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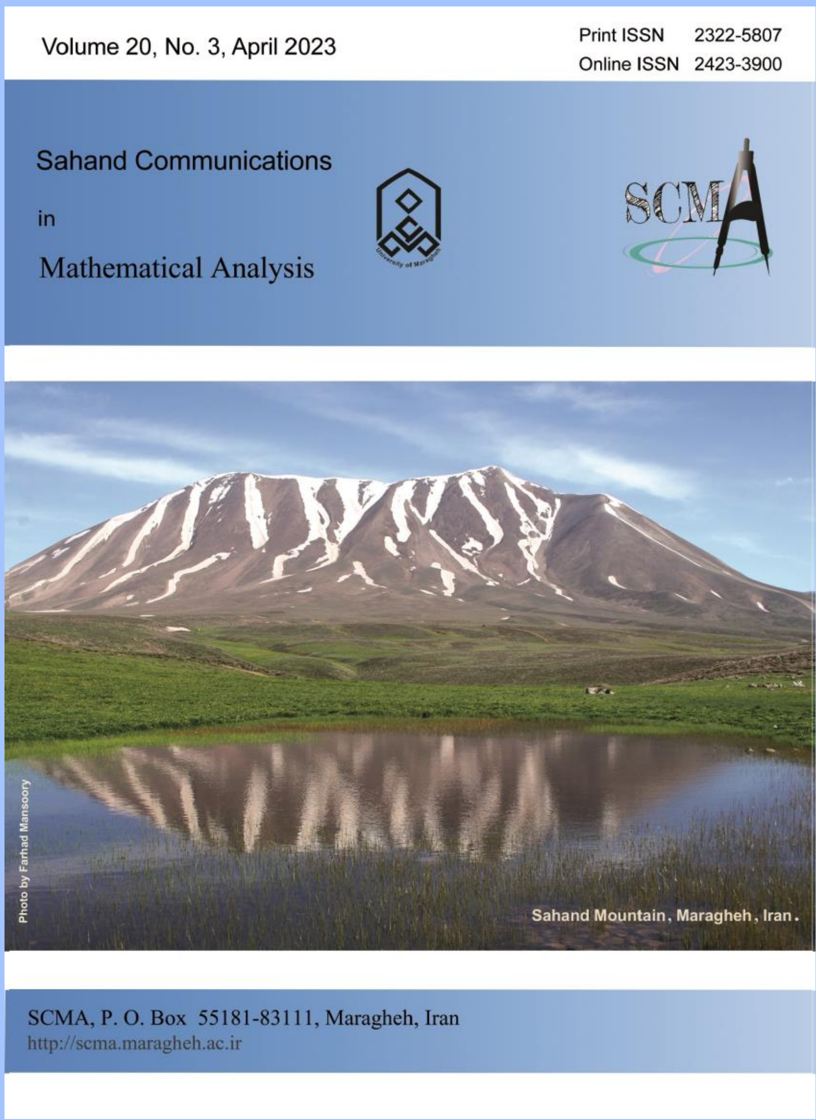

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Abstract. Let $\varphi : A \to B$ be an isomorphism of C^*-algebras and I be an ideal of A. Introducing the concepts of unitary equivalent and the implemented Finsler modules, we show that the $\frac{A}{I}$-module $\frac{E}{E_I}$ and the implemented $\frac{B}{\varphi(I)}$-module $\frac{F}{F_{\varphi(I)}}$ are unitary equivalent. We also, establish a one to one correspondence between the groups $U(E)$ and $U(F)$ of unitaries on full Finsler modules E and F, respectively. Finally, we explain regularized dynamical systems and apply the aforementioned one to one correspondence to prove that each regularized dynamical system in $U(E)$ implements a regularized dynamical system in $U(F)$.

1. Introduction

Let X be a Banach space. A one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded linear operators on X is called a regularized one parameter group if there exists an injective bounded linear operator c on X such that $\alpha_0 = c$ and $c\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$. For convenience, we say that such a regularized one parameter group is a c-one parameter group. A c-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is called strongly continuous if $\lim_{t\to 0} \alpha_t(x) = c(x)$, for each $x \in X$.

The infinitesimal generator δ of a c-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is a mapping $\delta : D(\delta) \subseteq X \to X$ such that $\delta(x) = c^{-1} \lim_{t\to 0} \frac{\alpha_t(x) - c(x)}{t}$ where $D(\delta) = \{x \in X : \lim_{t\to 0} \frac{\alpha_t(x) - c(x)}{t}$ exists in the range of $c\}$.

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The notion of regularized semigroups was introduced by Davies and Pang in 1987 [8]. The reader is referred to [8, 9, 15] for more details. Trivially, if c is the identity operator on X, then a regularized one parameter group is nothing than a one parameter group in the usual sense (see [25, p. 8]).

One parameter groups of bounded linear operators and their extensions are of more considerable magnitude because of their applications in the theory of dynamical systems. Such groups are applied widely to describe the dynamical systems appearing in quantum field theory and statistical mechanics [7, 10, 23, 30].

The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on C^*-algebras. On the other hand, the infinitesimal generator of a C^*-dynamical system is a closed densely defined $*$-derivation. Therefore, the theory of C^*-dynamical systems concerns the theory of derivations in C^*-algebras.

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras and Banach modules. Automatic continuity, approximately innerness and closability are some of important subjects which are investigated in the theory of derivations (see [1, 16, 20, 22] and references therein).

In each case of generalization of derivations, a noted point drawing the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamical system. Each extension of a C^*-dynamical system is usually provided by adjoining a suitable property to (an extension of) a uniformly (strongly) continuous one parameter group of bounded linear operators on (an extension of) a C^*-algebra. Some approaches to preparing new dynamical systems and their applications have been explained in [2, 3, 17–19, 21] and references therein.

As an extension of C^*-algebras it can be pointed to Hilbert C^*-modules. A (left) Hilbert C^*-module over a C^*-algebra A is an algebraic left A-module X equipped with an A-valued inner product $\langle .,. \rangle$ which is A-linear in the first and conjugate linear in the second variable such that X is Banach space with respect to the norm $\|x\| = \| \langle x, x\rangle \|^{\frac{1}{2}}$. The Hilbert A-module X is called full if $A_X := span\{\langle x, x\rangle : x, y \in X\}$ is dense in A. Note that A_X is an ideal in A, called the range ideal of X. We denote by $\langle X, X\rangle$ the closure of A_X and call it the support of X. Therefore, X is a full Hilbert A-module if $\langle X, X\rangle$ is equal to A.

