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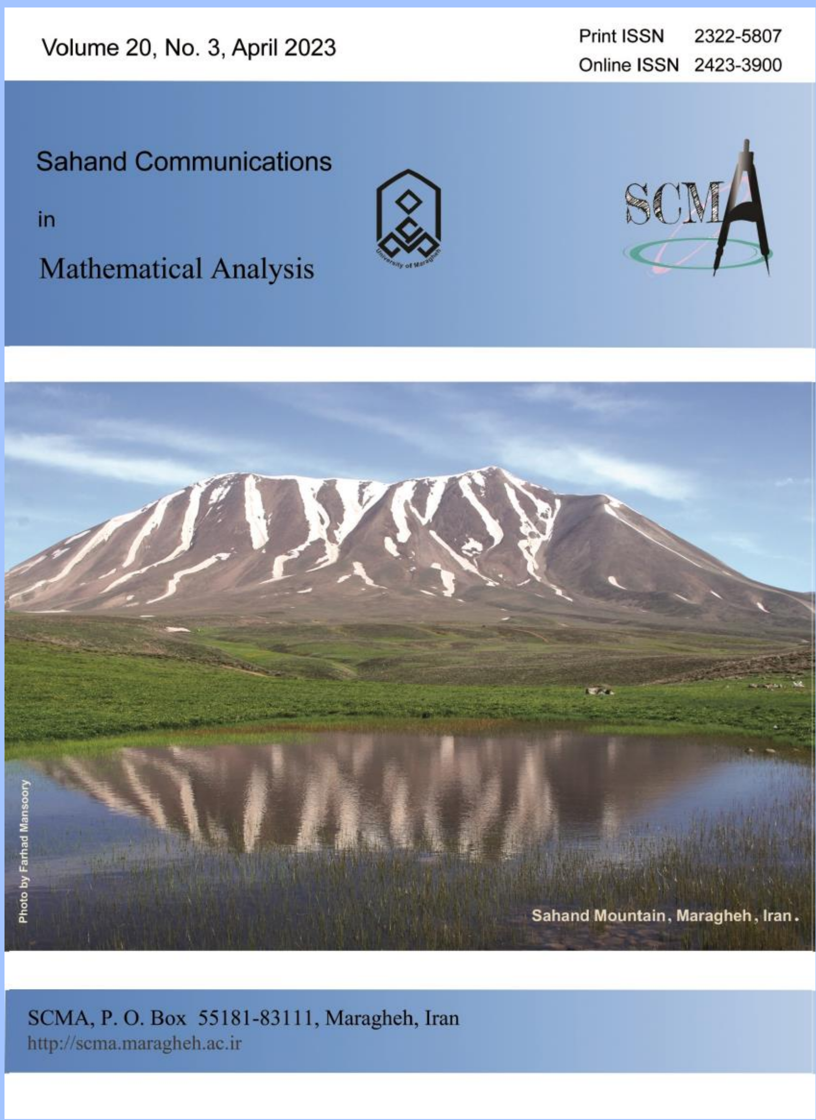

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Coefficient Bounds for a Family of Analytic Functions Linked with a Petal-Shaped Domain and Applications to Borel Distribution

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Abstract. In this paper, by employing sine hyperbolic inverse functions, we introduced the generalized subfamily $\mathcal{RK}_{\sinh}(\beta)$ of analytic functions defined on the open unit disk $\Delta := \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$ associated with the petal-shaped domain. The bounds of the first three Taylor-Maclaurin's coefficients, Fekete-Szegö functional and the second Hankel determinants are investigated for $f \in \mathcal{RK}_{\sinh}(\beta)$. We considered Borel distribution as an application to our main results. Consequently, a number of corollaries have been made based on our results, generalizing previous studies in this direction.

1. Introduction and Motivation

Let $\mathcal{A}$ represent the family of holomorphic functions $f(\xi)$ defined in the domain of an open unit disk $\Delta := \{\xi \in \mathbb{C} : |\xi| < 1\}$. Then the function $f(\xi)$ can have a Taylor-Maclaurin's series as:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad (\xi \in \Delta). \tag{1.1}$$

The subclass of $\mathcal{A}$ consists of all normalized univalent functions in Δ is denoted by $\mathcal{S}$. Let $f,\ g \in \mathcal{A}$. We say the function f is subordinate to g

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or g is superordinate to f, written as $f \prec g$ (see [17]) if there exists a Schwarz function $\omega(\xi)$ with $\omega(0) = 0$ and $|w(z)| < 1$ such that

$$f(\xi) = g(\omega(\xi)), \quad (\xi \in \Delta).$$

The subfamilies of the class $\mathcal{S}$ that play a dominant role in geometric function theory are the families of starlike functions ($\mathcal{S}^*$), convex functions ($\mathcal{K}$) and bounded turning functions ($\mathcal{R}$) defined in terms of subordination as:

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec \phi(\xi), (\xi \in \Delta) \right\},$$

$$\mathcal{K} := \left\{ f \in \mathcal{A} : 1 + \frac{\xi f''(\xi)}{f'(\xi)} \prec \phi(\xi), (\xi \in \Delta) \right\},$$

and

$$\mathcal{R} := \left\{ f \in \mathcal{A} : f'(\xi) \prec \phi(\xi), (\xi \in \Delta) \right\},$$

where

$$\begin{aligned} \phi(\xi) &= 1 + 2\sum_{n=2}^{\infty} \xi^n \\ &= \frac{1+\xi}{1-\xi}, \qquad (\xi \in \Delta). \end{aligned}$$

