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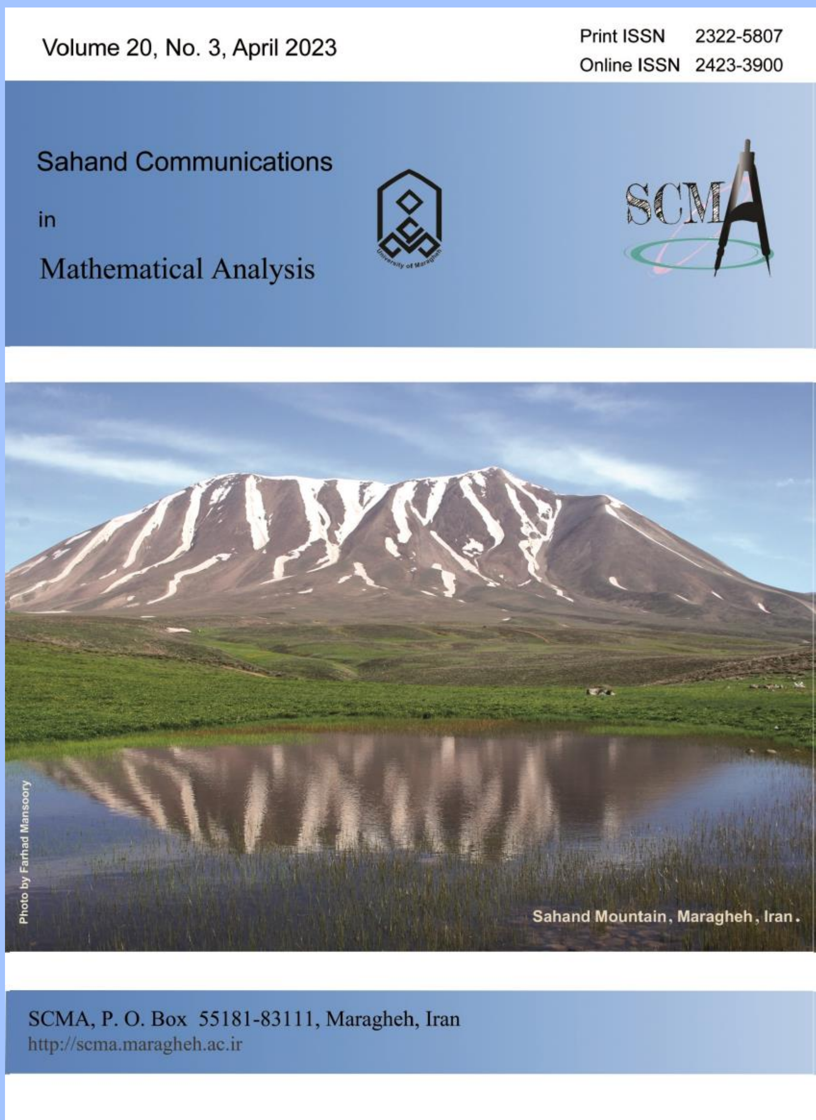

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Product-type Operators Between Minimal Möbius Invariant Spaces and Zygmund Type Spaces

Mostafa Hassanlou[1], Ebrahim Abbasi[2*], Mehdi Kanani Arpatapeh[3] and Sepideh Nasresfahani[4]

Abstract. In this work, we consider product-type operators $T^m_{u,v,\varphi}$ from minimal Möbius invariant spaces into Zygmund-type spaces. Firstly, some characterizations for the boundedness of these operators are given. Then some estimates of the essential norms of these operators are obtained. Therefore, some compactness conditions will be given.

1. Introduction

By $\mathbb{D}$ being the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ is denoted as the space of all analytic functions on $\mathbb{D}$. The classic Zygmund space $\mathcal{Z}$ consists of all functions $f \in H(\mathbb{D})$ which are continuous on the closed unit ball $\overline{\mathbb{D}}$ and

$$\sup \frac{\left|f\left(e^{i(\theta+h)}\right)+f\left(e^{i(\theta-h)}\right)-2f\left(e^{i\theta}\right)\right|}{h}<\infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h>0$. By [4, Theorem 5.3], an analytic function f belongs to $\mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f''(z)|<\infty$. Motivated by this, for each $\alpha>0$, the Zygmund type space $\mathcal{Z}_\alpha$ is defined to be the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{s\mathcal{Z}_\alpha}=\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|f''(z)\right|<\infty.$$