We recall that a linear map $T : X \to X$ is adjointable if there exists a map $T^* : X \to X$ that fulfills $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in X$. Such a map is called the adjoint map of T. This definition implies that each adjointable map is a bounded A-linear operator (see [14, p. 8]). An adjointable operator $T : X \to X$ is called unitary if $TT^* = T^*T = I$.

Hilbert C^*-modules were investigated by Kaplansky [13], Paschke [24], and Rieffel [29]. They are a generalization of Hilbert spaces, but there are some differences between these two classes. For example, each bounded operator on a Hilbert space has an adjoint, but a bounded A-linear map on a Hilbert A-module is not adjointable in general. For this and other general facts concerned with Hilbert C^*-modules we refer to [14, 28].

In 1995, Phillips and Weaver [27] demonstrated an interesting generalization of Hilbert C^*-modules entitled "Finsler modules" . They also, showed that if a C^*-algebra A has no nonzero commutative ideal, then any Finsler A-module is a Hilbert A-module.

In 2002, Bakić and B. Guljaš [6] introduced a canonical Hilbert C^*-module structure by applying ideal submodules. The inner product that turn the quotient of a Hilbert C^*-module over an ideal submodule into a Hilbert C^*-module motivated them to represent an interpretation of morphisms on Hilbert C^*-modules, as a class of module maps. Applying the aforementioned morphisms, they extended the notion of unitaries on Hilbert C^*-modules. One year later, Amyari and Niknam [5], implemented these new interpretations of morphisms and unitary operators under the frame of Finsler modules. In the present paper, we demonstrate necessary and sufficient conditions for a linear map between two full Finsler modules to be a unitary operator. We also, introduce the concepts of unitary equivalent and the implemented Finsler modules. Let $\varphi : A \to B$ be an isomorphism of C^*-algebras and I be an ideal of A. Considering the canonical Finsler module structure on the quotient of a Finsler module over its associated ideal submodule, we show that the canonical Finsler $\frac{A}{I}$-module $\frac{E}{E_I}$ and the implemented canonical Finsler $\frac{B}{\varphi(I)}$-module $\frac{F}{F_{\varphi(I)}}$ are unitary equivalent. Finally, as a main result of this section, we introduce an algebraic isomorphism Φ (in Proposition 3.13), which establishes a one to one correspondence between the groups $U(E)$ and $U(F)$ of unitary operators on full Finsler modules E and F, respectively.

In the final section, we closely examine the concepts of regularized C^*-dynamical systems and generalized derivations of Finsler modules. We also, explain regularized dynamical systems on full Finsler modules and show that generalized derivations of Finsler modules are appeared as the infinitesimal generator of regularized dynamical systems. In fact, we prove that if δ is the infinitesimal generator of a regularized dynamical system on a full Finsler A-module E, then there exists a unique derivation d of A for which d turns δ into a d-derivation. Applying this result, we show that if E and F are full unitary equivalent Finsler modules, then each regularized dynamical system on E implements a regularized dynamical system on F by the algebraic isomorphism Φ. More precisely, let E and F be full Finsler modules over C^*-algebras A and B, respectively. Suppose that $\varphi : A \to B$ is an injective homomorphism and $T : E \to F$ is a surjective φ-homomorphism. We prove that for a regularized dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ on E with the generator δ_1, one can correspond a regularized dynamical system $\{\beta_t\}_{t\in\mathbb{R}}$ on F and a unique regularized C^*-dynamical system $\{\beta'_t\}_{t\in\mathbb{R}}$ on B such that if δ_2 and d_2 are the generators of $\{\beta_t\}_{t\in\mathbb{R}}$ and $\{\beta'_t\}_{t\in\mathbb{R}}$ respectively, then δ_2 is a d_2-derivation.

2. Preliminaries

We start this section with the following basic definition.

Definition 2.1. Let A be a C^*-algebra and A^+ be the set of all positive elements of A. Suppose that E is complex linear space which is a left A-module (and $\lambda(a.x) = (\lambda a).x = a.(\lambda x)$ where $\lambda \in C, a \in A$ and $x \in E$) and there exists a map $\rho_A : E \to A^+$ such that

(*i*) The map $\|.\|_E : x \to \|\rho_A(x)\|$ is a norm on E which makes E into a Banach space; and
(*ii*) $\rho_A(a.x)^2 = a.\rho_A(x)^2.a^*$ for all $a \in A$ and $x \in E$.

Then, E is called a Finsler A-module under the map ρ_A. A Finsler A-module E is said to be full if the linear span $\{\rho_A(x)^2 : x \in E\}$, denoted by $\langle\rho_A(E)\rangle$, is dense in A.

It is easy to check that if E is a (full) Hilbert A-module, then E is a (full) Finsler A-module via $\rho_A(x) := \langle x, x\rangle^{\frac{1}{2}}$. The following theorem, which can be found in [11], provides a sufficient condition for a Finsler module to be a Hilbert module.

Theorem 2.2. *Let E be a full Finsler A-module under the map ρ_A such that ρ_A satisfies the parallelogram law on E. Then, E via* $\langle x, x'\rangle := \frac{1}{4}\sum_{k=0}^{3} i^k\rho_A(x + i^k x')^2$ *is a Hilbert A-module.*

Let E be a Finsler A-module and I be an ideal in A (throughout this paper, by an ideal we always mean a closed two-sided ideal). The associated ideal submodule E_I is defined by the closed linear span of the action of I on E. In the other words, $E_I := \overline{span\ \{a.x : \ a \in I,\ x \in E\}}$.

Clearly, E_I is a closed submodule of E and by the Hewitt-Cohen factorization theorem ([26, Theorem 4.1] and [28, Proposition 2.31]), it is easy to show that $E_I = \{a.x : a \in I, x \in E\}$. Also, it is known [5] that $E_I = \{x \in E : \rho_A(x) \in I\}$. It is notable that E_I can be regarded as a Finsler module over I via $\rho_I(a.x) := \rho_A(a.x)$ since $\rho_A(a.x)^2 = a.\rho_A(x)^2.a^* \in I$ whenever $a \in I$ and $x \in E$. Denote by $\pi : A \to \frac{A}{I}$ and $q : E \to \frac{E}{E_I}$ the quotient maps. A left action of $\frac{A}{I}$ on $\frac{E}{E_I}$ is defined by $\pi(a).q(x) := q(a.x)$.