Some subclasses of the set $\mathcal{S}$ can be generated by varying the function ϕ. For instance:

- If we take $\phi(\xi) = \frac{1+L\xi}{1+M\xi}$, $(-1 \leq M < L \leq 1)$, we get the class $\mathcal{S}^*(L, M) = \mathcal{S}^*\left(\frac{1+L\xi}{1+M\xi}\right)$, the function of Janowski starlike class studied by Janowski (see [9]).
- Letting $L = 1 - 2\alpha$ and $M = -1$, we get the class $\mathcal{S}^*(\alpha) = \mathcal{S}^*(1 - 2\alpha, -1)$, the familiar starlike functions of order α, $(0 \leq \alpha < 1)$.
- For $\phi(\xi) = \sqrt{1+\xi}$, the family $\mathcal{S}^*_L = \mathcal{S}^*(\phi)$ was studied by Sokòl and Stankiewicz [30]. The function $\phi(\xi)$ maps the region Δ onto the image domain which is bounded by $|\omega^2 - 1| < 1$.
- For $\phi(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2$, the class $\mathcal{S}^*_c = \mathcal{S}^*(\phi)$ was introduced in [27] and further studied by [28].
- By choosing $\phi(\xi) = e^{\xi}$, the class $\mathcal{S}^*_e = \mathcal{S}^*(\phi)$ was studied in [16] (also, see [29]).
- Taking $\phi(\xi) = \cos\xi$, we get the families $\mathcal{S}^*_{\cos} = \mathcal{S}^*(\phi)$ investigated by Bano and Raza [3].
- Taking $\phi(\xi) = \text{cosh}\xi$, we get the class $\mathcal{S}^*(\phi)$ studied by Alotaibi et al. [1]
- The family $\mathcal{S}^*_{\sin} = \mathcal{S}^*(\phi) = \mathcal{S}^*(1 + \sin\xi)$ was investigated in [6].

- Kumar and Arora [13] introduced the class $\mathcal{S}^*_\phi = \mathcal{S}^*(\phi)$ where $\phi(\xi) = 1 + \sinh^{-1}\xi$.

Note that, the function $\phi(\xi) = 1 + \sinh^{-1}\xi$ is a multivalued function and has the branch cuts about the line segments $(-i\infty, -i) \cup (i, i\infty)$ on the imaginary axis and hence it is analytical in Δ. Geometrically, the function $\phi(\xi)$ maps the unit disk Δ onto a petal-shaped domain Ω_ϕ where

$$\Omega_\phi = \{w \in \mathbb{C} : |\sinh w - 1| < 1\}.$$

Further, recently Barukab et al. [5] obtained the sharp bounds of the Hankel determinant of order three for the function class

$$\mathcal{BT}_s := \left\{f \in \mathcal{A} : f'(\xi) \prec 1 + \sinh^{-1}\xi, (\xi \in \Delta)\right\}.$$

Motivated by the above researchers, we introduce the following subclass of $\mathcal{A}$ as follows:

Definition 1.1. Let $0 \leq \beta \leq 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{RK}_{\sinh}(\beta)$ if it satisfies the following subordination condition:

$$\left[f'(\xi)\right]^\beta \left[\frac{(\xi f'(\xi))'}{f'(\xi)}\right]^{1-\beta} \prec 1 + \sinh^{-1}\xi, \quad (\xi \in \Delta). \tag{1.2}$$

Note that

$$\begin{aligned} RK_{\sinh}(0) &= K_{\sinh} \\ &= \left\{f \in \mathcal{A} : \frac{(\xi f'(\xi))'}{f'(\xi)} \prec 1 + \sinh^{-1}\xi, (\xi \in \Delta)\right\}, \end{aligned}$$

and

$$\begin{aligned} RK_{\sinh}(1) &= \mathcal{BT}_s \\ &= \left\{f \in \mathcal{A} : f'(\xi) \prec 1 + \sinh^{-1}\xi, (\xi \in \Delta)\right\}. \end{aligned}$$

Definition 1.2. For a function $f \in \mathcal{A}$ given by (1.1), Pommerenke [24, 25] stated the kth Hankel determinant as:

$$H_{k,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}, \quad (k, n \in \mathbb{N}, a_1 = 1).$$

In particular, for $k = 2,\ n = 1$ and $k = 2,\ n = 2$, respectively, we have

$$\begin{aligned} H_{2,1}(f) &= \begin{vmatrix} a_1 & a_2 \\ a_2 & a_2 \end{vmatrix} \\ &= a_3 - a_2^2, \end{aligned}$$

and

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

It may be noted that $H_{2,1}(f)$ is popularly known as Fekete-Szegö functional (see [7, 12, 18, 21, 22, 26]).

A significant amount of research papers have been devoted to determining the upper bounds for the second-order Hankel determinant $H_{2,2}(f)$ for different subclasses of $\mathcal{A}$ in the literature. For recent expository works on the second Hankel determinant, see ([4, 10, 11, 14, 19, 20]).

Specifically, we investigate the upper bounds of the coefficient inequality, Fekete-Szegö functional and Hankel determinant of order two for the function class $\mathcal{RK}_{\sinh}(\beta)$ associated with sine hyperbolic inverse function defined in Definition 1.1.

