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Convolution Product for Hilbert C^*-Module Valued Maps

Mawoussi Todjro[1] and Yaogan Mensah[2*]

Abstract. In this paper, we introduce a convolution-type product for strongly integrable Hilbert C^*-module valued maps on locally compact groups. We investigate various properties of this product related to uniform continuity, boundless, etc. For instance, we prove a convolution theorem. Also, we study the boundless of the related convolution operator in various settings.

1. Introduction

Convolution is a generalization of the notion of a moving average. It is the mathematical concept behind the linear filter in signal processing. It has many applications in engineering, probability theory, statistics, quantum mechanic, etc. The convolution interferes with the Fourier transform in solving some differential equations. Initially defined for complex compactly supported functions, the notion of convolution is extended to integrable functions, functions with rapid decay, distributions, measures, etc.

On the other hand, the idea of Hilbert modules is in vogue in mathematics. They are obtained similarly from the category of Hilbert spaces by allowing the inner product to take values in a C^*-algebra. The concept was first introduced by I. Kaplansky in the latter half of the 1950s to solve some problems involving the structure of derivations of AW^*-algebras [8]. Interested readers who want to learn more about the subject may consult [9, 10].

Analysis of Hilbert C^*-module valued functions on topological groups has been initiated in [13, 14]. This paper is the continuation of this

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work. It addresses some results about a convolution product defined with respect to the inner product of a Hilbert C^*-module.

The rest of the paper is organized as follows. Section 2 gives basic definitions and facts that will appear throughout this article. In Section 3, we define a convolution product in the framework of Hilbert C^*-module valued maps on groups and study some of its properties. We prove a convolution theorem in Section 4. Finally, in Section 5, we investigate some properties of a convolution operator.

2. Preliminaries

Let $\mathcal{A}$ be a C^*-algebra and $\|\cdot\|_{\mathcal{A}}$ its norm. We call a vector space $\mathcal{M}$ a *right pre-Hilbert module* over $\mathcal{A}$ (or a right pre-Hilbert $\mathcal{A}$-module) if for all $x \in \mathcal{M}$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, the equality $\lambda (xa) = (\lambda x) a = x (\lambda a)$ hold and there is a map $\langle \cdot, \cdot \rangle_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ satisfying the following properties:

(i) $\forall x, y, z \in \mathcal{M}$, $\alpha, \beta \in \mathbb{C}$, $\langle x, \alpha y + \beta z \rangle_{\mathcal{M}} = \alpha \langle x, y \rangle_{\mathcal{M}} + \beta \langle x, z \rangle_{\mathcal{M}}$.
(ii) $\forall x, y \in \mathcal{M}$, $\forall a \in \mathcal{A}$, $\langle x, ya \rangle_{\mathcal{M}} = \langle x, y \rangle_{\mathcal{M}} a$.
(iii) $\forall x, y \in \mathcal{M}$, $\langle y, x \rangle_{\mathcal{M}} = \langle x, y \rangle_{\mathcal{M}}^*$.
(iv) $\forall x \in \mathcal{M}$, $\langle x, x \rangle_{\mathcal{M}} \geq 0$ and if $\langle x, x \rangle_{\mathcal{M}} = 0$ then $x = 0$.

If $\mathcal{M}$ is a pre-Hilbert $\mathcal{A}$-module, then the formula

$$\|x\|_{\mathcal{M}} = \|\langle x, x \rangle_{\mathcal{M}}\|_{\mathcal{A}}^{\frac{1}{2}},$$

defines a norm in $\mathcal{M}$. This norm satisfies the following properties.

(i) $\forall x \in \mathcal{M}$, $\forall a \in \mathcal{A}$, $\|xa\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \|a\|_{\mathcal{A}}$.
(ii) $\forall x, y \in \mathcal{M}$, $\|\langle x, y \rangle_{\mathcal{M}}\|_{\mathcal{A}} \leq \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$.

The right pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ is said to be a *right Hilbert $\mathcal{A}$-module* if it is a Banach space relative to the norm $\|\cdot\|_{\mathcal{M}}$ defined above. Left Hilbert $\mathcal{A}$-modules are defined analogously. In the sequel, we consider the right Hilbert $\mathcal{A}$-modules and simply call them Hilbert $\mathcal{A}$-module (or Hilbert C^*-module over $\mathcal{A}$). Hilbert C^*-modules are generalizations of Hilbert spaces and C^*-algebras.

Now, let G be a locally compact group. Let $\mathcal{C}_0 (G, \mathcal{M})$ be the set of $\mathcal{M}$-valued continuous functions on G which vanish at infinity endowed with the supremum norm $\|f\|_\infty = \sup\limits_{x \in G} \|f(x)\|_{\mathcal{M}}$.