The following Theorem states that the quotient of a Finsler module over an ideal submodule admits a natural Finsler module structure.

Theorem 2.3 ([27, Lemma 12] and [11, Lemma 3.9])**.** *Let E be a Finsler A-module, let I be an ideal in A, and let E_I be the associated ideal submodule. Then, $\frac{E}{E_I}$ equipped with the aforementioned left action $\pi(a).q(x) := q(a.x)$ is a Finsler module over $\frac{A}{I}$ module via $\rho_{\frac{A}{I}}(q(x)) := \pi(\rho_A(x))$. Moreover, if E is full, then so is $\frac{E}{E_I}$.*

Let $\varphi : A \to B$ be a linear $*$-homomorphism of C^*-algebras. Suppose that E and F are Finsler modules over A and B, respectively.

A linear map $T : E \to F$ is called a φ-module map if

$$T(a.x) = \varphi(a).T(x), \quad \forall a \in A, x \in E.$$

A φ-module map $T : E \to F$ is called a φ-homomorphism if

$$\rho_B(T(x)) = \varphi(\rho_A(x)), \quad \forall a \in A, x \in E.$$

Following [5], a linear map $T : E \to F$ is said to be a unitary operator if there exists an injective homomorphism $\varphi : A \to B$ of C^*-algebras such that T is a surjective φ-homomorphism.

Remark 2.4. Let $\varphi : A \to B$ be an injective homomorphism of C^*-algebras and let $T : E \to F$ be a φ-homomorphism. It is known from [5] that T is an isometry. Thus, each unitary operator of Finsler modules is an isometry. Moreover, if F is a full Finsler B-module, then φ is surjective and so it is an isomorphism of C^*-algebras. We denote by $U(E)$ the group of all unitary operators from full Finsler module E onto E.

We end this section with introducing an important kind of derivations in the setting of Hilbert C^*-modules.

Definition 2.5. Let E be a Hilbert A-module. A densely defined linear map $\delta : D(\delta) \subseteq E \to E$ is said to be a ternary derivation if $\delta(\langle x, y\rangle .z) = \langle x, y\rangle .\delta(z) + (\langle \delta(x), y\rangle + \langle x, \delta(y)\rangle).z$ for each $x, y, z \in D(\delta)$ where, $D(\delta)$ is a ternary subalgebra of E in the sense that $\langle x, y\rangle .z \in D(\delta)$ for every $x, y, z \in D(\delta)$.

Example 2.6. Let E be a Hilbert A-module, and let $\delta : E \to E$ be an adjointable operator with the adjoint $-\delta$. Then, δ is a ternary derivation.

3. Some Results on Unitary Operators

In the following Theorem, we apply an isomorphism between C^*-algebras A and B, and a bijection between two different Finsler A-modules to construct a Finsler B-module.

Theorem 3.1. *Suppose that E and F are Finsler A-modules, $\varphi : A \to B$ is a linear isomorphism of C^*-algebras and $T : E \to F$ is a bijective linear operator. Define the module action $b.y := T(\varphi^{-1}(b).T^{-1}(y))$ on F. Then, F equipped with $\rho_B : F \to B^+$ defined by $\rho_B(y) := \varphi(\rho_A(T^{-1}(y)))$ $(y \in F)$ can also be regarded as a Finsler B-module. Moreover, if I is an ideal in A and E is a full A-module, then $\frac{F}{F_{\varphi(I)}}$ is a full Finsler $\frac{B}{\varphi(I)}$-module.*

Proof. Trivially, the map $\|.\|_F : y \to \|\rho_B(y)\|$ is a norm on F which makes F into a Banach space and

$$\begin{aligned}\rho_B(b.y)^2 &= \varphi\left(\varphi^{-1}(b)\right).\varphi\left(\rho_A\left(T^{-1}(y)\right)\right)^2.\varphi\left(\varphi^{-1}(b^*)\right)\\ &= b.\rho_B(y)^2.b^*.\end{aligned}$$

Hence, F is a Finsler B-module. Let $b \in B$ and take the unique element $a \in A$ for which $\varphi(a) = b$. It follows from fullness of E that there is a sequence $\{u_n\}$ in $\langle\rho_A(E)\rangle$ such that $u_n \to a$. Thus, $\{\varphi(u_n)\}$ is a sequence in $\langle\rho_B(F)\rangle$ satisfying $\varphi(u_n) \to b$ which means that F is full. The proof is completed by a direct application of Theorem 2.3. □

We call the above alternative Finsler B-module F the Finsler B-module implemented by (φ, T) or briefly the implemented Finsler B-module and denote it by $F^{(\varphi,T)}$. Trivially, the map T in the implemented Finsler B-module $F^{(\varphi,T)}$ is a φ-homomorphism. In fact, T is a unitary operator.

The following theorem demonstrates some conditions under which the range of a unitary operator of Finsler modules is a Hilbert module.

Theorem 3.2. *Let E be a full Finsler A-module under the map ρ_A, let F be a Finsler B-module under the map ρ_B, and let $T : E \to F$ be a unitary operator. If ρ_A fulfills the parallelogram law on E, then, F is a Hilbert B-module via $\langle y, y'\rangle := \frac{1}{4}\sum_{k=0}^{3} i^k \rho_B\left(y + i^k y'\right)^2$.*