2. A Set of Preliminaries

Let $\mathcal{P}$ denote the class of functions $q(\xi)$ which are holomorphic with a positive real part in the open unit disk Δ and have the following form:

$$q(\xi) = 1 + \sum_{n=1}^{\infty} q_n \xi^n, \quad (\xi \in \Delta). \tag{2.1}$$

We require the following lemmas for our investigation.

Lemma 2.1 ([23]). *If $q \in \mathcal{P}$ and has the form (2.1), then*

$$|q_n| \leq 2, \quad \text{for } n \geq 1, \tag{2.2}$$

$$|q_{n+k} - \delta q_n q_k| \leq \begin{cases} 2, & 0 \leq \delta \leq 1, \\ 2|2\delta - 1|, & \text{elsewhere.} \end{cases}$$

$$|q_n q_m - q_l q_k| \leq 4 \quad \text{for } n + m = l + k \tag{2.3}$$

$$\left|q_{n+2k} - \mu q_n q_k^2\right| \leq 2(1 + 2\mu) \quad \text{for } \mu \in \mathbb{R},$$

and

$$\left|q_2 - \frac{q_1^2}{2}\right| \leq 2 - \frac{|q_1|^2}{2}.$$

Lemma 2.2 (see [15]). *If $q \in \mathcal{P}$ and has the form (2.1), then for any complex number μ, we have*

$$\left|q_2 - \mu q_1^2\right| \leq 2 \max\{1, |2\mu - 1|\}.$$

Lemma 2.3 (see [2]). *Let $q \in \mathcal{P}$ and has of the form (2.1). Then*

$$\left|J q_1^3 - K q_1 q_2 + L q_3\right| \leq 2|J| + 2|K - 2J| + 2|J - K + L|, \tag{2.4}$$

where $J,\ K,\ L \in \mathbb{C}$.

Lemma 2.4 ([15])**.** *If $q \in \mathcal{P}$ is of the form (2.1), then there exists some $x,\ z$ with $|x| \leq 1,\ |z| \leq 1$ such that*

$$2q_2 = q_1^2 + x\left(4 - q_1^2\right),$$
$$4q_3 = q_1^3 + 2q_1 x\left(4 - q_1^2\right) - \left(4 - q_1^2\right) q_1 x^2 + 2\left(4 - q_1^2\right)\left(1 - |x|^2\right) z.$$

Lemma 2.5 ([15])**.** *If $q \in \mathcal{P}$ is of the form (2.1), then*

$$|q_2 - \nu q_1^2| \leq \begin{cases} -4\nu + 2 & \nu \leq 0, \\ 2 & 0 \leq \nu \leq 1, \\ 4\nu - 2 & \nu \geq 1. \end{cases}$$

3. Coefficient Estimates and Fekete-Szegö Functional

In the first theorem, we determine the bounds of the first three Taylor-Maclaurin's coefficients for the function class $\mathcal{RK}_{\sinh}(\beta)$.

Theorem 3.1. *Let the function $f \in \mathcal{A}$ of the form (1.1) be in the class $\mathcal{RK}_{\sinh}(\beta)$. Then*

$$|a_2| \leq \frac{1}{2}, \tag{3.1}$$
$$|a_3| \leq \frac{1}{3(2-\beta)},$$
$$|a_4| \leq \frac{1}{96(2-\beta)(3-2\beta)}\left[\left|12\beta^2 - 5\beta - 2\right| + \left|24\beta^2 - 22\beta - 16\right| + \left(12\beta^2 - 41\beta + 34\right)\right].$$

Proof. Let the function f given by (1.1) be in the class $\mathcal{RK}_{\sinh}(\beta)$. According to Definition 1.1, there exists an analytical function $\omega(\xi)$ satisfying the condition of the Schwarz lemma such that

$$[f'(\xi)]^{\beta}\left[\frac{(\xi f'(\xi))'}{f'(\xi)}\right]^{1-\beta} = 1 + \sinh^{-1}(\omega(\xi)), \quad (\xi \in \Delta). \tag{3.2}$$

Let $q \in \mathcal{P}$. Then, in terms of the Schwarz function $\omega(\xi)$, we can write

$$q(\xi) = \frac{1+\omega(\xi)}{1-\omega(\xi)} = 1 + q_1\xi + q_2\xi^2 + q_3\xi^3 + \cdots, \tag{3.3}$$

which implies

$$\begin{aligned} \omega(\xi) &= \frac{q(\xi)-1}{q(\xi)+1} \\ &= \frac{q_1\xi + q_2\xi^2 + q_3\xi^3 + \cdots}{2 + q_1\xi + q_2\xi^2 + \cdots} \\ &= \frac{1}{2}\left(q_1\xi + q_2\xi + q_3\xi^3 + \cdots\right)\left(1 + \frac{q_1\xi + q_2\xi^2 + \cdots}{2}\right)^{-1} \end{aligned} \tag{3.4}$$

$$= \frac{q_1}{2}\xi + \left(\frac{q_2}{2} - \frac{q_1^2}{4}\right)\xi^2 + \left(\frac{q_1^2}{8} - \frac{1}{2}q_1q_2 + \frac{q_3}{2}\right)\xi^3 + \cdots.$$

Using relation (3.4) in the series expansion of $sinh^{-1}(\omega(\xi))$, we get

$$1 + \sinh^{-1}(\omega(\xi)) = 1 + \omega(\xi) - \frac{(\omega(\xi))^3}{3!} + \frac{3}{40}(\omega(\xi))^5 - \cdots \tag{3.5}$$
$$= 1 + \frac{q_1}{2}\xi + \left(\frac{q_2}{2} - \frac{q_1^2}{4}\right)\xi^2$$
$$+ \left(\frac{q_3}{2} + \frac{5}{48}q_1^3 - \frac{q_1q_2}{2}\right)\xi^3 + \cdots.$$