Also, we designate by $\mathcal{C}_c (G, \mathcal{M})$ the subset of $\mathcal{C}_0 (G, \mathcal{M})$ consisting of $\mathcal{M}$-valued compactly y supported continuous functions on G. Throughout this paper, the Bochner (strong) integral is considered [4]. We denote by λ a left Haar measure of the locally compact group G. The Haar measure satisfies the relations

$$\int_G f(st)\, d\lambda(t) = \int_G f(t)\, d\lambda(t),$$

and

$$\int_G f(ts)\, d\lambda(t) = \Delta\left(s^{-1}\right) \int_G f(t)\, d\lambda(t), \quad f \in \mathcal{C}_c(G, \mathcal{M}),$$

where Δ is the modular function of G [4, 6]. We recall that G is said to be *unimodular* if $\Delta \equiv 1$. For instance, locally compact abelian groups and compact groups are unimodular [3, 7]. Let us mention that $\lambda(G)$ is finite if and only if G is compact.

Let $L^p(G, \mathcal{M}), \quad p \geq 1$, denote the set of Bochner strong p-integrable $\mathcal{M}$-valued functions on G [4]. The set $L^p(G, \mathcal{M})$ is endowed with the norm

$$\|f\|_p = \left(\int_G \|f(t)\|_{\mathcal{M}}^p\, d\lambda(t)\right)^{\frac{1}{p}}, \quad f \in L^p(G, \mathcal{M}).$$

It was proved in [13] that $L^2(G, \mathcal{M})$ is a pre-Hilbert $\mathcal{A}$-module under the inner product defined by

$$\langle f, g\rangle_{L^2(G,\mathcal{M})} = \int_G \langle f(t), g(t)\rangle_{\mathcal{M}}\, d\lambda(t), \quad f, g \in L^2(G, \mathcal{M}).$$

If G is a locally compact abelian group then we denote by $\widehat{G}$ its Pontrjagin dual group. The latter consists of continuous group homomorphisms $\chi : G \to \mathbb{T}$, where $\mathbb{T}$ is the unit circle. Such homomorphisms are called *characters* of group G [2, 3]. The Fourier transform of $f \in L^1(G, \mathcal{M})$ is defined by

$$\widehat{f}(\chi) = \int_G f(t)\, \overline{\chi(t)} d\lambda(t), \quad \chi \in \widehat{G}.$$

If G is a compact group, we denote by Σ its dual object. The latter is the set of equivalence classes of unitary irreducible representations of G. For $\sigma \in \Sigma$, fix an element in the class σ and still refer to it by σ. Denote by H_σ its representation space and by d_σ the dimension of H_σ [1, 7, 11]. Following [11], the Fourier transform of $f \in L^1(G, \mathcal{M})$ is defined by

$$\widehat{f}(\sigma)(\xi \otimes \eta) = \int_G \langle \sigma(t)^* \xi, \eta\rangle_{H_\sigma} f(t)\, d\lambda(t),$$

where $\xi, \eta \in H_\sigma$ and $\sigma(t)^*$ is the adjoint of the unitary operator $\sigma(t)$.

3. A Convolution Type Product

In this section, we consider a product that generalizes the usual convolution product to the framework of Hilbert C^*-module valued maps. The study of this product was initiated in [14]. We deepen this study here.

Definition 3.1. Let G be a locally compact group and $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. For $f, g \in L^1(G, \mathcal{M})$, define the convolution product $f \circledast g$ by

$$f \circledast g(t) = \int_G \left\langle f\left(xt^{-1}\right), g(x)\right\rangle_{\mathcal{M}} d\lambda(x).$$

Let $L^1(G, \mathcal{A})$ be the set of Bochner strong integrable functions on G with values in the C^*-algebra $\mathcal{A}$. It is a Banach space with respect to the norm defined by

$$\|f\|_{\mathcal{A}}^1 = \int_G \|f(t)\|_{\mathcal{A}} d\lambda(t).$$

For a function h, set $h^{\sim}(t) = h\left(t^{-1}\right)$. The following theorem addresses the relation between $f \circledast g$ and $g \circledast f$.

Theorem 3.2. *Let G be a locally compact group. If $f, g \in L^1(G, \mathcal{M})$, then for all $t \in G$,*

$$(f \circledast g)(t) = \Delta(t)\left[(g \circledast f)^{\sim}(t)\right]^*,$$

where Δ is the modular function of G.

