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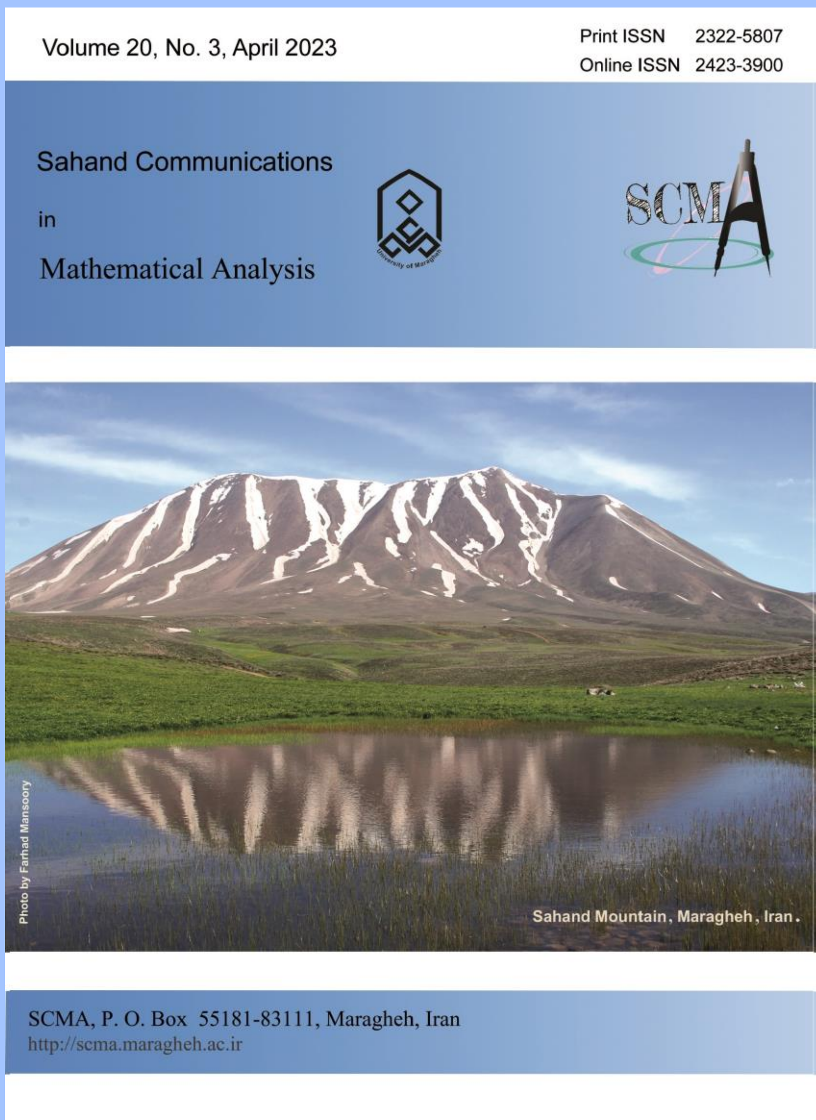

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Ostrowski Type Inequalities for n-Times Strongly m–MT-Convex Functions

Badreddine Meftah[1]* and Chayma Marrouche[2]

Abstract. In this paper, we introduce the class of strongly m–MT-convex functions based on the identity given in [P. Cerone et al., 1999]. We establish new inequalities of the Ostrowski-type for functions whose n^{th} derivatives are strongly m–MT-convex functions.

1. Introduction

In 1938, A. M. Ostrowski proved a significant integral inequality, given by the following theorem.

Theorem 1.1 ([24]). *Let $F : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior I° of I, and $\ell, \bar{e} \in I^\circ$, with $\ell < \bar{e}$. If $|F'| \leq M$ for all $x \in [\ell, \bar{e}]$, then*

$$\left|F(x) - \tfrac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}} F(t)\,dt\right| \leq M(\bar{e}-\ell)\left[\frac{1}{4} + \frac{\left(x-\frac{\ell+\bar{e}}{2}\right)^2}{(\bar{e}-\ell)^2}\right]. \tag{1.1}$$

In recent decades, inequality (1.1) has attracted much interest from many researchers. Considerable papers have appeared on the generalizations, variants and extensions of inequality (1.1).