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The space $\mathcal{Z}_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + \left|f'(0)\right| + \|f\|_{s\mathcal{Z}_\alpha}, \quad \forall f \in \mathcal{Z}_\alpha.$$

Let $Aut(\mathbb{D})$ be the group of all conformal automorphisms of $\mathbb{D}$ which is also called the Möbius group. It is well-known that each element of $Aut(\mathbb{D})$ is of the form

$$e^{i\theta}\sigma_a(z) = e^{i\theta}\frac{a-z}{1-\bar{a}z}, \quad a, z \in \mathbb{D}, \theta \in \mathbb{R}.$$

Let X be a linear space of analytic functions on $\mathbb{D}$, which is complete. X is called Möbius invariant if for each function f in X and each element ψ in $Aut(\mathbb{D})$, the composition function $f \circ \psi$ also lies in X and satisfies that $\|f \circ \psi\|_X = \|f\|_X$. For example, the space H^∞ of all bound analytic functions are Möbius invariant. Also, the Besov space $B_p(1 < p < \infty)$, is Möbius invariant which is the space of all $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^{p-2} dA(z) < \infty.$$

If $p = 2$, we have the well-known Dirichlet space. For $p = \infty$, $B_\infty = \mathcal{B}$ is the classic Bloch space. Space B_1 which is called the minimal Möbius invariant is defined separately. The function $f \in H(\mathbb{D})$ belongs to B_1 if and only if it has representation as $f(z) = \sum\limits_{k=1}^{\infty} c_k\sigma_{a_k}$ where $a_k \in \mathbb{D}$ and $\sum\limits_{k=1}^{\infty} |c_k| < \infty$. The norm on B_1 is defined as infimum of $\sum\limits_{k=1}^{\infty} |c_k|$ for which the above statement holds. B_1 is contained in any Möbius invariant space and it has been proved that it is the set of all analytic functions f on $\mathbb{D}$ such that f'' lies in $L^1(\mathbb{D}, dA)$. Also, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 \|f\|_{B_1} &\leq |f(0)| + \left|f'(0)\right| + \int_{\mathbb{D}} \left|f''(z)\right| dA(z) \\ &\leq C_2 \|f\|_{B_1}. \end{aligned}$$

Let $u, v, \varphi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$. The Stević-Sharma type operator is defined as follows

$$T_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Indeed $T_{u,v,\varphi} = uC_\varphi + vC_\varphi D$ where D is the differentiation operator and C_φ is the composition operator. More information about this operator can be found in [2, 3, 5–8, 11].

The generalized Stević-Sharma type operator $T^m_{u,v,\varphi}$ is defined by the second author of this paper and et al. in [1] as follows

$$T^m_{u,v,\varphi}f(z) = (uC_\varphi f)(z) + \left(D^m_{\varphi,v}f\right)(z)$$

$$= u(z)f(\varphi(z)) + v(z)f^{(m)}(\varphi(z)),$$

where $m \in \mathbb{N}$ and $D^m_{\varphi,u}$ is the generalized weighted composition operator. When $v = 0$, then $T^m_{u,0,\varphi} = uC_\varphi$ is the well-known weighted composition operator. If $u = 0$, then $T^m_{0,v,\varphi} = D^m_{\varphi,v}$ and for $m = 1$, $T^m_{u,v,\varphi} = T_{u,v,\varphi}$ is Stević-Sharma type operator. $T^m_{u,v,\varphi}$ also includes other operators as well as product type operators which have been studied in several papers in recent years. The results of the papers can be explained by many operators and obtained from the results of the papers published before.

For Banach spaces X and Y and a continuous linear operator $T : X \to Y$, the essential norm is the distance of T from the space of all compact operators, that is $\|T\|_e = \inf\{\|T - K\| : K : X \to Y \text{ is compact}\}$, so T is compact if and only if $\|T\|_e = 0$.

In this paper, we study the operator-theoretic properties in minimal Möbius invariant space. In Section 2, we first bring some lemmas on the space B_1 and then obtain some characterizations for the boundedness of operator $T^m_{u,v,\varphi} : B_1 \to \mathcal{Z}_\alpha$. In Section 3, some estimations of the essential norms of these operators are given. As a result, some new criteria for the compactness of $T^m_{u,v,\varphi}$ are presented.

By $A \succeq B$ we mean there exists a constant C such that $A \geq CB$ and $A \approx B$ means that $A \succeq B \succeq A$.