Proof.

$$\begin{aligned}
(f \circledast g)(t) &= \int_G \left\langle f\left(xt^{-1}\right), g(x)\right\rangle_{\mathcal{M}} d\lambda(x) \\
&= \Delta(t) \int_G \langle f(x), g(xt)\rangle_{\mathcal{M}} d\lambda(x) \\
&= \Delta(t) \int_G \langle g(xt), f(x)\rangle_{\mathcal{M}}^* d\lambda(x) \\
&= \Delta(t) \left(\int_G \langle g(xt), f(x)\rangle_{\mathcal{M}} d\lambda(x)\right)^* \\
&= \Delta(t)\left[(g \circledast f)\left(t^{-1}\right)\right]^* \\
&= \Delta(t)\left[(g \circledast f)^{\sim}(t)\right]^*.
\end{aligned}$$

□

Theorem 3.3 ([14])**.** *Let G be a locally compact unimodular group. If $f, g \in L^1(G, \mathcal{M})$ then $f \circledast g \in L^1(G, \mathcal{A})$. Moreover,*

$$\|f \circledast g\|_{\mathcal{A}}^1 \leq \|f\|_1 \|g\|_1.$$

Proof. Let $f, g \in L^1(G, \mathcal{M})$. Then

$$\begin{aligned}
\|f \circledast g\|_{\mathcal{A}}^1 &= \int_G \|f \circledast g(t)\|_{\mathcal{A}} d\lambda(t) \\
&= \int_G \left\| \int_G \left\langle f\left(xt^{-1}\right), g(x)\right\rangle_{\mathcal{M}} d\lambda(x)\right\|_{\mathcal{A}} d\lambda(t)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_G \int_G \left\| \left\langle f\left(xt^{-1}\right), g(x) \right\rangle_{\mathcal{M}} \right\|_{\mathcal{A}} d\lambda(x)\, d\lambda(t) \\
&\leq \int_G \int_G \left\| f\left(xt^{-1}\right) \right\|_{\mathcal{M}} \|g(x)\|_{\mathcal{M}} d\lambda(x)\, d\lambda(t) \\
&= \int_G \|g(x)\|_{\mathcal{M}} \left(\int_G \left\| f\left(xt^{-1}\right) \right\|_{\mathcal{M}} d\lambda(t) \right) d\lambda(x) \\
&= \int_G \|g(x)\|_{\mathcal{M}} \left(\int_G \|f(y)\|_{\mathcal{M}} d\lambda(y) \right) d\lambda(x) \\
&= \int_G \|f(y)\|_{\mathcal{M}} d\lambda(y) \int_G \|g(x)\|_{\mathcal{M}} d\lambda(x) \\
&= \|f\|_1 \|g\|_1 .
\end{aligned}
$$

□

For $s \in G$, consider the right translation τ_s defined by

$$\tau_s f(t) = f\left(ts^{-1}\right), \quad f \in L^1(G, \mathcal{M}).$$

We have $\tau_{st^{-1}} = \tau_s \tau_{t^{-1}}, \quad \forall s, t \in G$. Let $f, g \in L^1(G, \mathcal{M})$ then

$$
\begin{aligned}
f \circledast g(t) &= \int_G \left\langle f\left(xt^{-1}\right), g(x) \right\rangle_{\mathcal{M}} d\lambda(x) \\
&= \int_G \langle \tau_t f(x), g(x) \rangle_{\mathcal{M}} d\lambda(x) \\
&= \langle \tau_t f, g \rangle_{L^2(G,\mathcal{M})} .
\end{aligned}
$$

Theorem 3.4. *Let G be a locally compact group. If $f, g \in L^1(G, \mathcal{M})$ then*

$$\tau_s (f \circledast g) = (\tau_{s^{-1}} f) \circledast g = f \circledast \tau_s g.$$

Proof. Let $f, g \in L^1(G, \mathcal{M})$. For $s, t \in G$, we have

$$
\begin{aligned}
\tau_s (f \circledast g)(t) &= (f \circledast g)\left(ts^{-1}\right) \\
&= \int_G \langle \tau_{ts^{-1}} f(x), g(x) \rangle_{\mathcal{M}} d\lambda(x) \\
&= \int_G \langle \tau_t (\tau_{s^{-1}} f(x)), g(x) \rangle_{\mathcal{M}} d\lambda(x) \\
&= (\tau_{s^{-1}} f) \circledast g(t) .
\end{aligned}
$$

On the other hand

$$\tau_s (f \circledast g)(t) = \int_G \left\langle f\left(xst^{-1}\right), g(x) \right\rangle_{\mathcal{M}} d\lambda(x) .$$