Concerning some recent papers on integral inequalities, we refer to readers [1, 3–11, 13–22, 27, 31], and references therein.

In [20], Milovanović and Pěcarić gave the generalization of Theorem 1.1 when $\left|F^{(n)}\right| \leq M$.

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Theorem 1.2 ([20]). *Let $F : \mathbb{R} \to \mathbb{R}$ be n times differentiable function such that $\left|F^{(n)}\right| \leq M$ with $n > 1$. Then, for every $x \in [\ell, \bar{e}]$ one has*

$$\left|\frac{1}{n}\left(F(x)+\sum_{k=1}^{n-1}\frac{(n-k)\left(F^{(k-1)}(\ell)(x-\ell)^{k}+F^{(k-1)}(\bar{e})(\bar{e}-x)^{k}\right)}{(b-\ell)k!}\right)-\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}F(t)\,dt\right|$$
$$\leq \frac{(x-\ell)^{n+1}+(\bar{e}-x)^{n+1}}{(\bar{e}-\ell)n(n+1)!}M.$$

Recently, Tunç [30], gave the following results concerning Ostrowski's inequality for differentiable MT-convex functions.

Theorem 1.3 ([30]). *Let $F : [\ell, \bar{e}] \to \mathbb{R}$ be a differentiable mapping on $(\ell, \bar{e})$ with $0 \leq \ell < \bar{e}$ such that $F' \in L^1[\ell, \bar{e}]$. If $|F'|$ be MT-convex and $|F'(x)| \leq \widetilde{M}$ for all $x \in [\ell, \bar{e}]$, then we have*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}F(u)\,du-F(x)\right| \leq \frac{\pi}{4}\frac{(x-\ell)^2+(\bar{e}-x)^2}{\bar{e}-\ell}\widetilde{M}.$$

Theorem 1.4 ([30]). *Let $F : [\ell, \bar{e}] \to \mathbb{R}$ be a differentiable mapping on $(\ell, \bar{e})$ with $0 \leq \ell < \bar{e}$ such that $F' \in L^1[\ell, \bar{e}]$. If $|F'|^q$ be MT-convex where $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $|F'(x)| \leq \widetilde{M}$ for all $x \in [\ell, \bar{e}]$, then we have*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}F(u)\,du-F(x)\right| \leq \frac{\widetilde{M}}{(1+p)^{\frac{1}{p}}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-\ell)^2+(\bar{e}-x)^2}{\bar{e}-\ell}.$$

Theorem 1.5 ([30]). *Let $F : [\ell, \bar{e}] \to \mathbb{R}$ be a differentiable mapping on $(\ell, \bar{e})$ with $0 \leq \ell < \bar{e}$ such that $F' \in L^1[\ell, \bar{e}]$. If $|F'|^q$ be MT-convex where $q \geq 1$ and $|F'(x)| \leq \widetilde{M}$ for all $x \in [\ell, \bar{e}]$, then we have*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}F(u)\,du-F(x)\right| \leq \left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-\ell)^2+(\bar{e}-x)^2}{\bar{e}-\ell}\frac{\widetilde{M}}{2}.$$

Motivated by the above results, in this study, we first introduce the class of strongly m-MT-convex functions, and by using the identity given in [2] we establish some new Ostrowski-type inequalities whose n^{th} derivatives are strongly m-MT-convex functions.

2. Preliminaries

In this section, we recall some definitions of certain classes of convex functions and lemma.

Definition 2.1 ([25]). A function $\mathcal{C} : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I, if

$$\mathcal{C}(\varepsilon v + (1-\varepsilon)w) \leq \varepsilon\mathcal{C}(v) + (1-\varepsilon)\mathcal{C}(w),$$

holds for all $v, w \in I$ and all $\varepsilon \in [0, 1]$.