From (1.1), it can be easily derived that

$$[f'(\xi)]^\beta \left[\frac{(\xi f'(\xi))'}{f'(\xi)}\right]^{1-\beta} \tag{3.6}$$
$$= 1 + 2a_2\xi + \left[3(2-\beta)a_3 - 4(1-\beta)a_2^2\right]\xi^2$$
$$+ \left[4(3-2\beta)a_4 - 18(1-\beta)a_2a_3 + 8(1-\beta)a_2^3\right]\xi^3 + \cdots.$$

Using (3.5) and (3.6) in (3.2) and then comparing the coefficients of ξ, ξ^2 and ξ^3 on both sides, we get

$$a_2 = \frac{q_1}{4}, \tag{3.7}$$

$$3(2-\beta)a_3 - 4(1-\beta)a_2^2 = \frac{q_2}{2} - \frac{q_1^2}{4}, \tag{3.8}$$

then

$$a_3 = \frac{1}{3(2-\beta)}\left(\frac{q_2}{2} - \frac{\beta}{4}q_1^2\right)$$
$$= \frac{1}{6(2-\beta)}\left(q_2 - \frac{\beta}{2}q_1q_1\right),$$

and

$$4(3-2\beta)a_4 - 18(1-\beta)a_2a_3 + 8(1-\beta)a_2^3 = \frac{q_3}{2} + \frac{5}{48}q_1^3 - \frac{q_1q_2}{2},$$

then

$$a_4 = \frac{1}{4(3-2\beta)}\left[\frac{q_3}{2} + \frac{12\beta^2 - 5\beta - 2}{48(2-\beta)}q_1^3 - \frac{1+\beta}{4(2-\beta)}q_1q_2\right]. \tag{3.9}$$

For a_2, utilizing (2.2) in (3.7), we obtain

$$|a_2| \le \frac{1}{2}.$$

For a_3, applying (1.2) of Lemma 2.1 one will get

$$|a_3| \le \frac{1}{3(2-\beta)}.$$

Taking the modulus on both sides of (3.9) and the application of (2.4) of Lemma 2.3, we get

$$\begin{aligned}|a_4| &\le \frac{1}{2(3-2\beta)}\left[\left|\frac{12\beta^2-5\beta-2}{48(2-\beta)}\right| + \left|\frac{1+\beta}{4(2-\beta)} - \frac{12\beta^2-5\beta-2}{24(2-\beta)}\right|\right.\\ &\quad \left. + \left|\frac{12\beta^2-5\beta-2}{48(2-\beta)} - \frac{1+\beta}{4(2-\beta)} + \frac{1}{2}\right|\right]\\ &= \frac{1}{96(2-\beta)(3-2\beta)}\left[|12\beta^2-5\beta-2| + |24\beta^2-22\beta-16|\right.\\ &\quad \left. + (12\beta^2-41\beta+34)\right].\end{aligned}$$

The proof of Theorem 3.1 is thus completed. □

Taking $\beta = 1$ in Theorem 3.1 we get the following result as a corollary for the class $\mathcal{BT}_s$ due to Barukab (see [5])

Corollary 3.2 ([5, Theorem 4]). *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{BT}_s$. Then*

$$|a_2| \le \frac{1}{2}, \qquad |a_3| \le \frac{1}{3}, \qquad |a_4| \le \frac{1}{4}.$$

Putting $\beta = 0$ we get the result for the class of K_{sinh}

Corollary 3.3. *Let $f \in K_{\text{sinh}}$. Then*

$$|a_2| \le \frac{1}{2}, \qquad |a_3| \le \frac{1}{3}, \qquad |a_4| \le \frac{13}{144}.$$

The next theorem gives a bound for Fekete-Szegö inequality when μ is complex.

Theorem 3.4. *Let the function $f \in \mathcal{A}$ belongs to the class $\mathcal{RK}_{\text{sinh}}(\beta)$. Then, for any complex number ν, we have*

$$|a_3 - \nu a_2^2| \le \frac{1}{3(2-\beta)} \max\left\{1, \left|\frac{3(2-\beta)\nu - 4(1-\beta)}{4}\right|\right\}.$$

Proof. Relations (3.7) and (3.8) yield

$$\begin{aligned}|a_3 - \nu a_2^2| &= \left|\frac{q_2}{6(2-\beta)} - \frac{\beta}{12(2-\beta)}q_1^2 - \nu\frac{q_1^2}{16}\right| \\ &= \frac{1}{6(2-\beta)}|q_2 - \mu q_1^2|\end{aligned} \tag{3.10}$$

where

$$\mu = \frac{4\beta + 3(2-\beta)\mu}{8}.$$

An application of Lemma 2.2 to relation (3.10) gives

$$|a_3 - \nu a_2^2| \le \frac{1}{3(2-\beta)} \max\left\{1, \left|\frac{3(2-\beta)\nu - 4(1-\beta)}{4}\right|\right\}.$$

This completes the proof of Theorem 3.4. □

Taking $\beta = 1$ in the above theorem, we get the result of Barukab et al.(see [5]) as

Corollary 3.5. *Let $f \in \mathcal{BT}_s$. Then for any complex number ν, we have*

$$|a_3 - \nu a_2^2| \leq \frac{1}{3} \max\left\{1, \frac{3}{4}|\nu|\right\}.$$

Letting $\beta = 0$ in Theorem 3.4 we get Fekete-Szegö functional for the class $K_{\sinh}$.