Proof. Consider the injective homomorphism $\varphi : A \to B$ which makes T into a surjective φ-homomorphism. Similar to the proof of the previous theorem, it follows from fullness of E that F is a full Finsler B-module. Since, ρ_A fulfills the parallelogram law on E, E is a Hilbert A-module by Theorem 2.2. For every $y, y' \in F$, there exist $x, x' \in E$ such that $T(x) = y$ and $T(x') = y'$. Therefore,

$$\begin{aligned}
\rho_B(y+y')^2+\rho_B(y-y')^2 &= \varphi(\rho_A(x+x'))^2+\varphi(\rho_A(x-x'))^2\\
&= \varphi(\rho_A(x+x')^2)+\varphi(\rho_A(x-x')^2)\\
&= \varphi(\rho_A(x+x')^2+\rho_A(x-x')^2)\\
&= \varphi(2\rho_A(x)^2+2\rho_A(x')^2)\\
&= 2\varphi(\rho_A(x))^2+2\varphi(\rho_A(x'))^2\\
&= 2\rho_B(T(x))^2+2\rho_B(T(x'))^2\\
&= 2\rho_B(y)^2+2\rho_B(y')^2.
\end{aligned}$$

Thus, ρ_B satisfies the parallelogram law on F. Using Theorem 2.2 once more, we conclude that F equipped with the inner product $\langle y, y'\rangle := \frac{1}{4}\sum_{k=0}^{3} i^k \rho_B\left(y+i^k y'\right)^2$ is a Hilbert B-module. □

Substituting the implemented Finsler B-module $F^{(\varphi,T)}$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *Let E be a full Finsler A-module under the map ρ_A such that ρ_A fulfills the parallelogram law on E. Then, the implemented Finsler B-module $F^{(\varphi,T)}$ is a Hilbert B-module via*

$$\langle y, y'\rangle := \frac{1}{4}\sum_{k=0}^{3} i^k \rho_B\left(y+i^k y'\right)^2.$$

We are ready to establish the converse of Theorem 3.1.

Theorem 3.4. *Let E be a full Finsler A-module, let F be a Finsler B-module, and let $T : E \to F$ be a bijective linear operator. If there exists a map $\varphi : A \to B$ such that $a.x = T^{-1}(\varphi(a).T(x))$ and $\varphi(\rho_A(x)) = \rho_B(T(x))$ $(a \in A,\ x \in E)$, then φ is a $*$-isomorphism of C^*-algebras if and only if F is full.*

Proof. The proof is similar to that of [4, Main Theorem]. □

The next result is an immediate consequence of the preceding theorem and the final assertion of Theorem 2.3.

Corollary 3.5. *Let E be a full Finsler A-module and let F be a Finsler B-module. If $\varphi : A \to B$ is a $*$-isomorphism and and $T : E \to F$ be a surjective φ-homomorphism, then $\frac{F}{F_{\varphi(I)}}$ is a full Finsler $\frac{B}{\varphi(I)}$-module.*

Applying Theorem 3.4, we now present a necessary and sufficient condition for a linear map between two full Finsler modules to be a unitary operator.

Theorem 3.6. *Let E be a full Finsler A-module, let F be a full Finsler B-module, and let $T : E \to F$ be a linear operator. Then, T is a unitary operator if and only if T is bijective and there exists a map $\varphi : A \to B$ such that $T(a.x) = \varphi(a).T(x)$ and $\varphi(\rho_A(x)) = \rho_B(T(x))$ for all $a \in A$ and $x \in E$.*

Proof. Suppose that F is a full Finsler B-module. If T is bijective and there exists a map $\varphi : A \to B$ such that $T(a.x) = \varphi(a).T(x)$ and $\varphi(\rho_A(x)) = \rho_B(T(x))$ for all $a \in A$ and $x \in E$, then by Theorem 3.4, φ is a $*$-isomorphism and therefore, T is a surjective φ-homomorphism. The converse is evident. □

Before we state another characterization theorem for unitary operators of Finsler modules, we need the following useful lemma which can be found in [14].

Lemma 3.7. *Suppose that b and c are positive elements of C^*-algebra $\mathcal{A}$ such that $\|ac\| = \|ab\|$ for all $a \in A$. Then $c = b$.*

Theorem 3.8. *Let E be a full Finsler A-module, let F be a full Finsler B-module, and let $\varphi : A \to B$ be a linear $*$-isomorphism of C^*-algebras. Then, a surjective φ-module map $T : E \to F$ is a unitary operator if and only if T is an isometry.*

Proof. By assumption, $\varphi : A \to B$ is an isomorphism of C^*-algebras. Suppose that T, as a surjective φ-module map, is an isometry. We have to show that T is a φ-homomorphism. Take $b \in B$. Then, there exists $a \in A$ such that $\varphi(a) = b$. So, for each $x \in E$ we have

$$\begin{aligned}
\|b.\rho_B(T(x))\|^2 &= \|(b.\rho_B(T(x))).(b.\rho_B(T(x)))^*\| \\
&= \|b.\rho_B(T(x))^2.b^*\| \\
&= \|\varphi(a).\rho_B(T(x))^2.\varphi(a)^*\| \\
&= \|\rho_B(\varphi(a).T(x))^2\| \\
&= \|\rho_B(T(a.x))^2\| \\
&= \|\rho_B(T(a.x))\|^2 \\
&= \|T(a.x)\|^2 \\
&= \|a.x\|^2 \\
&= \|\rho_A(a.x)\|^2 \\
&= \|\rho_A(a.x)^2\| \\
&= \|\varphi(a.\rho_A(x)^2.a^*)\| \\
&= \|b.\varphi(\rho_A(x))^2.b^*\|
\end{aligned}$$

$$= \|(b.\varphi(\rho_A(x))).(b.\varphi(\rho_A(x)))^*\||$$
$$= \|b.\varphi(\rho_A(x))\|^2.$$

By the previous lemma, we conclude that $\rho_B(T(x)) = \varphi(\rho_A(x))$ for every $x \in E$. The converse is obvious. □

Definition 3.9. Two Finsler modules E and F are said to be unitary equivalent if there is a unitary operator from E to F.

Applying Theorem 3.1 and Remark 2.4, we have the following result.