2. Boundedness

In this section, we give some necessary and sufficient conditions for the generalized Stević-Sharma type operators to be bound. Firstly, we state some lemmas which are needed for proving the main results.

According to the definition of the norm in minimal Möbius invariant space, for each $f \in B_1$, $\|f\|_\infty \leq \|f\|_{B_1}$. Thus, from [9, Proposition 5.1.2] and [10, Proposition 8] we have the following lemma.

Lemma 2.1. *Let $n \in \mathbb{N}$. Then there exists a positive constant C such that for each $f \in B_1$, $\left(1 - |z|^2\right)^n \left|f^{(n)}(z)\right| \leq C \|f\|_{B_1}$.*

As a similar proof in Lemma 2.5 of [12] we get the following lemma.

Lemma 2.2. *Let*

$$f_{j,a}(z) = \left(\frac{1 - |a|^2}{1 - \overline{a}z}\right)^j, \quad j \in \mathbb{N}, a \in \mathbb{D}.$$

Then $f_{j,a} \in B_1$, $\sup_{a\in\mathbb{D}} \|f_{j,a}\|_{B_1} < \infty$.

Moreover, $\{f_{j,a}\} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$.

The proof of the following lemmas are similar to the proof of Lemmas 2.6 and 2.5 [1], so they are omitted.

Lemma 2.3. *For any $m \in \mathbb{N}-\{1,2\}$, $0 \neq a \in \mathbb{D}$ and $i,k \in \{0,1,2,m,m+1,m+2\}$, there exists a function $g_{i,a} \in B_1$ such that*

$$g_{i,a}^{(k)}(a) = \frac{\delta_{ik}\overline{a}^k}{(1-|a|^2)^k},$$

where δ_{ik} is Kronecker delta. For each $i \in \{0,1,2\}$ and $i \in \{m,m+1,m+2\}$, respectively

$$g_{i,a}(z) = \sum_{j=1}^{3} c_j^i f_{j,a}(z), \qquad g_{i,a}(z) = \sum_{j=m+1}^{m+3} c_j^i f_{j,a}(z),$$

where c_j^i is independent of a.

Lemma 2.4. *Let $m = 1$ or 2, $0 \neq a \in \mathbb{D}$ and $i,k \in \{0,1,\dots,m+2\}$, there exists a function $g_{i,a} \in B_1$ such that*

$$g_{i,a}^{(k)}(a) = \frac{\delta_{ik}\overline{a}^k}{(1-|a|^2)^k}.$$

Let $f \in B_1$. Then

$$\begin{aligned}\left\|T_{u,v,\varphi}^m f\right\|_{\mathcal{Z}_\alpha} = &\left|T_{u,v,\varphi}^m f(0)\right| \\ &+ \left|(T_{u,v,\varphi}^m f)'(0)\right| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|(T_{u,v,\varphi}^m f)''(z)\right|.\end{aligned}$$

We compute the above sentences separately. We have

$$\begin{aligned}\left(T_{u,v,\varphi}^m f\right)'(0) =&u'(0)f(\varphi(0)) + u(0)\varphi'(0)f'(\varphi(0)) + v'(0)f^{(m)}(\varphi(0)) \\ &+ v(0)\varphi'(0)f^{(m+1)}(\varphi(0)),\end{aligned}$$

and

$$\left(T_{u,v,\varphi}^m f\right)''(z) = \sum_{i=0}^{2}\left(I_i(z)f^{(i)}(\varphi(z)) + I_{i+m}(z)f^{(i+m)}(\varphi(z))\right),$$

where,

$$\begin{aligned}I_0 &= u'', & I_1 &= 2u'\varphi' + u\varphi'', & I_2 &= u\varphi'^2, \\ I_m &= v'', & I_{m+1} &= 2v'\varphi' + v\varphi'', & I_{m+2} &= v\varphi'^2.\end{aligned}$$

Theorem 2.5. *Let $\alpha > 0$, $u,v,\varphi \in H(\mathbb{D})$, $\varphi : \mathbb{D} \to \mathbb{D}$ and $m > 2$ be an integer. Then the following conditions are equivalent:*

(i) *The operator $T_{u,v,\varphi}^m : B_1 \to \mathcal{Z}_\alpha$ is bounded.*

(ii) *For each $j \in \{0,1,2,m,m+1,m+2\} = \mathfrak{Q}$*

$$\max\left\{\sup_{a\in\mathbb{D}}\left\|T_{u,v,\varphi}^m f_{j,a}\right\|_{\mathcal{Z}_\alpha},\ \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha |I_j(z)|\right\} < \infty.$$

