqrt{2}}{2}e_1-\frac{\sqrt{2}}{2}e_2+\kappa e_3$ and then, we have $P_2e_1=\left(\kappa^2-\frac{1}{2}\right)e_1-\frac{1}{2}e_2-\frac{\sqrt{2}}{2}\kappa e_3$, $P_2e_2=\frac{1}{2}e_1+\left(\kappa^2+\frac{1}{2}\right)e_2+\frac{\sqrt{2}}{2}\kappa e_3$ and $P_2e_3=\frac{\sqrt{2}}{2}\kappa e_1+\frac{\sqrt{2}}{2}\kappa e_2+\kappa^2e_3$.

Using the polar decomposition $\nabla H_2=\sum_{i=1}^{3}\epsilon_ie_i(H_2)e_i$, from condition (2.9)(ii) we get

(3.29)

$$\text{(i) } \epsilon_1\mathrm{e}_1(\mathrm{H}_2)\left[\left(\kappa^2-\frac{1}{2}\right)-\frac{9}{2}\mathrm{H}_2\right]+\frac{1}{2}\epsilon_2\mathrm{e}_2(\mathrm{H}_2)+\frac{\sqrt{2}}{2}\epsilon_3\mathrm{e}_3(\mathrm{H}_2)\kappa=0$$

$$\text{(ii) } \frac{-1}{2}\epsilon_1\mathrm{e}_1(\mathrm{H}_2)+\epsilon_2\mathrm{e}_2(\mathrm{H}_2)\left[\left(\kappa^2+\frac{1}{2}\right)-\frac{9}{2}\mathrm{H}_2\right]+\frac{\sqrt{2}}{2}\epsilon_3\mathrm{e}_3(\mathrm{H}_2)\kappa=0$$

$$\text{(iii) } \epsilon_1\mathrm{e}_1(\mathrm{H}_2)\frac{-\sqrt{2}}{2}\kappa+\epsilon_2\mathrm{e}_2(\mathrm{H}_2)\frac{\sqrt{2}}{2}\kappa+\epsilon_3\mathrm{e}_3(\mathrm{H}_2)\left(\kappa^2-\frac{9}{2}\mathrm{H}_2\right)=0.$$

Now, we prove some simple claims.

Claim: $e_1(H_2)=e_2(H_2)=e_3(H_2)=0$.
If $e_1(H_2)\neq 0$, then by dividing both sides of equalities (3.27)(i,ii,iii) by $\epsilon_1e_1(H_2)$, and using the identity $H_2=\kappa^2$ in Case III, we get

$$\text{(3.30)} \qquad \text{(i) } -\frac{1}{2}-\frac{7}{2}\kappa^2+\frac{1}{2}\mathrm{u}_1+\frac{\sqrt{2}}{2}\mathrm{u}_2\kappa=0$$

$$\text{(ii) } \frac{-1}{2}+\mathrm{u}_1\left(\frac{1}{2}-\frac{7}{2}\kappa^2\right)+\frac{\sqrt{2}}{2}\mathrm{u}_2\kappa=0$$

$$\text{(iii) } \frac{-\sqrt{2}}{2}\kappa+\frac{\sqrt{2}}{2}\mathrm{u}_1\kappa-\frac{7}{2}\kappa^2\mathrm{u}_2=0,$$

where $u_1:=\frac{\epsilon_2e_2(H_2)}{\epsilon_1e_1(H_2)}$ and $u_2:=\frac{\epsilon_3e_3(H_2)}{\epsilon_1e_1(H_2)}$, which, by comparing (i) and (ii), gives $\kappa^2(u_1-1)=0$. If $\kappa=0$, then $H_2=0$. Assuming $\kappa\neq 0$, we get $u_1=1$, which, using (3.30)(iii), gives $u_2=0$. Substituting $u_1=1$ and $u_2=0$ in (3.30)(i), we obtain again $\kappa=0$, which is a contradiction. Hence $e_1(H_2)\equiv 0$.

Therefore, using the result $e_1(H_2)\equiv 0$, the system of equations (3.29) gives

$$\text{(3.31)} \qquad \text{(i) } \frac{1}{2}\epsilon_2\mathrm{e}_2(\mathrm{H}_2)+\frac{\sqrt{2}}{2}\epsilon_3\mathrm{e}_3(\mathrm{H}_2)\kappa=0$$

$$\text{(ii) } \epsilon_2 \mathrm{e}_2(\mathrm{H}_2)\left(\frac{1}{2}-\frac{7}{2}\kappa^2\right)+\frac{\sqrt{2}}{2}\epsilon_3\mathrm{e}_3(\mathrm{H}_2)\kappa=0$$

$$\text{(iii) } \epsilon_2 \mathrm{e}_2(\mathrm{H}_2)\frac{\sqrt{2}}{2}\kappa-\epsilon_3\mathrm{e}_3(\mathrm{H}_2)\frac{7}{2}\kappa^2=0.$$

Comparing (i) and (ii), we get $\kappa e_2(H_2)=0$, which using (iii) gives $\kappa e_3(H_2)=0$, and then, using (i), gives $e_2(H_2)=0$. Then, the second claim (i.e. $e_2(H_2)=0$) is proved.