Corollary 3.10. *The Finsler A-module E and the implemented Finsler B-module $F^{(\varphi,T)}$ are unitary equivalent. Conversely, if E and F are full unitary equivalent Finsler modules over C^*-algebras A and B, respectively, then A and B are isomorphic C^*-algebras.*

Theorem 3.11. *Let I be in ideal in A, let E be a Finsler A-module, and let $F^{(\varphi,T)}$ be the implemented Finsler B-module. Then, the canonical Finsler $\frac{A}{I}$-module $\frac{E}{E_I}$ and the implemented canonical Finsler $\frac{B}{\varphi(I)}$-module $\frac{F}{F_{\varphi(I)}}$ are unitary equivalent.*

Proof. Denote by $\pi' : B \to \frac{B}{\varphi(I)}$ and $q' : F \to \frac{F}{F_{\varphi(I)}}$ the quotient maps. Define $\widehat{\varphi} : \frac{A}{I} \to \frac{B}{\varphi(I)}$ and $\widehat{T} : \frac{E}{E_I} \to \frac{F}{F_{\varphi(I)}}$ by $\widehat{\varphi}(\pi(a)) := \pi'(\varphi(a))$ and $\widehat{T}(q(x)) := q'(T(x))$, respectively. It is easy to observe that $\widehat{\varphi}$ is an injective homomorphism of C^*-algebras and $\widehat{T}$ is a surjective $\widehat{\varphi}$-homomorphism. □

Proposition 3.12. *Unitary equivalence in the set of full Finsler modules is an equivalence relation.*

Proof. Let E be a full Finsler A-module. Then, i_A (the identity operator on A) is an injective homomorphism and i_E (the identity operator on E) is a surjective i_A-homomorphism. Therefore, the relation is reflexive. To show that the relation is symmetric, let $T : E \to F$ be a unitary operator. Hence, there exists an injective homomorphism $\varphi : A \to B$ such that T is a surjective φ-homomorphism. From Remark 2.4, it follows that T is an isometry and φ is an isomorphism of C^*-algebras. It is not difficult to observe that T^{-1} is a unitary operator from F onto E.

Finally, assume that G to be a Finsler module over a C^*-algebra C. If $T : E \to F$ and $S : F \to G$ are unitary operators, then trivially, ST is a unitary operator from E onto G and consequently, the relation is transitive . □

The following proposition establishes a one to one correspondence between unitary operators groups of possibly different full Finsler modules.

Proposition 3.13. *Let E and F be full Finsler modules over C^*-algebras A and B, respectively. Suppose that $\varphi : A \to B$ is an injective homomorphism and $T : E \to F$ is a surjective φ-homomorphism. Then, the map $\Phi : U(E) \to U(F)$ defined by $\Phi(S) = TST^{-1}$ is a group isomorphism. Furthermore, if ρ_A fulfills the parallelogram law on E, then, there is an isomorphism $\phi : A \to A$ such that TST^{-1} is an adjointable operator (mod $\varphi\phi\varphi^{-1}$) in the sense that $\langle TST^{-1}(y_1), y_2\rangle = \varphi\phi\varphi^{-1}\left(\langle y_1, TS^{-1}T^{-1}(y_2)\rangle\right)$ for each $y_1, y_2 \in F$.*

Proof. First, note that φ is an isomorphism and T is a bijective φ-homomorphism by Remark 2.4. It is easy to check that Φ is a group isomorphism. If ρ_A fulfills the parallelogram law, then Theorem 2.2 admits an inner product on E which turns E into a Hilbert A-module. Also, it follows from Theorem 3.2 that F is a Hilbert B-module. Let $x' \in E$, and $y \in F$. Putting $T(x') = y'$ and $T^{-1}(y) = x$, we have

$$\begin{aligned}
4\left\langle y, T(x')\right\rangle &= 4\left\langle y, y'\right\rangle \\
&= \sum_{k=0}^{3} i^k \rho_B\left(y + i^k y'\right)^2 \\
&= \sum_{k=0}^{3} i^k \varphi\left(\rho_A\left(x + i^k x'\right)\right)^2 \\
&= \sum_{k=0}^{3} i^k \varphi\left(\rho_A\left(x + i^k x'\right)^2\right) \\
&= \varphi\left(\sum_{k=0}^{3} i^k \rho_A\left(x + i^k x'\right)^2\right) \\
&= \varphi\left(4\left\langle x, x'\right\rangle\right) \\
&= 4\varphi\left(\left\langle x, x'\right\rangle\right) \\
&= 4\varphi\left(\left\langle T^{-1}(y), x'\right\rangle\right).
\end{aligned}$$

This relation implies that, $\langle T^{-1}(y'), x\rangle = \varphi^{-1}\left(\langle y', T(x)\rangle\right)$ for each $x' \in E$, and $y \in F$. We apply this fact to prove the adjointability of TST^{-1} (mod $\varphi\phi\varphi^{-1}$). Let $S \in U(E)$. Remark 2.4 ensures that there exists an isomorphism $\phi : A \to A$ such that S is a bijective ϕ-homomorphism. Additionally, a direct calculation, similar as stated for T, shows that $\langle S(x_1), x_2\rangle = \phi(\langle x_1, S^{-1}(x_2)\rangle)$ for all $x_1, x_2 \in E$. Assume that y_1 and y_2 to be arbitrary elements of F. Taking $x_1 = T^{-1}(y_1)$, $x' = S(x_1)$ and $x_2 = T^{-1}(y_2)$, we have

$$\langle TST^{-1}(y_1), y_2\rangle = \langle TS(x_1), y_2\rangle$$

$$
\begin{aligned}
&= \left\langle T(x'), y_2 \right\rangle \\
&= \left\langle y_2, T(x') \right\rangle^* \\
&= \varphi\left(\left\langle T^{-1}(y_2), x' \right\rangle\right)^* \\
&= \varphi\left(\left\langle T^{-1}(y_2), x' \right\rangle^*\right) \\
&= \varphi\left(\left\langle x', T^{-1}(y_2) \right\rangle\right) \\
&= \varphi\left(\langle S(x_1), x_2 \rangle\right) \\
&= \varphi\phi\left(\left\langle x_1, S^{-1}(x_2) \right\rangle\right) \\
&= \varphi\phi\left(\left\langle T^{-1}(y_1), S^{-1}T^{-1}(y_2) \right\rangle\right) \\
&= \varphi\phi\varphi^{-1}\left(\left\langle y_1, T\left(S^{-1}\left(T^{-1}(y_2)\right)\right) \right\rangle\right). \qquad \square
\end{aligned}
$$

A discussion similar to what was stated in the proof of the preceding theorem, will lead to the following result for the group of unitary operators on the quotient of a Finsler modules over their associated ideal submodules.