(iii)
$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^{\alpha}|I_k(z)|}{\left(1-|\varphi(z)|^2\right)^k} < \infty, \quad k \in \mathfrak{Q}.$$

Proof. $(iii) \Rightarrow (i)$ Suppose that $f \in B_1$. By using Lemma 2.1

$$\begin{aligned}\left(1-|z|^2\right)^{\alpha}\left|(T^m_{u,v,\varphi}f)''(z)\right| &= \left(1-|z|^2\right)^{\alpha}\left|\sum_{k\in\mathfrak{Q}} I_k(z) f^{(k)}(\varphi(z))\right| \\ &\leq \sum_{k\in\mathfrak{Q}} |I_k(z)|\left(1-|z|^2\right)^{\alpha}\left|f^{(k)}(\varphi(z))\right| \\ &\leq C\sum_{k\in\mathfrak{Q}} \frac{\left(1-|z|^2\right)^{\alpha}|I_k(z)|}{\left(1-|\varphi(z)|^2\right)^k}\|f\|_{B_1}.\end{aligned}$$

Also, using the fact that $\|f\|_\infty \leq \|f\|_{B_1}$ and Lemma 2.1, we have

$$\begin{aligned}\left|T^m_{u,v,\varphi}f(0)\right| &\leq |u(0)f(\varphi(0))| + \left|v(0)f^{(m)}(\varphi(0))\right| \\ &\leq |u(0)|\,\|f\|_{B_1} + C\frac{|v(0)|}{(1-|\varphi(0)|^2)^m}\|f\|_{B_1},\end{aligned}$$

and

$$\begin{aligned}\left|\left(T^m_{u,v,\varphi}f\right)'(0)\right| \leq C\Bigg(&|u'(0)| + \frac{|u(0)\varphi'(0)|}{1-|\varphi(0)|^2} \\ &+\frac{|v'(0)|}{(1-|\varphi(0)|^2)^m} + \frac{|v(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{m+1}}\Bigg)\|f\|_{B_1}.\end{aligned}$$

Therefore $T^m_{u,v,\varphi}: B_1 \to \mathcal{Z}_\alpha$ is bounded.

$(i) \Rightarrow (ii)$ Suppose that $T^m_{u,v,\varphi}: B_1 \to \mathcal{Z}_\alpha$ be a bounded operator. Lemma 2.2 implies that $\|f_{j,a}\|_{B_1} < \infty$. So

$$\left\|T^m_{u,v,\varphi}f_{j,a}\right\|_{\mathcal{Z}_\alpha} \leq \left\|T^m_{u,v,\varphi}\right\|\,\|f_{j,a}\|_{B_1} < \infty.$$

Then

$$\begin{aligned}\sup_{a\in\mathbb{D}}\left\|T^m_{u,v,\varphi}f_{j,a}\right\|_{\mathcal{Z}_\alpha} &\leq \left\|T^m_{u,v,\varphi}\right\|\sup_{a\in\mathbb{D},j\in\mathfrak{Q}}\|f_{j,a}\|_{B_1} \\ &< \infty.\end{aligned}$$

Define $f_0(z) = 1 \in B_1$. The boundedness of the operator implies that

$$\begin{aligned}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}|I_0(z)| &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}\left|u''(z)\right| \\ &\leq \|T^m_{u,v,\varphi}f_0\|_{\mathcal{Z}_\alpha} \\ &\leq \|T^m_{u,v,\varphi}\|\|f_0\|_{B_1} \\ &< \infty.\end{aligned}$$

Take $f_1(z) = z \in B_1$. Then, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|u''(z)\varphi(z) + 2u'(z)\varphi'(z) + u(z)\varphi''(z)\right| &\le \|T^m_{u,v,\varphi} f_1\|_{\mathcal{Z}_\alpha} \\ &\le \|T^m_{u,v,\varphi}\| \|f_1\|_{B_1} \\ &< \infty.\end{aligned}$$

Using the previous equations, we can get that

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |I_1(z)| < \infty.$$

Similarly by employing the functions $f_2(z) = z^2$, $f_m(z) = z^m$, $f_{m+1}(z) = z^{m+1}$ and $f_{m+2}(z) = z^{m+2}$ for the operator $T^m_{u,v,\varphi}$ we get the other part of (ii).