By setting $y = xs$ and using the fact that λ is a left-invariant measure on G, we obtain

$$\begin{aligned}\tau_s (f \circledast g)(t) &= \int_G \left\langle f\left(yt^{-1}\right), g\left(ys^{-1}\right)\right\rangle_{\mathcal{M}} d\lambda(y) \\ &= \int_G \left\langle f\left(yt^{-1}\right), \tau_s g(y)\right\rangle_{\mathcal{M}} d\lambda(y) \\ &= [f \circledast (\tau_s g)](t),\end{aligned}$$

□

Theorem 3.5. *Let G be a locally compact group. If $f \in L^1(G, \mathcal{M})$ and $g \in \mathcal{C}_c(G, \mathcal{M})$ then $f \circledast g$ is bounded and*

$$\|f \circledast g\|_\infty \leq \|g\|_\infty \|f\|_1 .$$

Proof. Let $f \in L^1(G, \mathcal{M})$, $g \in \mathcal{C}_c(G, \mathcal{M})$ and $t \in G$. Then

$$\begin{aligned}\|f \circledast g(t)\|_{\mathcal{A}} &= \left\| \int_G \left\langle f\left(xt^{-1}\right), g(x)\right\rangle_{\mathcal{M}} d\lambda(x) \right\|_{\mathcal{A}} \\ &\leq \int_G \left\| \left\langle f\left(xt^{-1}\right), g(x)\right\rangle_{\mathcal{M}} \right\|_{\mathcal{A}} d\lambda(x) \\ &\leq \int_G \left\| f\left(xt^{-1}\right)\right\|_{\mathcal{M}} \|g(x)\|_{\mathcal{M}} d\lambda(x) \\ &\leq \|g\|_\infty \int_G \left\| f\left(xt^{-1}\right)\right\|_{\mathcal{M}} d\lambda(x) \\ &\leq \|g\|_\infty \int_G \|f(x)\|_{\mathcal{M}} d\lambda(x) \\ &\leq \|g\|_\infty \|f\|_1 .\end{aligned}$$

Taking the supremum, we obtain

$$\begin{aligned}\|f \circledast g\|_\infty &= \sup_{t \in G} \|f \circledast g(t)\|_{\mathcal{A}} \\ &\leq \|g\|_\infty \|f\|_1 .\end{aligned}$$

□

Theorem 3.6. *Let G be a locally compact group. If $f \in L^1(G, \mathcal{M})$ then the map $s \mapsto \tau_s f$ from G into $L^1(G, \mathcal{M})$ is uniformly continuous on G.*

Proof. Let $\varepsilon > 0$ be given and $f \in L^1(G, \mathcal{M})$. The set $\mathcal{C}_c(G, \mathcal{M})$ is dense in $L^1(G, \mathcal{M})$. Therefore we can find $g \in \mathcal{C}_c(G, \mathcal{M})$ such that $\|f - g\|_1 < \dfrac{\varepsilon}{3}$. Denoted by K the support of g. The function g is uniformly continuous on K. Therefore there is an open subset V of G included in K such that

$$\forall s \in V, \quad \|g - \tau_s g\|_\infty < \frac{\varepsilon}{3\lambda(K)}.$$

Hence

$$\begin{aligned}\|g-\tau_s g\|_1 &= \int_G \|g(t)-\tau_s g(t)\|_{\mathcal{M}}\, d\lambda(t)\\ &= \int_K \|g(t)-\tau_s g(t)\|_{\mathcal{M}}\, d\lambda(t)\\ &\le \frac{\varepsilon}{3\lambda(K)}\int_K d\lambda(t)\\ &= \frac{\varepsilon}{3}.\end{aligned}$$

So, we have

$$\begin{aligned}\|f-\tau_s f\|_1 &\le \|f-g\|_1 + \|g-\tau_s g\|_1 + \|\tau_s g-\tau_s f\|_1\\ &\le \|f-g\|_1 + \|g-\tau_s g\|_1 + \|\tau_s(g-f)\|_1\\ &\le \|f-g\|_1 + \|g-\tau_s g\|_1 + \|g-f\|_1\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

Finally, for $y\in G$ such that $ys^{-1}\in V$, we have

$$\|\tau_s f-\tau_y f\|_1 = \left\|\tau_s\left(f-\tau_{ys^{-1}}f\right)\right\|_1 = \left\|f-\tau_{ys^{-1}}f\right\|_1 < \varepsilon.$$

Thus the map $s\mapsto \tau_s f$ is uniformly continuous on G. □

Theorem 3.7. *Let G be a locally compact group. If $f\in L^1(G,\mathcal{M})$ and $g\in\mathcal{C}_c(G,\mathcal{M})$ then $\forall s,t\in G$,*

$$\|f\circledast g(t)-f\circledast g(s)\|_{\mathcal{A}} \le \|\tau_t f-\tau_s f\|_1\,\|g\|_\infty.$$