Corollary 3.14. *Let I be in ideal in A, let E be a Finsler A-module, and let $F^{(\varphi,T)}$ be the implemented Finsler B-module. Then, the group of unitary operators on the canonical Finsler $\frac{A}{I}$-module $\frac{E}{E_I}$ and the group of the implemented canonical Finsler $\frac{B}{\varphi(I)}$-module $\frac{F}{F_{\varphi(I)}}$ are isomorphic.*

4. Dynamical Systems Implemented by Isomorphic Groups of Unitaries

Our scope in this section, is obtaining a suitable dynamical system under the frame of Finsler modules, which covers the available extensions. To achieve this goal, first we base our discussion on the concept of regularized one parameter groups as the desired extension of ordinary one parameter groups. We represent the definition of a regularized C^*-dynamical system as follows.

Definition 4.1. Let c' be an automorphism on a C^*-algebra A. A regularized C^*-dynamical system is one parameter family $\{\alpha'_t\}_{t\in\mathbb{R}}$ of linear $*$-automorphisms on A such that $\alpha'_0 = c'$ and $c'\alpha'_{t+s} = \alpha'_t\alpha'_s$ for every $t, s \in \mathbb{R}$, and $\lim\limits_{t\to 0} \alpha'_t(a) = c'(a)$ for each $a \in \mathcal{A}$.

We define the infinitesimal generator d of a regularized C^*-dynamical system $\{\alpha'_t\}_{t\in\mathbb{R}}$ as a mapping $d : D(d) \subseteq A \to A$ such that $d(a) = c'^{-1} \lim\limits_{t\to 0} \dfrac{\alpha'_t(a) - c'(a)}{t}$ where $D(d) = \left\{ a \in A : \lim\limits_{t\to 0} \dfrac{\alpha'_t(a) - c'(a)}{t} \text{ exists} \right\}$.

In the case when c' is the identity operator on A, then a regularized C^*-dynamical system $\{\alpha'_t\}_{t\in\mathbb{R}}$ is nothing but a classical C^*-dynamical system.

Remark 4.2. If $\{\alpha'_t\}_{t\in\mathbb{R}}$ is a regularized C^*-dynamical system with the infinitesimal generator d, then it is easily seen that

(i) $\alpha'_s\alpha'_t = \alpha'_t\alpha'_s$ for every $t, s \in \mathbb{R}$.
(ii) $c'\alpha'_t = \alpha'_t c'$ and hence, $c'^{-1}\alpha'_t = \alpha'_t c'^{-1}$ for every $t, s \in \mathbb{R}$.
(iii) $c'(d(a)) = d(c'(a))$ and $c'^{-1}(d(a)) = d\left(c'^{-1}(a)\right)$ for each $a \in D(d)$.
(iv) To each regularized C^*-dynamical system $\{\alpha'_t\}_{t\in\mathbb{R}}$ with the infinitesimal generator d one can associate a C^*-dynamical system $\{\alpha'_t c'^{-1}\}_{t\in\mathbb{R}}$ on A. One easily verifies that the infinitesimal generators of $\{\alpha'_t\}_{t\in\mathbb{R}}$ and $\{\alpha'_t c'^{-1}\}_{t\in\mathbb{R}}$ have the same domain. Moreover, d is the infinitesimal generator of $\{\alpha'_t c'^{-1}\}_{t\in\mathbb{R}}$.

Definition 4.3. A linear mapping δ from a dense subspace $D(\delta)$ of a full Finsler A-module E into E is called a generalized derivation if there exists a mapping d from a dense subalgebra $D(d)$ of A into A for which $D(\delta)$ is an algebraic left $D(d)$-module, and $\delta(a.x) = a.\delta(x) + d(a).x$ for each $a \in D(d)$, and $x \in D(\delta)$.

The method has been used in [2] shows that d is a derivation. For convenience, we say that such a generalized derivation δ is a d-derivation.

Example 4.4. Let E be a full Hilbert A-module and $\delta : D(\delta) \subseteq E \to E$ be a ternary derivation. Then, E together with $\rho_A(x) = \langle x, x\rangle^{\frac{1}{2}}$ is a full Finsler A-module. Additionally, it has been proved in [3, Theorem 3.5] that there exists a unique derivation d of A such that d turns δ into a d-derivation. Hence, δ is a generalized derivation.

Attaching unitary operators to regularized one parameter groups, we obtain regularized dynamical systems in the setting of Finsler modules as follows.

Definition 4.5. Let c be a unitary operator on a full Finsler A-module E. A regularized dynamical system is a mapping $t \to \alpha_t$ from the additive group $\mathbb{R}$ into the group $U(E)$ of unitary operators on E for which $\alpha_0 = c$ and $c\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$, and $\lim_{t\to 0}\alpha_t(x) = c(x)$ for each $x \in E$.

We define the *infinitesimal generator* δ of a regularized dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq E \to E$ such that $\delta(x) = c^{-1}\lim_{t\to 0}\dfrac{\alpha_t(x) - c(x)}{t}$ where $D(\delta) = \left\{x \in E : \lim_{t\to 0}\dfrac{\alpha_t(x) - c(x)}{t} \text{ exists}\right\}$.

Before we state the next theorem, we need the following useful lemma which can be found in [4].

Lemma 4.6. *Let E be a full Finsler module over C^*-algebra A and $a \in A$. Then, $a.z = 0$ for all $z \in E$ iff $a = 0$.*

The following theorem shows that generalized derivations of Finsler modules are appeared as the infinitesimal generator of regularized dynamical systems.

Theorem 4.7. *Let E be a full Finsler A-module and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a regularized dynamical system on E with the infinitesimal generator δ. Then, $D(\delta)$ is a dense subspace of E and there exists a derivation $d : D(d) \subseteq A \to A$ such that $D(\delta)$ is a left $D(d)$-module and d turns δ into a d-derivation.*

Proof. Consider the corresponding one parameter group $\left\{\alpha_t c^{-1}\right\}_{t\in\mathbb{R}}$ of unitary operators on E. Similar as stated in Remark 4.2, the infinitesimal generators of $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\left\{\alpha_t c^{-1}\right\}_{t\in\mathbb{R}}$ have the same domain. So, by Hille-Yosida theorem [25], $D(\delta)$ is a dense subspace of E. Since $\{\alpha_t\}_{t\in\mathbb{R}}$ is a regularized dynamical system, there exists a unitary operator c from E onto E such that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a c-one parameter group. On the other hand, by Remark 2.4, there exists a $*$-automorphism c' on A such that c is a bijective c'-module map. A similar argument for $\{\alpha_t\}_{t\in\mathbb{R}}$ shows that for each $t \in \mathbb{R}$ there exists a $*$-automorphism ${\alpha'}_t$ on A such that $\alpha_t(a.x) = {\alpha'}_t(a).\alpha_t(x)$ $(a \in A, x \in E)$.