$(ii) \Rightarrow (iii)$ For any, $i \in \mathfrak{Q}$ and $a \in \mathbb{D}$, by applying Lemma 2.3, we have

$$\begin{aligned}\frac{\left(1-|z|^2\right)^\alpha |I_i(a)||\varphi(a)|^i}{(1-|\varphi(a)|^2)^i} &\le \left(1-|z|^2\right)^\alpha |(T^m_{u,v,\varphi} g_{i,\varphi(a)})''(a)| \\ &\le \sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} g_{i,\varphi(a)}\|_{\mathcal{Z}_\alpha} \\ &\le \sum_{j=1}^{3} c^i_j \sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathcal{Z}_\alpha}, \\ &\quad + \sum_{j=m+1}^{m+3} c^i_j \sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathcal{Z}_\alpha} \\ &< \infty.\end{aligned}$$

So, for any $i \in \mathfrak{Q}$

$$\sup_{|\varphi(a)|>1/3} \frac{\left(1-|z|^2\right)^\alpha |I_i(a)|}{\left(1-|\varphi(a)|^2\right)^i} < \infty.$$

On the other hand

$$\begin{aligned}\sup_{|\varphi(a)|\le 1/3} \frac{\left(1-|z|^2\right)^\alpha |I_i(a)|}{\left(1-|\varphi(a)|^2\right)^i} &\le C \sup_{a\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |I_i(a)| \\ &< \infty.\end{aligned}$$

From the last inequalities, we get the desired result. □

In the special case $m \le 2$, by using Lemma 2.4, we have the following theorems which are stated without proof.

Theorem 2.6. *Let $\alpha > 0$, $u, v, \varphi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$. Then the following conditions are equivalent:*

(i) *The operator $T^2_{u,v,\varphi} : B_1 \to \mathcal{Z}_\alpha$ is bounded.*

(ii) *For* $j \in \{1, ..., 5\}$, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathcal{Z}_\alpha} < \infty$ *and*

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(|u''(z)| + |(2u'\varphi' + u\varphi'')(z)| + |(u\varphi'^2 + v'')(z)|\right) < \infty,$$
$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(|(2v'\varphi' + v\varphi'')(z)| + |(v\varphi'^2)(z)|\right) < \infty.$$

(iii)

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(|u''(z)| + \frac{|(2u'\varphi' + u\varphi'')(z)|}{1-|\varphi(z)|^2} + \frac{|(u\varphi'^2 + v'')(z)|}{(1-|\varphi(z)|^2)^2}\right) < \infty,$$
$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(\frac{|(2v'\varphi' + v\varphi'')(z)|}{(1-|\varphi(z)|^2)^3} + \frac{|(v\varphi'^2)(z)|}{(1-|\varphi(z)|^2)^4}\right) < \infty.$$

Theorem 2.7. *Let* $\alpha > 0$, $u, v, \varphi \in H(\mathbb{D})$ *and* $\varphi : \mathbb{D} \to \mathbb{D}$. *Then the following conditions are equivalent:*

(i) *The operator* $T_{u,v,\varphi} : B_1 \to \mathcal{Z}_\alpha$ *is bounded.*
(ii) *For* $j \in \{1, \ldots, 4\}$, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{j,a}\|_{\mathcal{Z}_\alpha} < \infty$ *and*

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(|u''(z)| + |(2u'\varphi' + u\varphi'' + v'')(z)|\right) < \infty,$$
$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(|(u\varphi'^2 + 2v'\varphi' + v\varphi'')(z)| + |(v\varphi'^2)(z)|\right) < \infty.$$

(iii)

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(|u''(z)| + \frac{|(2u'\varphi' + u\varphi'' + v'')(z)|}{1-|\varphi(z)|^2}\right) < \infty,$$
$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left(\frac{|(u\varphi'^2 + 2v'\varphi' + v\varphi'')(z)|}{(1-|\varphi(z)|^2)^2} + \frac{|(v\varphi'^2)(z)|}{(1-|\varphi(z)|^2)^3}\right) < \infty.$$

3. Essential Norm

In this section, some estimations for the essential norm of the operator $T^m_{u,v,\varphi}$ from minimal Möbius invariant spaces into Zygmund-type spaces are given.