Proof. Let $f,g\in L^1(G,\mathcal{M})$ and $s,t\in G$, we have:

$$\begin{aligned}f\circledast g(t)-f\circledast g(s) &= \langle \tau_t f, g\rangle_{L^2(G,\mathcal{M})} - \langle \tau_s f, g\rangle_{L^2(G,\mathcal{M})}\\ &= \langle \tau_t f-\tau_s f, g\rangle_{L^2(G,\mathcal{M})}\\ &= \int_G \langle (\tau_t f-\tau_s f)(x), g(x)\rangle_{\mathcal{M}}\, d\lambda(x).\end{aligned}$$

Hence

$$\begin{aligned}\|f\circledast g(t)-f\circledast g(s)\|_{\mathcal{A}} &= \left\|\int_G \langle (\tau_t f-\tau_s f)(x), g(x)\rangle_{\mathcal{M}}\, d\lambda(x)\right\|_{\mathcal{A}}\\ &\le \int_G \|\langle (\tau_t f-\tau_s f)(x), g(x)\rangle_{\mathcal{M}}\|_{\mathcal{A}}\, d\lambda(x)\\ &\le \int_G \|(\tau_t f-\tau_s f)(x)\|_{\mathcal{M}}\,\|g(x)\|_{\mathcal{M}}\, d\lambda(x)\\ &\le \|g\|_\infty \int_G \|(\tau_t f-\tau_s f)(x)\|_M\, d\lambda(x)\end{aligned}$$

$$= \|\tau_t f - \tau_s f\|_1 \|g\|_\infty . \qquad \square$$

Corollary 3.8. *Let G be a locally compact group. If $f \in L^1(G, \mathcal{M})$ and $g \in \mathcal{C}_c(G, \mathcal{M})$ then $f \circledast g$ is uniformly continuous on G.*

Proof. The result is trivial for $g = 0$. Now assume $g \neq= 0$. Let $\varepsilon > 0$. From the proof of Theorem 3.7, there exists a subset V of G such that for $t, s \in G$, if $ts^{-1} \in V$ then $\|\tau_t f - \tau_s f\|_1 < \dfrac{\varepsilon}{\|g\|_\infty}$. Hence

$$\begin{aligned}\|f \circledast g(t) - f \circledast g(s)\|_{\mathcal{A}} &\leq \|\tau_t f - \tau_s f\|_1 \|g\|_\infty \\ &< \frac{\varepsilon}{\|g\|_\infty} \|g\|_\infty \\ &= \varepsilon.\end{aligned} \qquad \square$$

Theorem 3.9. *Let G be a locally compact unimodular group. Let $p, q > 0$ such that $1 < p < +\infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $f \in L^p(G, \mathcal{M})$ and $g \in L^q(G, \mathcal{M})$ then $f \circledast g \in \mathcal{C}_0(G, \mathcal{M})$.*

Proof. Let $f \in L^p(G, \mathcal{M})$ and $g \in L^q(G, \mathcal{M})$. By density of $\mathcal{C}_c(G, \mathcal{M})$ in $L^p(G, \mathcal{M})$ and in $L^q(G, \mathcal{M})$ there exists sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_c(G, \mathcal{M})$ such that

$$\begin{aligned}&\|f_n - f\|_p \to 0 \text{ when } n \to +\infty, \\ &\|g_n - g\|_q \to 0 \text{ when } n \to +\infty.\end{aligned}$$

For $t \in G$ set $\alpha_n(t) = \|(f_n \circledast g_n)(t) - (f \circledast g)(t)\|_{\mathcal{A}}$. Then

$$\begin{aligned}\alpha_n(t) &= \|(f_n \circledast g_n)(t) - (f_n \circledast g)(t) + (f_n \circledast g)(t) - (f \circledast g)(t)\|_{\mathcal{A}} \\ &\leq \|(f_n \circledast g_n)(t) - (f_n \circledast g)(t)\|_{\mathcal{A}} + \|(f_n \circledast g)(t) - (f \circledast g)(t)\|_{\mathcal{A}} \\ &= \|f_n \circledast (g_n - g)(t)\|_{\mathcal{A}} + \|(f_n - f) \circledast g(t)\|_{\mathcal{A}} \\ &\leq \int_G \left\|f_n\left(xt^{-1}\right)\right\|_{\mathcal{M}} \|(g_n - g)(x)\|_{\mathcal{M}} \, d\lambda(x) \\ &\quad + \int_G \left\|(f_n - f)\left(xt^{-1}\right)\right\|_{\mathcal{M}} \|g(x)\|_{\mathcal{M}} \, d\lambda(x) \\ &= \int_G \|\tau_t f_n(x)\|_{\mathcal{M}} \|(g_n - g)(x)\|_{\mathcal{M}} \, d\lambda(x) \\ &\quad + \int_G \|\tau_t(f_n - f)(x)\|_{\mathcal{M}} \|g(x)\|_{\mathcal{M}} \, d\lambda(x) \\ &\leq \|\tau_t f_n\|_p \|g_n - g\|_q + \|\tau_t(f_n - f)\|_p \|g\|_q \\ &= \|f_n\|_p \|g_n - g\|_q + \|f_n - f\|_p \|g\|_q .\end{aligned}$$