We show that $\{{\alpha'}_t\}_{t\in\mathbb{R}}$ is a regularized C^*-dynamical system. For this aim, let $a \in A, z \in E$. Since c is bijective, there exists $x \in E$ such that $c(x) = z$. We have $c'(a).c(x) = c(a.x) = \alpha_0(a.x) = {\alpha'}_0(a).c(x)$. Thus, by the previous lemma $\alpha'_0(a) = c'(a)$ for all $a \in A$. Therefore, $\alpha'_0 = c'$. To justify regularized group property note that for all $t, s \in \mathbb{R}$ we have

$$\begin{aligned} c'\left({\alpha'}_{t+s}(a)\right).c\left(\alpha_{t+s}(x)\right) &= c\left({\alpha'}_{t+s}(a).\alpha_{t+s}(x)\right) \\ &= c\left(\alpha_{t+s}(a.x)\right) \\ &= \alpha_t(\alpha_s(a.x)) \\ &= \alpha_t({\alpha'}_s(a).\alpha_s(x)) \\ &= {\alpha'}_t({\alpha'}_s(a)).\alpha_t(\alpha_s(x)) \\ &= {\alpha'}_t\left({\alpha'}_s(a)\right).c(\alpha_{t+s}(x)), \end{aligned}$$

and so, $c'\left({\alpha'}_{t+s}(a)\right) = {\alpha'}_t\left({\alpha'}_s(a)\right)$. Thus, $c'\alpha_{t+s} = {\alpha'}_t{\alpha'}_s$.

It follows from strong continuity of $\{{\alpha'}_t\}_{t\in\mathbb{R}}$ that

$$\begin{aligned} \left\|{\alpha'}_t(a).c(x) - c'(a).c(x)\right\| &\leq \left\|\alpha'_t(a).c(x) - \alpha'_t(a).\alpha_t(x)\right\| \\ &\quad + \left\|\alpha'_t(a).\alpha_t(x) - c(a.x)\right\|. \end{aligned}$$

Therefore, $\lim\limits_{t\to 0} \alpha'_t(a).c(x) = c'(a).c(x)$ for all $a \in A$, $x \in E$. Hence, $\lim\limits_{t\to 0} {\alpha'}_t(a) = c'(a)$ for all $a \in A$. Consequently, $\{{\alpha'}_t\}_{t\in\mathbb{R}}$ is a regularized C^*-dynamical system on A.

Let d be the infinitesimal generator of $\{\alpha'{}_t\}_{t\in\mathbb{R}}$. Then, for each $a \in D(d)$, and each $x \in D(\delta)$ we have

$$\begin{aligned} c^{-1}\left(\lim_{t\to 0}\frac{\alpha_t(a.x)-c(a.x)}{t}\right) &= c^{-1}\left(\lim_{t\to 0}\frac{c'(a).\alpha_t(x)-c(a.x)}{t}\right) \\ &\quad + c^{-1}\left(\lim_{t\to 0}\frac{\alpha'{}_t(a).\alpha_t(x)-c'(a).\alpha_t(x)}{t}\right) \\ &= c^{-1}\left(c'(a).\lim_{t\to 0}\frac{\alpha_t(x)-c(x)}{t}\right) \\ &\quad + c^{-1}\left(\lim_{t\to 0}\frac{\alpha'_t(a)-c'(a)}{t}.\alpha_t(x)\right) \\ &= c'^{-1}(c'(a)).c^{-1}(\lim_{t\to 0}\frac{\alpha_t(x)-c(x)}{t}) \\ &\quad + c'^{-1}\left(\lim_{t\to 0}\frac{\alpha'_t(a)-c'(a)}{t}\right).c^{-1}\left(\lim_{t\to 0}\alpha_t(x)\right) \\ &= a.\delta(x)+d(a).x, \end{aligned}$$

which means that, $a.x \in D(\delta)$, and $D(\delta)$ is a left $D(d)$-module. Furthermore, $\delta(a.x) = a.\delta(x) + d(a).x$ for all $a \in D(d)$, and $x \in D(\delta)$. $\square$

Remark 4.8. Let E be a full Finsler A-module and $\{\alpha_t\}_{t\in\mathbb{R}}$ be a regularized dynamical system on E. As in the preceding proof, there exists a regularized C^*-dynamical system $\{\alpha'_t\}_{t\in\mathbb{R}}$ such that α'_t turns α_t into a α''_t-module map ($t \in \mathbb{R}$). Let $t \in \mathbb{R}$ and $z \in E$. So, there exists a unique element $x \in E$ such that $\alpha_t(x) = z$. Suppose that $\{\alpha''_t\}_{t\in\mathbb{R}}$ is also a regularized C^*-dynamical system such that α''_t turns α_t into a α''_t-module map ($t \in \mathbb{R}$). The equality $\alpha''_t(a).z = \alpha_t(a.x) = \alpha'_t(a).z$ ($a \in A$) together with Lemma 4.6 imply that $\alpha'_t = \alpha''_t$. This means that the regularized C^*-dynamical system $\{\alpha'_t\}_{t\in\mathbb{R}}$ in the proof of the previous theorem is unique.

Now, we prove that each regularized dynamical system on E implements a regularized dynamical system on F by the algebraic isomorphism Φ (introduced in Proposition 3.13).