Theorem 3.1. *Let* $u, v, \varphi \in H(\mathbb{D})$, $\varphi : \mathbb{D} \to \mathbb{D}$ *and* $2 < m \in \mathbb{N}$. *Let the operator* $T^m_{u,v,\varphi} : B_1 \to \mathcal{Z}_\alpha$ *be bounded. Then*

$$\begin{aligned}\left\|T^m_{u,v,\varphi}\right\|_e &\approx \max\{E_i\}_{i=1}^6 \\ &\approx \max\{F_k\}_{k\in\{0,1,2,m,m+1,m+2\}},\end{aligned}$$

where,

$$E_i = \limsup_{|a|\to 1} \left\|T^m_{u,v,\varphi} f_{i,a}\right\|_{\mathcal{Z}_\alpha}, \qquad F_k = \limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)^\alpha |I_k(z)|}{(1-|\varphi(z)|^2)^k}.$$

Proof. First, we prove the lower estimates. Suppose that $K : B_1 \to \mathcal{Z}_\alpha$ be an arbitrary compact operator. Since $\{f_{i,a}\}$ is a bounded sequence in B_1 and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$, we have $\limsup_{|a|\to 1} \|Kf_{i,a}\|_{\mathcal{Z}_\alpha} = 0$. So

$$\begin{aligned}\left\|T^m_{u,v,\varphi} - K\right\|_{B_1\to\mathcal{Z}_\alpha} &\succeq \limsup_{|a|\to 1} \left\|(T^m_{u,v,\varphi} - K)f_{i,a}\right\|_{\mathcal{Z}_\alpha}\\ &= E_i.\end{aligned}$$

Then

$$\begin{aligned}\left\|T^m_{u,v,\varphi}\right\|_e &= \inf_K \left\|T^m_{u,v,\varphi} - K\right\|_{B_1\to\mathcal{Z}_\alpha}\\ &\succeq \max\{E_i\}_{i=1}^6 .\end{aligned}$$

For the other part, let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Since $T^m_{u,v,\varphi} : B_1 \to \mathcal{Z}_\alpha$ is bounded, using Lemmas 2.2 and 2.3 for any compact operator $K : B_1 \to \mathcal{Z}_\alpha$ and $i \in \{0, 1, 2, m, m+1, m+2\}$, we obtain

$$\begin{aligned}\|T^m_{u,v,\varphi} - K\|_{B_1\to\mathcal{Z}_\alpha} &\succeq \limsup_{j\to\infty} \|T^m_{u,v,\varphi}(g_{i,\varphi(z_j)})\|_{\mathcal{Z}_\alpha} - \limsup_{j\to\infty} \|K(g_{i,\varphi(z_j)})\|_{\mathcal{Z}_\alpha}\\ &\succeq \limsup_{j\to\infty} \frac{(1-|z_j|^2)^\alpha |\varphi(z_j)|i|I_i(z_j)|}{(1-|\varphi(z_j)|^2)^i}\\ &= F_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\|T^m_{u,v,\varphi}\|_e &= \inf_K \|T^m_{u,v,\varphi} - K\|_{B_1\to\mathcal{Z}_\alpha}\\ &\succeq \max\{F_i\}.\end{aligned}$$

Now, we prove the upper estimates. Consider the operators K_r on B_1, $K_r f(z) = f_r(z) = f(rz)$, where $0 < r < 1$. K_r is a compact operator and $\|K_r\| \le 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. For any positive integer j, the operator $T^m_{u,v,\varphi}K_{r_j} : B_1 \to \mathcal{Z}_\alpha$ is compact. Thus $\|T^m_{u,v,\varphi}\|_e \le \limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j}\|$.

So it will be sufficient to prove that

$$\limsup_{j\to\infty} \|T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j}\| \preceq \min\{\max\{E_i\}, \max\{F_i\}\}.$$

For any $f \in B_1$ such that $\|f\|_{B_1} \le 1$,

$$\left\|(T^m_{u,v,\varphi} - T^m_{u,v,\varphi}K_{r_j})f\right\|_{\mathcal{Z}_\alpha} = A_1 + A_2 + A_3,$$

where

$$\begin{aligned}A_1 &=: \left|T^m_{u,v,\varphi}f(0) - T^m_{u,v,\varphi}f_{r_j}(0)\right|\\ &= \left|u(0)(f - f_{r_j})(\varphi(0)) + v(0)(f - f_{r_j})^{(m)}(\varphi(0))\right|,\end{aligned}$$