Definition 2.2 ([26])**.** A function $\mathcal{C}: I \rightarrow \mathbb{R}$ is called strongly convex with modulus c, if

$$\mathcal{C}\left(\varepsilon v+(1-\varepsilon) w\right) \leq \varepsilon \mathcal{C}\left(v\right)+(1-\varepsilon) \mathcal{C}\left(w\right)-c \varepsilon(1-\varepsilon)\left|v-w\right|^{2},$$

holds for all $v, w \in I$ and $\varepsilon \in[0,1]$.

Definition 2.3 ([28])**.** A function $\mathcal{C}:[0, b] \rightarrow \mathbb{R}$ is said to be m-convex, where $m \in(0,1]$, if

$$\mathcal{C}\left(\varepsilon v+m\left(1-\varepsilon\right) w\right) \leq \varepsilon \mathcal{C}\left(v\right)+m\left(1-\varepsilon\right) \mathcal{C}(w),$$

holds for all $v, w \in I$, and $\varepsilon \in[0,1]$.

Definition 2.4 ([12])**.** A function $\mathcal{C}: I \subset[0, \infty) \rightarrow \mathbb{R}$ is called strongly m-convex with modulus c where $m \in[0,1]$, if

$$\mathcal{C}\left(\varepsilon v+m\left(1-\varepsilon\right) w\right) \leq \varepsilon \mathcal{C}\left(v\right)+m\left(1-\varepsilon\right) \mathcal{C}\left(w\right)-c m \varepsilon(1-\varepsilon)\left|v-w\right|^{2},$$

holds for all $v, w \in I$ and $\varepsilon \in[0,1]$.

Definition 2.5 ([29])**.** A function $\mathcal{C}: K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be an MT -convex function

$$\mathcal{C}\left(\varepsilon v+\left(1-\varepsilon\right) w\right) \leq \tfrac{\sqrt{\varepsilon}}{2 \sqrt{1-\varepsilon}} \mathcal{C}\left(v\right)+\tfrac{\sqrt{1-\varepsilon}}{2 \sqrt{\varepsilon}} \mathcal{C}\left(w\right),$$

for all $v, w \in K$, and $\varepsilon \in(0,1)$.

Definition 2.6 ([23])**.** A function $\mathcal{C}: K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be an m-MT -convex function

$$\mathcal{C}\left(\varepsilon v+\left(1-\varepsilon\right) w\right) \leq \tfrac{\sqrt{\varepsilon}}{2 \sqrt{1-\varepsilon}} \mathcal{C}\left(v\right)+\tfrac{m \sqrt{1-\varepsilon}}{2 \sqrt{\varepsilon}} \mathcal{C}\left(w\right),$$

for all $v, w \in K$, and $t \in(0,1)$.

Lemma 2.7 ([2])**.** *Let $\mathcal{C}:[\ell, \bar{e}] \subset \mathbb{R} \rightarrow \mathbb{R}$ be n-times differentiable mapping such that $\mathcal{C}^{(n-1)}(x)$ is absolutely continuous on $[\ell, \bar{e}]$, then for $n \in \mathbb{N}$ one has the following identity*

$$\begin{aligned}
&\int_{\ell}^{\bar{e}} \mathcal{C}\left(u\right) d u-\sum_{k=0}^{n-1} \tfrac{(\bar{e}-x)^{k+1}+(-1)^{k}(x-\ell)^{k+1}}{(k+1)!} \mathcal{C}^{(k)}\left(x\right) \\
&=\tfrac{(-1)^{n}}{n!}\left((x-\ell)^{n+1} \int_{0}^{1}\left(1-t\right)^{n} \mathcal{C}^{(n)}\left(t \ell+\left(1-t\right) x\right) d t\right. \\
&\quad\left.+\left(\bar{e}-x\right)^{n+1} \int_{0}^{1} t^{n} \mathcal{C}^{(n)}\left(t x+\left(1-t\right) \bar{e}\right) d t\right),
\end{aligned}$$

where an empty sum is understood to be nil.