Hence

$$\|f_n \circledast g_n - f \circledast g\|_\infty = \sup_{t \in G} \alpha_n(t)$$

$$\leq \|f_n\|_p \|g_n - g\|_q + \|f_n - f\|_p \|g\|_q .$$

Therefore, the function $f \circledast g$ is the uniform limit of a sequence of functions in $\mathcal{C}_c(G, \mathcal{M})$. Thus $f \circledast g \in \mathcal{C}_0(G, \mathcal{M})$. ☐

4. Convolution Theorem

It is well-known that convolution interferes with the Fourier transform. For instance, the Fourier transform of the convolution product of two functions on $\mathbb{R}^n$ is the product of the Fourier transform of these functions. This result extends to other more general frameworks with of course some resulting modifications. The following theorem was obtained for functions in $L^1(G, \mathcal{M})$ when G is a locally compact abelian group.

Theorem 4.1 ([14])**.** *Let G be a locally compact abelian group. If $f, g \in L^1(G, \mathcal{M})$ then*

$$\widehat{f \circledast g}(\chi) = \left\langle \widehat{f}(\chi), \widehat{g}(\chi) \right\rangle_{\mathcal{M}}, \quad \chi \in \widehat{G}.$$

In what follows, we aim to obtain the analog of Theorem 4.1 for compact groups which are not necessarily commutative. Let G be a compact group with dual object Σ. For a (class of) representation $\sigma \in \Sigma$, denote by H_σ its representation space and by d_σ its dimension. Fix a basis $\left(\xi_1^\sigma, \cdots, \xi_{d_\sigma}^\sigma\right)$ of H_σ. Let $f, g \in L^1(G, \mathcal{M})$ and set

$$\left[\widehat{f}(\sigma) \odot \widehat{g}(\sigma)\right]\left(\xi_i^\sigma \otimes \xi_j^\sigma\right) = \sum_{k=1}^{d_\sigma} \left\langle \widehat{f}(\sigma)\left(\xi_i^\sigma \otimes \xi_k^\sigma\right), \widehat{g}(\sigma)\left(\xi_k^\sigma \otimes \xi_j^\sigma\right) \right\rangle_{\mathcal{M}}.$$

We have the following theorem.

Theorem 4.2. *Let G be a compact group. If $f, g \in L^1(G, \mathcal{M})$ then*

$$\widehat{f \circledast g}(\sigma) = \widehat{f}(\sigma) \odot \widehat{g}(\sigma), \quad \sigma \in \Sigma.$$

Proof. Let $f, g \in L^1(G, \mathcal{M})$ and $\sigma \in \Sigma$. Set $X = \widehat{f \circledast g}(\sigma)\left(\xi_i^\sigma \otimes \xi_j^\sigma\right)$. We have

$$\begin{aligned}
X &= \int_G \left\langle \sigma(t)^* \xi_i^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma} (f \circledast g)(t)\, d\lambda(t) \\
&= \int_G \left\langle \sigma(t)^* \xi_i^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma} \left(\int_G \left\langle f\left(xt^{-1}\right), g(x) \right\rangle_{\mathcal{M}} d\lambda(x) \right) d\lambda(t) \\
&= \int_G \int_G \left\langle \sigma(t)^* \xi_i^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma} \left\langle f\left(xt^{-1}\right), g(x) \right\rangle_{\mathcal{M}} d\lambda(x)\, d\lambda(t) \\
&= \int_G \int_G \left\langle \sigma\left(y^{-1}x\right)^* \xi_i^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma} \left\langle f(y), g(x) \right\rangle_{\mathcal{M}} d\lambda(x)\, d\lambda(y)
\end{aligned}$$

$$= \int_G \int_G \left\langle \sigma(y)\, \xi_i^\sigma, \sigma(x)\, \xi_j^\sigma \right\rangle_{H_\sigma} \langle f(y), g(x)\rangle_{\mathcal{M}}\, d\lambda(x)\, d\lambda(y)$$