Corollary 3.6. *If $f \in K_{\sinh}$, then for $\nu \in \mathbb{C}$, we have*

$$|a_3 - \nu a_2^2| \leq \frac{1}{6} \max\left\{1, \left|\frac{3\nu - 2}{2}\right|\right\}.$$

Remark 3.7. Letting $\nu = 1$ in Theorem 3.4 we get $|a_3 - a_2^2| \leq \frac{1}{3(2-\beta)}$.

Now, we discuss the result based on Fekete-Szegö functional $|a_3 - \nu a_2^2|$ when ν is real.

Theorem 3.8. *If the function $f \in \mathcal{A}$ belongs to the function class $\mathcal{RK}_{\sinh}(\beta)$, then any real number ν, we have*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{4(1-\beta)-3(2-\beta)\nu}{12(2-\beta)}, & \nu \leq \frac{-4\beta}{3(2-\beta)}, \\ \\ \frac{1}{3(2-\beta)}, & -\frac{4\beta}{3(2-\beta)} \leq \nu \leq \frac{4}{3}, \\ \\ \frac{3(2-\beta)\nu-4(1-\beta)}{12(2-\beta)}, & \nu \geq \frac{4}{3}. \end{cases}$$

Proof. From (3.10), we obtain

$$|a_3 - \nu a_2^2| = \frac{1}{6(2-\beta)}|q_2 - \mu q_1^2|,$$

where $\mu = \frac{4\beta+3(2-\beta)\nu}{8}$. The result was followed by of Lemma 2.5. This proves the result of Theorem 3.8. □

4. Coefficient Inequalities for the Function f^{-1}

Theorem 4.1. *If the function $f \in \mathcal{RK}_{\sinh}(\beta)$ given by (1.1) and $f^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n$ is the analytic continuation to Δ of the inverse function of f with $|w| < r_0$ where $r_0 > \frac{1}{4}$, the radius of the Koebe domain, then for any complex number ν, we have*

$$|l_2| \leq \frac{1}{2},$$

$$|l_3| \leq \frac{4-\beta}{6(2-\beta)},$$

and

$$|l_3 - \nu l_2^2| \leq \frac{1}{3(2-\beta)} \max\left\{1, \left|\frac{2(4-\beta)-3(2-\beta)\nu}{4}\right|\right\}.$$

Proof. Since

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} l_n w^n, \tag{4.1}$$

is the inverse of f, we have

$$f^{-1}(f(\xi)) = f\left(f^{-1}(\xi)\right) = \xi. \tag{4.2}$$

From (4.2), we have

$$f^{-1}\left(\xi + \sum_{n=2}^{\infty} a_n \xi^n\right) = \xi. \tag{4.3}$$

From (4.1) and (4.3), we have

$$\xi + (a_2 + l_2)\xi^2 + (a_3 + 2a_2 l_2 + l_3)\xi^3 + \cdots = \xi. \tag{4.4}$$

Equating the coefficients of ξ^2 and ξ^3 on both sides of (4.4), we get

$$l_2 = -a_2, \tag{4.5}$$

and

$$l_3 = -a_3 - 2a_2 l_2 = 2a_2^2 - a_3. \tag{4.6}$$

Using (3.7), (3.8) in (4.5) and (4.6), we obtain

$$l_2 = -\frac{q_1}{4},$$

and

$$\begin{aligned} l_3 &= \frac{q_1^2}{8} - \frac{q_2}{6(2-\beta)} + \frac{\beta}{12(2-\beta)} q_1^2 \\ &= -\frac{1}{6(2-\beta)}\left(q_2 - \frac{6-\beta}{4} q_1^2\right). \end{aligned}$$

The bound for l_2 can be obtained by using (2.2) of Lemma 2.1. Further, an application of Lemma 2.2 gives

$$\begin{aligned} |l_3| &\leq \frac{1}{3(2-\beta)} \max\left\{1, \left|\frac{4-\beta}{2}\right|\right\} \\ &= \frac{4-\beta}{6(2-\beta)}. \end{aligned}$$

Moreover, for any complex number ν, we have

$$|l_3 - \nu l_2^2| = \frac{1}{6(2-\beta)} \left| q_2 - \frac{2(6-\beta) - 3(2-\beta)\nu}{8} q_1^2 \right|.$$

When Lemma 2.2 is used,

$$|l_3 - \nu l_2^2| \leq \frac{1}{3(2-\beta)} \max\left\{1, \left|\frac{2(4-\beta) - 3(2-\beta)\nu}{4}\right|\right\}.$$

The proof of Theorem 4.1 is complete. □

5. Coefficient Functional Associated with $\frac{\xi}{f(\xi)}$

In this section, we obtain Fekete-Szegö functional estimate related to the function $\frac{\xi}{f(\xi)}$ defined as

$$N(\xi) = \frac{\xi}{f(\xi)} = 1 + \sum_{n=1}^{\infty} k_n \xi^n, \quad (\xi \in \Delta), \tag{5.1}$$

where the function f belongs to the class $\mathcal{RK}_{\sinh}(\beta)$.