Now, using the results $e_1(H_2)=e_2(H_2)=0$, we get $\kappa e_3(H_2)=0$, which, using $H_2=\kappa^2$, implies $\kappa e_3(\kappa^2)=0$ and then $e_3(\kappa^3)=0$, and finally $e_3(H_2)=0$. □

Theorem 3.5. *Let $x: M_1^3 \to \mathbb{E}_1^4$ be an $\mathcal{C}$-bi-conservative connected orientable timelike hypersurface with shape operator of type IV in $\mathbb{E}_1^4$. If M_1^3 has constant mean curvature and a constant real principal curvature, then its second and third mean curvatures are constant.*

Proof. Assuming H_2 to be non-constant on M, we show that the open subset $\mathcal{U}=\{p\in M : \nabla H_2^2(p)\neq 0\}$ is an empty set. By the assumption M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1,e_2,e_3\}$ on M, the shape operator S has the matrix form B_4, such that $Se_1=\kappa e_1-\lambda e_2$, $Se_2=\lambda e_1+\kappa e_2$, $Se_3=\eta e_3$ and then, we have $P_2e_1=\kappa\eta e_1+\lambda\eta e_2$, $P_2e_2=-\lambda\eta e_1+\kappa\eta e_2$ and $P_2e_3=(\kappa^2+\lambda^2)e_3$.

Using the polar decomposition $\nabla H_2=\sum_{i=1}^{3}\epsilon_i e_i(H_2)e_i$, from condition (2.9)(ii) we get

$$\text{(i) } \epsilon_1\mathrm{e}_1(\mathrm{H}_2)\left(\kappa\eta-\frac{9}{2}\mathrm{H}_2\right)=\epsilon_2\mathrm{e}_2(\mathrm{H}_2)\lambda\eta, \tag{3.32}$$

$$\text{(ii) } \epsilon_2\mathrm{e}_2(\mathrm{H}_2)\left(\kappa\eta-\frac{9}{2}\mathrm{H}_2\right)=-\epsilon_1\mathrm{e}_1(\mathrm{H}_2)\lambda\eta,$$

$$\text{(iii) } \epsilon_3\mathrm{e}_3(\mathrm{H}_2)\left(\kappa^2+\lambda^2-\frac{9}{2}\mathrm{H}_2\right)=0.$$

Now, we prove three simple claims.

Claim 1: $e_1(H_2)=e_2(H_2)=0$.

If $e_1(H_2)\neq 0$, then by dividing both sides of equalities (3.32)(i,ii) by $\epsilon_1 e_1(H_2)$ we get

$$\text{(i) } \kappa\eta-\frac{9}{2}\mathrm{H}_2=\frac{\epsilon_2\mathrm{e}_2(\mathrm{H}_2)}{\epsilon_1\mathrm{e}_1(\mathrm{H}_2)}\lambda\eta, \tag{3.33}$$

$$\text{(ii) } \frac{\epsilon_2\mathrm{e}_2(\mathrm{H}_2)}{\epsilon_1\mathrm{e}_1(\mathrm{H}_2)}\left(\kappa\eta-\frac{9}{2}\mathrm{H}_2\right)=-\lambda\eta,$$

which, by substituting (i) in (ii), gives $\lambda\eta\left(1+\left(\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\right)^2\right)=0$, then $\lambda\eta=0$. Since by assumption $\lambda\neq 0$, we get $\eta=0$. So, by $(3.33)(i)$ we have $H_2=0$, which is a contradiction.

Similarly, if $e_2(H_2)\neq 0$, then by dividing both sides of equalities $(3.32(i,ii))$ by $\epsilon_2 e_2(H_2)$ we get

$$\text{(3.34)}\qquad \begin{aligned} &\text{(i)}\ \frac{\epsilon_1 \mathrm{e}_1(\mathrm{H}_2)}{\epsilon_2 \mathrm{e}_2(\mathrm{H}_2)}\left(\kappa\eta-\frac{9}{2}\mathrm{H}_2\right)=\lambda\eta,\\ &\text{(ii)}\ \kappa\eta-\frac{9}{2}\mathrm{H}_2=-\frac{\epsilon_1 \mathrm{e}_1(\mathrm{H}_2)}{\epsilon_2 \mathrm{e}_2(\mathrm{H}_2)}\lambda\eta, \end{aligned}$$

which, by substituting (i) in (ii), gives $\lambda\eta\left(1+\left(\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\right)^2\right)=0$, then $\lambda\eta=0$. Since by assumption $\lambda\neq 0$, we get $\eta=0$. So, by $(3.34)(ii)$ we have $H_2=0$, which is a contradiction.

Claim 2: $e_3(H_2)=0$.

If $e_3(H_2)\neq 0$, then from equality $(3.32)(iii)$ we have $\kappa^2+\lambda^2=\frac{9}{2}H_2$, which gives $\kappa^2+\lambda^2=-6\kappa\eta$, where $\eta=3H_1-2\kappa$ and η and H_1 are assumed to be constant on $\mathcal{U}$. So, κ is also constant on $\mathcal{U}$, and then,we get that $H_2=\frac{-4}{3}\kappa\eta=\frac{8}{3}\kappa^2-4H_1\kappa$ is constant on $\mathcal{U}$, which contradicts with the assumption. □

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Department of Mathematics, Faculty of Science, University of Maragheh, P.O.Box 55181-83111, Maragheh, Iran.

Email address: f_pashaie@maragheh.ac.ir