$$= \int_G \int_G \sum_{k=1}^{d_\sigma} \langle \sigma(y)\, \xi_i^\sigma, \xi_k^\sigma \rangle_{H_\sigma} \left\langle \sigma(x)^* \xi_k^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma}$$
$$\times \langle f(y), g(x)\rangle_{\mathcal{M}}\, d\lambda(x)\, d\lambda(y)$$

$$= \sum_{k=1}^{d_\sigma} \int_G \int_G \left\langle \langle \sigma(y)^* \xi_i^\sigma, \xi_k^\sigma \rangle_{H_\sigma} f(y), \left\langle \sigma(x)^* \xi_k^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma} g(x) \right\rangle_{\mathcal{M}}$$
$$\times\, d\lambda(y)\, d\lambda(x)$$

$$= \sum_{k=1}^{d_\sigma} \langle A, B\rangle_{\mathcal{M}},$$

where

$$A = \int_G \langle \sigma(y)^* \xi_i^\sigma, \xi_k^\sigma \rangle_{H_\sigma} f(y)\, d\lambda(y),$$
$$B = \int_G \left\langle \sigma(x)^* \xi_k^\sigma, \xi_j^\sigma \right\rangle_{H_\sigma} g(x)\, d\lambda(x).$$

Then

$$X = \sum_{k=1}^{d_\sigma} \left\langle \widehat{f}(\sigma)\left(\xi_i^\sigma \otimes \xi_k^\sigma\right), \widehat{g}(\sigma)\left(\xi_k^\sigma \otimes \xi_j^\sigma\right) \right\rangle_{\mathcal{M}}$$
$$= \left[\widehat{f}(\sigma) \odot \widehat{g}(\sigma)\right]\left(\xi_i^\sigma \otimes \xi_j^\sigma\right).$$

Hence $\widehat{f \circledast g}(\sigma) = \widehat{f}(\sigma) \odot \widehat{g}(\sigma)$. □

5. Convolution Operators

For $\varphi \in L^1(G, \mathcal{M})$ we consider the convolution operator

$$T_\varphi : L^1(G, \mathcal{M}) \to L^1(G, \mathcal{A}), \quad f \mapsto \varphi \circledast f.$$

Theorem 5.1. *The map T_φ is a $\mathbb{C}$-linear and an $\mathcal{A}$-linear bounded operator.*

Proof. Let $f, g \in L^1(G, \mathcal{M})$, $\alpha \in \mathbb{C}$ and $a \in \mathcal{A}$. We have, for all $t \in G$

$$T_\varphi(f+g)(t) = (\varphi \circledast (f+g))(t)$$
$$= \int_G \left\langle \varphi\left(xt^{-1}\right), f(x) + g(x) \right\rangle d\lambda(x)$$
$$= \int_G \left\langle \varphi\left(xt^{-1}\right), f(x) \right\rangle d\lambda(x) + \int_G \left\langle \varphi\left(xt^{-1}\right), g(x) \right\rangle d\lambda(x)$$
$$= \varphi \circledast f(t) + \varphi \circledast g(t)$$

$$= T_\varphi f(t) + T_\varphi g(t),$$

and

$$\begin{aligned} T_\varphi(\alpha f)(t) &= \varphi \circledast (\alpha f)(t) \\ &= \int_G \left\langle \varphi\left(xt^{-1}\right), \alpha f(x)\right\rangle_{\mathcal{M}} d\lambda(x) \\ &= \alpha \int_G \left\langle \varphi\left(xt^{-1}\right), f(x)\right\rangle_{\mathcal{M}} d\lambda(x) \\ &= \alpha T_\varphi f(t). \end{aligned}$$

Also

$$\begin{aligned} T_\varphi(fa)(t) &= \int_G \left\langle \varphi\left(xt^{-1}\right), (fa)(x)\right\rangle_{\mathcal{M}} d\lambda(x) \\ &= \int_G \left\langle \varphi\left(xt^{-1}\right), f(x)\, a\right\rangle d\lambda(x) \\ &= \int_G \left\langle \varphi\left(xt^{-1}\right), f(x)\right\rangle_{\mathcal{M}} a d\lambda(x) \\ &= \left(\int_G \left\langle \varphi\left(xt^{-1}\right), f(x)\right\rangle_{\mathcal{M}} d\lambda(x)\right) a \\ &= ((T_\varphi f)\, a)(t). \end{aligned}$$

Hence T_φ is $\mathcal{A}$-linear and $\mathbb{C}$-linear.