Theorem 5.1. *Let $f \in \mathcal{A}$ given by (1.1) be in the class $\mathcal{RK}_{\sinh}(\beta)$ and $N(\xi)$ is given by (5.1). Then for any complex number ν, we have*

$$|k_2 - \nu k_1^2| \leq \frac{1}{3(2-\beta)} \max\left\{1, \left|\frac{(2+\beta) - 3(2-\beta)\nu}{4}\right|\right\}.$$

Proof. It is straightforward to write

$$\begin{aligned} N(\xi) &= \frac{\xi}{f(\xi)} \\ &= 1 - a_2\xi + \left(a_2^2 - a_3\right)\xi^2 + \cdots . \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we obtain

$$k_1 = -a_2, \tag{5.3}$$

and

$$k_2 = a_2^2 - a_3. \tag{5.4}$$

Using (3.7), (3.8) in (5.3) and (5.4), respectively, we get

$$k_1 = -\frac{q_1}{4},$$

and

$$\begin{aligned} k_2 &= \frac{q_1^2}{16} - \left(\frac{q_2}{6(2-\beta)} - \frac{\beta}{12(2-\beta)} q_1^2\right) \\ &= \frac{6+\beta}{48(2-\beta)} q_1^2 - \frac{q_2}{6(2-\beta)}. \end{aligned}$$

Thus, for any complex number ν, we have

$$|k_2 - \nu k_1^2| = \frac{1}{6(2-\beta)} \left| q_2 - \frac{(6+\beta) - 3(2-\beta)\nu}{8} q_1^2 \right|. \tag{5.5}$$

Relation (5.5) gives a desired estimate by Lemma 2.2. The proof of Theorem 5.1 is complete. □

6. Second Hankel Determinant for the Class $\mathcal{RK}_{\sinh}(\beta)$

Theorem 6.1. *Let the function $f \in \mathcal{A}$ given by (1.1) be in the class $\mathcal{RK}_{\sinh}(\beta)$. Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{9(2-\beta)^2}.$$

Proof. Using relations (3.7), (3.8) and (3.9) in the functional $\left|a_2 a_4 - a_3^2\right|$ and after simplification, we get

$$\begin{aligned} a_2 a_4 - a_3^2 = {} & \frac{1}{32(3-2\beta)} q_1 q_3 - \frac{4\beta^3 - 39\beta^2 + 24\beta + 12}{2304(2-\beta)^2(3-2\beta)} q_1^4 \\ & - \frac{23\beta^2 - 39\beta + 18}{576(2-\beta)^2(3-2\beta)} q_1^2 q_2 - \frac{1}{36(2-\beta)^2} q_2^2. \end{aligned} \tag{6.1}$$

Since $q \in \mathcal{P}$, it follows that $q(e^{-i\theta}\xi) \in \mathcal{P}$ $(\theta \in \mathbb{R})$. Therefore, we may assume without loss of any generality that $q_1 = q \geq 0$. Substituting the values of q_2 and q_3 from Lemma 2.3 in the relation (6.1), we obtain

$$\begin{aligned} a_2 a_4 - a_3^2 = {} & -\frac{4\beta^3 - 11\beta^2 - 14\beta + 24}{2304(2-\beta)^2(3-2\beta)} q^4 \\ & + \frac{6 - \beta - 5\beta^2}{1152(2-\beta)^2(3-2\beta)} q^2 x \left(4 - q^2\right) \\ & + \frac{1}{64(3-2\beta)} q \left(4 - q^2\right) \left(1 - |x|^2\right) z \\ & - \frac{1}{128(3-2\beta)} \left(4 - q^2\right) q^2 x^2 - \frac{\left(4 - q^2\right)^2 x^2}{144(2-\beta)^2}. \end{aligned}$$

With the help of triangle inequality and replacing $|z| \leq 1$ and $|x| = \rho \leq 1$, we obtain

$$\begin{aligned} \left|a_2 a_4 - a_3^2\right| \leq {} & \frac{4\beta^3 - 11\beta^2 - 14\beta + 24}{2304(2-\beta)^2(3-2\beta)} q^4 \\ & + \frac{6 - \beta - 5\beta^2}{1152(2-\beta)^2(3-2\beta)} q^2 \left(4 - q^2\right) \rho \\ & + \frac{1}{64(3-2\beta)} q \left(4 - q^2\right) \left(1 - \rho^2\right) \end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{128(3-2\beta)}\left(4-q^2\right)q^2\rho^2 + \frac{\left(4-q^2\right)^2\rho^2}{144(2-\beta)^2} \\
&= H(q,\rho)(say).
\end{aligned}$$

Differentiating $H(q,\rho)$ partially with respect to ρ, we get

$$\begin{aligned}
\frac{\partial H}{\partial \rho} &= \frac{6-\beta-5\beta^2}{1152(2-\beta)^2(3-2\beta)}q^2\left(4-q^2\right) \\
&\quad + \left\{\frac{q^2}{64(3-2\beta)} - \frac{q}{32(3-2\beta)} + \frac{4-q^2}{72(2-\beta)^2}\right\}\left(4-q^2\right)\rho \\
&= \frac{6-\beta-5\beta^2}{1152(2-\beta)^2(3-\beta)}q^2\left(4-q^2\right) \\
&\quad + \left\{\frac{9(2-\beta)^2q^2 - 18q(2-\beta)^2 + 8(3-2\beta)\left(4-q^2\right)}{576(2-\beta)^2(3-2\beta)}\right\}\left(4-q^2\right)\rho \\
&= \frac{6-\beta-5\beta^2}{1152(2-\beta)^2(3-2\beta)}q^2\left(4-q^2\right) \\
&\quad + \frac{(2-q)(-9q\beta^2+20q\beta-32\beta-12q+48)}{576(2-\beta)^2(3-2\beta)}\left(4-q^2\right)\rho.
\end{aligned}$$