Theorem 4.9. *Let E and F be full Finsler modules over C^*-algebras A and B, respectively. Suppose that $\varphi : A \to B$ is an injective homomorphism and $T : E \to F$ is a surjective φ-homomorphism. If $\{\alpha_t\}_{t\in\mathbb{R}}$ is a regularized dynamical system on E with the infinitesimal generator δ_1, then one can correspond a regularized dynamical system $\{\beta_t\}_{t\in\mathbb{R}}$ on F and a unique regularized C^*-dynamical system $\{\beta'_t\}_{t\in\mathbb{R}}$ on the C^*-algebra B such that if δ_2 and d_2 are the infinitesimal generators of $\{\beta_t\}_{t\in\mathbb{R}}$ and $\{\beta'_t\}_{t\in\mathbb{R}}$ respectively, then δ_2 is a d_2-derivation.*

Proof. First note that, as stated in the preceding proof, there exist an automorphism c'_1 and a c'_1-one parameter group $\{\alpha'_t\}_{t\in\mathbb{R}}$ on A such that $\{\alpha'_t\}_{t\in\mathbb{R}}$ is a regularized C^*-dynamical system and α'_t turns α_t into a α'_t-homomorphism ($t \in \mathbb{R}$). Also, there is a bijective c'_1-homomorphism c_1 from E onto E such that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a c_1-one parameter group. Consider an injective homomorphism between C^*-algebras A and B, and a surjective φ-homomorphism T from E onto F. It follows from Remark 2.4 that φ is an isomorphism between C^*-algebras A and B and a unitary map T establishes a unitary equivalence between E and F. We apply Proposition 3.13 to define a one parameter family $\{\beta_t\}_{t\in\mathbb{R}}$ on Finsler module F. For this aim, let $b \in B$ and $y \in F$. So, there are unique elements $a \in A$, and $x \in E$ such that $\varphi(a) = b$, and $T(x) = y$. Now, define $c'_2(\varphi(a)) := \varphi(c'_1(a))$, $\beta'_t(\varphi(a)) := \varphi(\alpha'_t(a))$, $c_2(T(x)) := T(c_1(x))$, and $\beta_t(T(x)) := T(\alpha_t(x))$. In fact, $c'_2 = \varphi c_1 \varphi^{-1}$, $\beta'_t = \varphi\alpha'_t\varphi^{-1}$, $c_2 = Tc_1T^{-1}$, and $\beta_t = T\alpha_tT^{-1}$ (in the other words, $\beta_t = \Phi(\alpha_t)$). Trivially, c'_2 is an automorphism on B and c_2 is a bijective c'_1-homomorphism. Also, β'_t is an automorphism on B that turns the bijective operator β_t into a β'_t-homomorphism ($t \in \mathbb{R}$). We have $\beta_0 = c_2$ and for all $t, s \in \mathbb{R}$ we have

$$\begin{aligned}\beta_t\beta_s &= \beta_t T\alpha_s T^{-1}\\ &= T\alpha_t T^{-1}T\alpha_s T^{-1}\\ &= Tc_1\alpha_{t+s}T^{-1}\\ &= c_2T\alpha_{t+s}T^{-1}\\ &= c_2\beta_{t+s}.\end{aligned}$$

It follows from strong continuity of $\{\alpha_t\}_{t\in\mathbb{R}}$ that

$$\begin{aligned}\|\beta_t(T(x)) - c_2(T(x))\| &= \|T(\alpha_t(x)) - T(c_1(x))\|\\ &\le \|T\|.\|\alpha_t(x) - c_1(x)\| \to 0,\end{aligned}$$

which means that $\{\beta_t\}_{t\in\mathbb{R}}$ is a regularized dynamical system. In fact, $\{\beta_t\}_{t\in\mathbb{R}}$ is a strongly continuous c_2-one parameter group of unitary operators on F. The method has been used in the proof of Theorem 4.7 and the previous remark show that $\{\beta'_t\}_{t\in\mathbb{R}}$ is the unique regularized C^*-dynamical system on B associated with $\{\beta_t\}_{t\in\mathbb{R}}$. Finally, suppose that δ_2 and d_2 are the infinitesimal generators of $\{\beta_t\}_{t\in\mathbb{R}}$ and $\{\beta'_t\}_{t\in\mathbb{R}}$, respectively. An easy calculation shows that $D(d_2) = \varphi\left(D(d_1)\right)$ and $D(\delta_2) = T\left(D(\delta_1)\right)$. Furthermore, $d_2(\varphi(a)) = \varphi(d_1(a))$ and $\delta_2(T(x)) = T(\delta_1(x))$ for every $a \in D(d_1)$, $x \in D(\delta_1)$. Hence,

$$\begin{aligned}\delta_2(\varphi(a).T(x)) &= \delta_2(T(a.x))\\ &= T(\delta_1(a.x))\\ &= T\left(a.\delta_1(x) + d_1(a).x\right)\end{aligned}$$

$$= \varphi(a).T(\delta_1(x)) + \varphi(d_1(a)).T(x)$$
$$= \varphi(a).\delta_2(T(x)) + d_2(\varphi(a)).T(x).$$

Similarly, one can verify that

$$d_2(\varphi(a).\varphi(c)) = d_2(\varphi(a)).\varphi(c) + \varphi(a).d_2(\varphi(c)),$$

and consequently, δ_2 is a d_2-derivation. □

Corollary 4.10. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a regularized dynamical system on E and δ_1 be its infinitesimal generator. Then, $\{\alpha_t\}_{t\in\mathbb{R}}$ implements the regularized dynamical system $\{\Phi(\alpha_t)\}_{t\in\mathbb{R}}$ on the implemented Finsler module $F^{(\varphi,T)}$. Consequently, there exists a unique derivation d_2 of C^*-algebra B such that if δ_2 is the infinitesimal generators of $\{\Phi(\alpha_t)\}_{t\in\mathbb{R}}$, then δ_2 is a d_2-derivation.*

We end the paper with the following corollary on quotients.

Corollary 4.11. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a regularized dynamical system on $\frac{E}{E_I}$ and δ_1 be its infinitesimal generator. Then, one can correspond a regularized dynamical system $\{\beta_t\}_{t\in\mathbb{R}}$ on the implemented canonical Finsler module $\frac{F}{F_{\varphi(I)}}$ and a unique regularized C^*-dynamical system $\{\beta'_t\}_{t\in\mathbb{R}}$ on the C^*-algebra $\frac{B}{\varphi(I)}$ such that if δ_2 and d_2 are the infinitesimal generators of $\{\beta_t\}_{t\in\mathbb{R}}$ and $\{\beta'_t\}_{t\in\mathbb{R}}$ respectively, then δ_2 is a d_2-derivation.*

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