On the other hand, T is bounded on $L^1(G, \mathcal{M})$ since

$$\|T_\varphi f\|_{L^1(G,\mathcal{A})} \leq \|\varphi\|_1 \|f\|_1. \qquad \square$$

The operator norm of the operator T_φ is defined by

$$\|T_\varphi\| = \sup\{\|T_\varphi f\|_{L^1(G,\mathcal{A})} : \|f\|_1 \leq 1\}.$$

From Theorem 3.5, we deduce:

Theorem 5.2. *Let G is a locally compact group. If $\varphi \in L^1(G, \mathcal{M})$ and $g \in \mathcal{C}_c(G, \mathcal{M})$ then*

$$\|T_\varphi g\|_{L^1(G,\mathcal{A})} \leq \|g\|_\infty \|\varphi\|_1 \ \text{ and } \ \|T_\varphi\| \leq \|\varphi\|_1.$$

Theorem 5.3. *Let G be a compact group and let φ be in $L^2(G, \mathcal{M})$. If $f \in L^2(G, \mathcal{M})$ then $T_\varphi f \in L^2(G, \mathcal{A})$.*

Proof. Let $\varphi, f \in L^2(G, \mathcal{M})$. Set $X = \int_G \|T_\varphi f(x)\|_{\mathcal{A}}^2 d\lambda(x)$. Then

$$\begin{aligned} X &= \int_G \left\| \int_G \left\langle \varphi\left(tx^{-1}\right), f(t)\right\rangle_{\mathcal{M}} d\lambda(t) \right\|_{\mathcal{A}}^2 d\lambda(x) \\ &\leq \int_G \left(\int_G \left\| \left\langle \varphi\left(tx^{-1}\right), f(t)\right\rangle_{\mathcal{M}} d\lambda(t) \right\|_{\mathcal{A}} \right)^2 d\lambda(x) \end{aligned}$$

$$\begin{aligned}
&\leq \int_G \left(\int_G \left\| \varphi\left(tx^{-1}\right)\right\|_{\mathcal{M}} \|f(t)\|_{\mathcal{M}}\, d\lambda(t) \right)^2 d\lambda(x) \\
&\leq \int_G \left(\int_G \left\| \varphi\left(tx^{-1}\right)\right\|_{\mathcal{M}}^2 d\lambda(t) \right) \left(\int_G \|f(t)\|_{\mathcal{M}}^2\, d\lambda(t) \right) d\lambda(x) \\
&\leq \|f\|_2^2 \int_G \int_G \left\| \varphi\left(tx^{-1}\right)\right\|_{\mathcal{M}}^2 d\lambda(t)\, d\lambda(x) \\
&= \|f\|_2^2 \int_G \left(\int_G \|\varphi(y)\|_{\mathcal{M}}^2\, d\lambda(y) \right) d\lambda(x) \quad \left(\text{ set } y = tx^{-1} \right) \\
&= \|f\|_2^2 \|\varphi\|_2^2\, \lambda(G) \\
&= \|f\|_2^2 \|\varphi\|_2^2 < +\infty.
\end{aligned}$$

Hence $T_\varphi f \in L^2(G, \mathcal{A})$. □

Theorem 5.4. *Assume that G is a compact group. If $\varphi \in L^2(G, \mathcal{M})$ then $\|T_\varphi\| \leq \|\varphi\|_2$.*

Proof. Let $f \in L^1(G, \mathcal{M})$. Set $X = \|T_\varphi f\|_{L^1(G,\mathcal{A})}$. Then

$$\begin{aligned}
X &= \int_G \|T_\varphi f(x)\|_{\mathcal{A}}\, d\lambda(x) \\
&= \int_G \left\| \int_G \left\langle \varphi\left(tx^{-1}\right), f(t) \right\rangle_{\mathcal{M}} d\lambda(t) \right\|_{\mathcal{A}} d\lambda(x) \\
&\leq \int_G \left(\int_G \left\| \left\langle \varphi\left(tx^{-1}\right), f(t) \right\rangle_{\mathcal{M}} \right\|_{\mathcal{A}} d\lambda(t) \right) d\lambda(x) \\
&\leq \int_G \left(\int_G \left\| \varphi\left(tx^{-1}\right)\right\|_{\mathcal{M}} \|f(t)\|_{\mathcal{M}}\, d\lambda(t) \right) d\lambda(x) \\
&\leq \int_G \left(\int_G \left\| \varphi\left(tx^{-1}\right)\right\|_{\mathcal{M}}^2 d\lambda(t) \right)^{\frac{1}{2}} \left(\int_G \|f(t)\|_{\mathcal{M}}^2\, d\lambda(t) \right)^{\frac{1}{2}} d\lambda(x) \\
&= \|\varphi\|_2 \|f\|_2 .
\end{aligned}$$

Therefore $\|T_\varphi\| \leq \|\varphi\|_2$. □

Conclusion

In this paper, we introduced a convolution product in the framework of Hilbert C^*-modules. Many properties are obtained among which are convolution theorems and the boundedness of the related convolution operator.

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