For $0 \le \rho \le 1$ and for any fixed $q \in [0,2]$ we observe that $\frac{\partial H}{\partial \rho} > 0$. Thus $H(q,\rho)$ is an increasing function of ρ and for $q \in [0,2]$, $H(q,\rho)$ has a maximum value at $\rho = 1$. Therefore

$$\begin{aligned}
\max_{0\le\rho\le1} H(q,\rho) &= H(q,1) \\
&= G(q)(say),
\end{aligned}$$

where

$$\begin{aligned}
G(q) &= \frac{4\beta^3-11\beta^2-14\beta+24}{2304(2-\beta)^2(3-2\beta)}q^4 + \frac{6-\beta-5\beta^2}{1152(2-\beta)^2(3-2\beta)}q^2\left(4-q^2\right) \\
&\quad + \frac{1}{128(3-2\beta)}\left(4-q^2\right)q^2 + \frac{1}{144(2-\beta)^2}\left(4-q^2\right)^2.
\end{aligned}$$

Now

$$\begin{aligned}
G'(q) &= q\left[\frac{4\beta^3-11\beta^2-14\beta+24}{576(2-\beta)^2(3-2\beta)}q^2 + \frac{6-\beta-5\beta^2}{288(2-\beta)^2(3-2\beta)}(2-q^2)\right. \\
&\quad \left. + \frac{2-q^2}{32(3-2\beta)} - \frac{\left(4-q^2\right)q}{36(2-\beta)^2}\right] \\
&= q\left[\frac{4\beta^3-11\beta^2-14\beta+24}{576(2-\beta)^2(3-2\beta)}q^2 + \frac{4\beta^2-37\beta+42}{288(2-\beta)^2(3-2\beta)}(2-q^2)\right.
\end{aligned}$$

$$- \frac{\left(4-q^2\right)}{36(2-\beta)^2}\Bigg].$$

$G'(q) = 0$ then $q = 0$. Also

$$G''(q) = \frac{4\beta^3 - 11\beta^2 - 14\beta + 24}{192(2-\beta)^2(3-2\beta)}q^2 + \frac{(4\beta^2 - 37\beta + 42)(2-3q^2)}{288(2-\beta)^2(3-2\beta)} - \frac{4-3q^2}{36(2-\beta)^2}.$$

Now

$$[G''(q)]_{q=0} = -\frac{4\beta+3}{144(2-\beta)(3-2\beta)} < 0.$$

This implies that the function $G(q)$ can take the maximum value at $q = 0$. The maximum value is

$$\max_{q\in[0,2]} G(q) = G(0) = \frac{1}{9(2-\beta)^2}.$$

The proof of Theorem 6.1 is completed. □

Letting $\beta = 1$ in Theorem 6.1, we get the following result due to Barubak (see [5]).

Corollary 6.2. *Let $f \in \mathcal{BT}_s$. Then*

$$|H_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}.$$

$\beta = 0$ in Theorem 6.1 gives the result for class K_{sinh} as follows:

Corollary 6.3. *Let the function $f \in \mathcal{A}$ belongs to the class K_{sinh}. Then*

$$|H_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| \leq \frac{1}{36}.$$

7. Application of Borel Distribution

The distributions such as Binomial, Poisson, Pascal, logarithm, hypergeometric and their applications to the class of univalent functions have been intensively studied by various researchers from a different perspectives. Now, we discuss the application of the Borel distribution to the results obtained for the function class $\mathcal{RK}_{\text{sinh}}(\beta)$.

A discrete random variable X is said to follow a Borel distribution with parameter μ if its probability mass function $p(x)$ is given by

$$p(x=r)=\frac{(\mu r)^{r-1}e^{-\mu r}}{r!},\quad r=1,2,3,\cdots. \tag{7.1}$$

Recently, Wanas and Khuttar [32] introduced a power series whose coefficients are the probabilities of the Borel distribution i.e.

$$M(\mu,\xi)=\xi+\sum_{n=2}^{\infty}\frac{(\mu(n-1))^{n-2}e^{-\mu(n-1)}}{(n-1)!}\xi^n,\quad (\xi\in\Delta),$$

where $0\le\mu\le1$. By using the ratio test, it can be shown that the radius of convergence of the above series is infinite.

Let us introduce a linear operator $L_\mu:\mathcal{A}\longrightarrow\mathcal{A}$ defined by

$$\begin{aligned}L_\mu f(\xi)&=M(\mu,\xi)*f(\xi)\\&=\xi+\sum_{n=2}^{\infty}\frac{(\mu(n-1))^{n-2}e^{-\mu(n-1)}}{(n-1)!}a_n\xi^n\\&=\xi+\sum_{n=2}^{\infty}\alpha_n(\mu)a_n\xi^n\\&=\xi+\alpha_2a_2\xi^2+\alpha_3a_3\xi^3+\cdots,\end{aligned} \tag{7.2}$$

where $\alpha_n=\alpha_n(\mu)=\frac{(\mu(n-1))^{n-2}e^{-\mu(n-1)}}{(n-1)!}$.