$$
\begin{aligned}
A_2 :=& \left|(T^m_{u,v,\varphi}f - T^m_{u,v,\varphi}f_{r_j})'(0)\right| \\
=& \Big|u'(0)(f-f_{r_j})(\varphi(0)) + u(0)\varphi'(0)(f-f_{r_j})'(\varphi(0)) \\
&+ v'(0)(f-f_{r_j})^{(m)}(\varphi(0)) + v(0)\varphi'(0)(f-f_{r_j})^{(m+1)}(\varphi(0))\Big|
\end{aligned}
$$

$$
\begin{aligned}
A_3 :=& \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|(T^m_{u,v,\varphi}f - T^m_{u,v,\varphi}f_{r_j})''(z)\right| \\
=& \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|\sum_{k\in\{0,1,2,m,m+1,m+2\}} I_k(z)(f-f_{r_j})^{(k)}(\varphi(z))\right| \\
\le& \sup_{|\varphi(z)|\le r_N} \left(1-|z|^2\right)^\alpha \sum_{k\in\{0,1,2,m,m+1,m+2\}} \left|I_k(z)(f-f_{r_j})^{(k)}(\varphi(z))\right| \\
&+ \sup_{|\varphi(z)|> r_N} \left(1-|z|^2\right)^\alpha \sum_{k\in\{0,1,2,m,m+1,m+2\}} \left|I_k(z)(f-f_{r_j})^{(k)}(\varphi(z))\right|, \\
:=& A_4 + A_5,
\end{aligned}
$$

Since $(f-f_{r_j})^{(i)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, for any nonnegative integer i, then using Theorem 2.5, we get

$$
\begin{aligned}
\limsup_{j\to\infty} A_1 &= \limsup_{j\to\infty} A_2 \\
&= \limsup_{j\to\infty} A_4 \\
&= 0.
\end{aligned}
$$

About A_5, we get

$$
\begin{aligned}
A_5 \le& \sum_{k\in\{0,1,2,m,m+1,m+2\}} \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)^\alpha |I_k(z)| \left|f^{(k)}(\varphi(z))\right| \\
&+ \sum_{k\in\{0,1,2,m,m+1,m+2\}} \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right)^\alpha |I_k(z)| \left|r_j^k f^{(k)}(r_j\varphi(z))\right| \\
=& \sum_{k\in\{0,1,2,m,m+1,m+2\}} A_{k,6} + \sum_{k\in\{0,1,2,m,m+1,m+2\}} A_{k,7}.
\end{aligned}
$$

For $A_{k,6}$, using Lemmas 2.1, 2.2 and 2.3, we obtain

$$
\begin{aligned}
A_{k,6} =& \sup_{|\varphi(z)|>r_N} \frac{(1-|\varphi(z)|^2)^k|f^{(k)}(\varphi(z))|}{|\varphi(z)|^k} \frac{\left(1-|z|^2\right)^\alpha |I_k(z)||\varphi(z)|^k}{(1-|\varphi(z)|^2)^k} \\
\le& \|f\|_{B_1} \sup_{|\varphi(z)|>r_N} \|(T^m_{u,v,\varphi}g_{k,\varphi(z)}\|_{\mathcal{Z}_\alpha}
\end{aligned}
$$

$$\preceq \sum_{j=1}^{6} c_j^k \sup_{|a|>r_N} \|T_{u,v,\varphi}^m f_{j,a}\|_{\mathcal{Z}_\alpha},$$

where $k \in \{0, 1, 2, m, m+1, m+2\}$. As $N \to \infty$,

$$\begin{aligned}\limsup_{j\to\infty} A_{k,6} &\preceq \sum_{j=1}^{6} \limsup_{|a|\to 1} \|T_{u,v,\varphi}^m f_{j,a}\|_{\mathcal{Z}_\alpha} \\ &\preceq \max\{E_j\}_{j=1}^{6}.\end{aligned}$$

Also, for $A_{k,6}$, we can write

$$\begin{aligned}A_{k,6} &= \sup_{|\varphi(z)|>r_N} \left(1-|\varphi(z)|^2\right)^k \left|f^{(k)}(\varphi(z))\right| \frac{\left(1-|z|^2\right)^\alpha |I_k(z)|}{\left(1-|\varphi(z)|^2\right)^k} \\ &\preceq \|f\|_{B_1} \sup_{|\varphi(z)|>r_N} \frac{\left(1-|z|^2\right)^\alpha |I_k(z)|}{\left(1-|\varphi(z)|^2\right)^k},\end{aligned}$$

which can be deduced that

$$\begin{aligned}\limsup_{j\to\infty} A_{k,6} &\preceq \limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)^\alpha |I_k(z)|}{\left(1-|\varphi(z)|^2\right)^k} \\ &\leq \max\{F_k\}_1^6.\end{aligned}$$