We define the class $RK^{\mu}_{\sinh}$ as follows:

$$RK^{\mu}_{\sinh}(\beta)=\{f\in\mathcal{A}:L_\mu f\in\mathcal{RK}_{\sinh}(\beta)\}. \tag{7.3}$$

In the same way, as in Theorem 3.4 and Theorem 3.8 we can obtain the coefficient bounds and Fekete-Szegö functional for the class $RK^{\mu}_{\sinh}(\beta)$ from the corresponding estimates for the function of the class $\mathcal{RK}_{\sinh}(\beta)$.

Theorem 7.1. *Let $0\le\beta\le1$ and $L_\mu f$ given by (7.2). If $f\in RK^{\mu}_{\sinh}(\beta)$, then for any complex number ν, we have*

$$|a_3-\nu a_2^2|\le\frac{1}{3(2-\beta)\alpha_3}\max\left\{1,\left|\frac{(\beta\alpha_3-4\alpha_2^2)+3(2-\beta)\nu\alpha_3}{4\alpha_2^2}\right|\right\}.$$

Proof. Since $f\in RK^{\mu}_{\sinh}(\beta)$, it follows from (7.3) that

$$[L_\mu f(\xi)]^\beta\left[\frac{(\xi(L_\mu f(\xi))')'}{(L_\mu f(\xi))'}\right]^{1-\beta}=1+\sinh^{-1}(\omega(\xi)). \tag{7.4}$$

From (7.2), we obtain

(7.5)

$$[L_\mu f(\xi)]^\beta\left[\frac{(\xi(L_\mu f(\xi))')'}{(L_\mu f(\xi))'}\right]^{1-\beta}$$

$$= 1 + 2\alpha_2 a_2 \xi + \left[3(1-\beta)\alpha_3 a_3 - 4(1-\beta)\alpha_2^2 a_2^2\right] \xi^2$$
$$+ \left[4(3-2\beta)\alpha_4 a_4 - 18(1-\beta)\alpha_2\alpha_3 a_2 a_3 + 8(1-\beta)\alpha_2^3 a_2^3\right] \xi^3 + \cdots .$$

Using (3.5) and (7.5) in (7.4) and equating the corresponding coefficients of ξ and ξ^2, we get

$$a_2 = \frac{q_1}{4\alpha_2},$$

and

$$a_3 = \frac{1}{3(2-\beta)\alpha_3}\left[\frac{q_2}{2} - \frac{\beta}{4}q_1^2\right].$$

Thus, for any complex number ν, we have

$$|a_3 - \nu a_2^2| = \left|\frac{q_2}{6(2-\beta)\alpha_3} - \frac{\beta}{12(2-\beta)\alpha_3}q_1^2 - \nu\frac{q_1^2}{16\alpha_2^2}\right| \tag{7.6}$$
$$= \frac{1}{6(2-\beta)\alpha_3}\left[q_2 - \left(\frac{\beta\alpha_3}{2} + \frac{3(2-\beta)\nu\alpha_3}{8\alpha_2^2}\right)q_1^2\right].$$

An application of Lemma 2.2 to (7.6) yields

$$|a_3 - \nu a_2^2| \le \frac{1}{3(2-\beta)\alpha_3}\ \max\left\{1, \left|\frac{(\beta\alpha_3 - 4\alpha_2^2) + 3(2-\beta)\nu\alpha_3}{4\alpha_2^2}\right|\right\}.$$

The proof of Theorem 7.1 is thus completed. □

The next theorem gives Fekete-Szegö inequality for the class $RK^{\mu}_{\sinh}(\beta)$ when ν is real. We omit the proof as the proof follows the same line as in Theorem 3.8.

Theorem 7.2. *Let $0 \le \beta \le 1$ and $L_\mu f$ given by (7.2). For any real number ν, we have*

$$|a_3 - \nu a_2^2| \le \begin{cases} \frac{4\alpha_2^2(1-\alpha_3) - 3(2-\beta)\alpha_3\nu}{12\alpha_2^2(2-\beta)\alpha_3}, & \nu \le \frac{-4\beta\alpha_2^2}{3(2-\beta)}, \\ \frac{1}{3(2-\beta)\alpha_3}, & -\frac{4\beta\alpha_2^2}{3(2-\beta)} \le \nu \le \frac{4\alpha_2^2(2-\beta\alpha_3)}{3(2-\beta)\alpha_3}, \\ \frac{3(2-\beta)\nu\alpha_3 - 4\alpha_2^2(1-\alpha_3)}{12\alpha_2^2(2-\beta)\alpha_3}, & \nu \ge \frac{4(2-\beta\alpha_3)\alpha_2^2}{3(2-\beta)\alpha_3}. \end{cases}$$

Concluding Remarks: In our current paper we investigated the coefficient bounds, Fekete-Szegö functional, second Hankel determinant for $f \in \mathcal{RK}_{\sinh}(\beta)$ associated with the petal-shaped domain. Further, we determined the coefficient estimate and Fekete-Szegö inequalities to the inverse function class f^{-1} and $\frac{\xi}{f(\xi)}$. We also established an application for Borel distribution to our main results. The class defined in this paper generalizes the class considered by Barukab et al.. The results in [5] are a special case of our results ($\beta = 1$). In recent years, the application of (p, q)-calculus or more specifically $q-$calculus has played a dominant role in the theory of geometric function theory of complex analysis (see [31]). Researchers can make use of $q-$calculus to modify the class $\mathcal{RK}_{\sinh}(\beta)$ and all the results of this paper can be extended to the study of analytic or meromorphic functions[8].

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