A similar argument can be made for $A_{k,7}$. Thus, we prove that

$$\sup_{\|f\|_{B_1}\leq 1} \|(T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j})f\|_{\mathcal{Z}_\alpha} \preceq \max\{E_j\}_1^6,$$

and

$$\sup_{\|f\|_{B_1}\leq 1} \|(T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j})f\|_{\mathcal{Z}_\alpha} \preceq \max\{F_k\}_1^6.$$

Finally, we have

$$\limsup_{j\to\infty} \|T_{u,v,\varphi}^m - T_{u,v,\varphi}^m K_{r_j}\| \preceq \min\left\{\max\{E_i\}_1^6, \max\{F_k\}_1^6\right\}. \qquad \square$$

In the case $m \leq 2$, a similar result can be stated using Theorems 2.6 and 2.7.

Theorem 3.2. *Let $\alpha > 0$, $u, v, \varphi \in H(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ and the operator $T_{u,v,\varphi}^2 : B_1 \to \mathcal{Z}_\alpha$ be bounded. Then*

$$\begin{aligned}&\|T_{u,v,\varphi}^2\|_e \\ &\approx \max\left\{\limsup_{|a|\to 1} \|T_{u,v,\varphi}^2 f_{j,a}\|_{\mathcal{Z}_\alpha}\right\}_1^5 \\ &\approx \limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right)^\alpha \left(|u''(z)| + \frac{|(2u'\varphi' + u\varphi'')(z)|}{(1-|\varphi(z)|^2)} + \frac{|(u\varphi'^2 + v'')(z)|}{(1-|\varphi(z)|^2)^2}\right)\end{aligned}$$

$$+\limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)^\alpha\left(\frac{|(2v'\varphi'+v\varphi'')(z)|}{(1-|\varphi(z)|^2)^3}+\frac{|(v\varphi'^2)(z)|}{(1-|\varphi(z)|^2)^4}\right).$$

Theorem 3.3. *Let $\alpha>0$, $u,v,\varphi\in H(\mathbb{D})$ and $\varphi:\mathbb{D}\to\mathbb{D}$ and the operator $T_{u,v,\varphi}:B_1\to\mathcal{Z}_\alpha$ be bounded. Then*

$$\|T_{u,v,\varphi}\|_e$$
$$\approx \max\left\{\limsup_{|a|\to 1}\|T_{u,v,\varphi}f_{j,a}\|_{\mathcal{Z}_\alpha}\right\}_1^4$$
$$\approx \limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)^\alpha\left(|u''(z)|+\frac{|(2u'\varphi'+u\varphi''+v'')(z)|}{(1-|\varphi(z)|^2)}\right)$$
$$+\limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)^\alpha\left(\frac{|(u\varphi'^2+2v'\varphi'+v\varphi'')(z)|}{(1-|\varphi(z)|^2)^2}+\frac{|(v\varphi'^2)(z)|}{(1-|\varphi(z)|^2)^3}\right).$$

By using Theorem 3.1, we have the following Corollary.

Corollary 3.4. *Let $u,v,\varphi\in H(\mathbb{D})$, $\varphi:\mathbb{D}\to\mathbb{D}$ and $2<m\in\mathbb{N}$. Let operator $T^m_{u,v,\varphi}:B_1\to\mathcal{Z}_\alpha$ be bounded. Then the following conditions are equivalent:*

(i) *The operator $T^m_{u,v,\varphi}:B_1\to\mathcal{Z}_\alpha$ is compact.*
(ii) $\limsup_{|a|\to 1}\|T^m_{u,v,\varphi}f_{i,a}\|_{\mathcal{Z}_\alpha}=0,\quad i=1,2,\dots,6$
(iii) $\limsup_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)^\alpha|I_k(z)|}{(1-|\varphi(z)|^2)^k}=0,\quad k\in\{0,1,2,m,m+1,m+2\}.$

Remark 3.5. From Theorems 3.2 and 3.3, we obtain similar results for the compactness of operator $T^2_{u,v,\varphi}:B_1\to\mathcal{Z}_\alpha$ and $T_{u,v,\varphi}:B_1\to\mathcal{Z}_\alpha$, respectively.

Remark 3.6. By taking $u=0\,(v=0)$, we can get the results of the paper for generalized (weighted) composition operators.

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