We recall that the gamma and beta functions for any complex numbers and non-positive integers x, y such that $\operatorname{Re}(x) > 0$ and $\operatorname{Re}(y) > 0$ are defined, respectively

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp t dt,$$

and

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Some properties

(i) $\Gamma(x+1) = x\Gamma(x)$.
(ii) $B(x, x) = 2^{1-2x} B\left(\frac{1}{2}, x\right)$.
(iii) $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

3. Main Results

Definition 3.1. A nonnegative function $\mathcal{C} : \left[0, \frac{\bar{e}}{m}\right] \subset I \to \mathbb{R}$ is said to be strongly m-MT-convex functions with modulus c on I, if

$$\mathcal{C}(t\ell + m(1-t)\bar{e}) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\mathcal{C}(\ell) + m\frac{\sqrt{1-t}}{2\sqrt{t}}\mathcal{C}(\bar{e}) - cmt(1-t)\left|\ell - \bar{e}\right|^2,$$

holds for all $\ell, \bar{e} \in I, m \in (0, 1]$ and $t \in (0, 1)$.

Theorem 3.2. *Let $\mathcal{C} : [\ell, \bar{e}^*] \to \mathbb{R}$ be n-times differentiable mapping such that $\left|\mathcal{C}^{(n)}\right|^q \in L^1[\ell, \bar{e}^*]$ where $0 \leq \ell < \bar{e} \leq \frac{\bar{e}}{m} \leq \bar{e}^*$. If $\left|\mathcal{C}^{(n)}\right|$ is strongly m-MT-convex and $n \in \mathbb{N}$ with $n \geq 1$, then the following inequality holds*

$$\begin{aligned}
&\left|\int_\ell^{\bar{e}} \mathcal{C}(u)\, du - \sum_{k=0}^{n-1} \frac{(\bar{e}-x)^{k+1} + (-1)^k (x-\ell)^{k+1}}{(k+1)!} \mathcal{C}^{(k)}(x)\right| \\
&\quad \leq \frac{\Gamma\left(n+\frac{1}{2}\right)}{4(n+1)\times(n!)^2}\sqrt{\pi} \\
&\qquad \left((x-\ell)^{n+1}\left|\mathcal{C}^{(n)}(\ell)\right| + (2n+1)\, m\, (x-\ell)^{n+1}\left|\mathcal{C}^{(n)}\left(\tfrac{x}{m}\right)\right|\right. \\
&\qquad \left. + (2n+1)(\bar{e}-x)^{n+1}\left|\mathcal{C}^{(n)}(x)\right| + m(\bar{e}-x)^{n+1}\left|\mathcal{C}^{(n)}\left(\tfrac{\bar{e}}{m}\right)\right|\right) \\
&\qquad - \frac{c(n+1)}{(n+3)!m}\left((x-\ell)^{n+1}(x-m\ell)^2 + (\bar{e}-x)^{n+1}(\bar{e}-mx)^2\right),
\end{aligned}$$

where Γ is the gamma function.

Proof. Using Lemma 2.7, the property of modulus and the strong m-MT-convexity of $\left|\mathcal{C}^{(n)}\right|$, we deduce

$$
\begin{aligned}
&\left|\int_a^b \mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\tfrac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|\\
&\le \tfrac{1}{n!}\Bigg((x-\ell)^{n+1}\int_0^1(1-t)^n\left|\mathcal{C}^{(n)}(t\ell+(1-t)x)\right|dt\\
&\quad+(\bar{e}-x)^{n+1}\int_0^1 t^n\left|\mathcal{C}^{(n)}(tx+(1-t)\bar{e})\right|dt\Bigg)\\
&\le \tfrac{1}{n!}\Bigg((x-a)^{n+1}\int_0^1(1-t)^n\Big(\tfrac{\sqrt{t}}{2\sqrt{1-t}}\left|\mathcal{C}^{(n)}(\ell)\right|+m\tfrac{\sqrt{1-t}}{2\sqrt{t}}\left|\mathcal{C}^{(n)}\left(\tfrac{x}{m}\right)\right|\\
&\quad-cmt(1-t)\left(\ell-\tfrac{x}{m}\right)^2\Big)dt\\
&\quad+(\bar{e}-x)^{n+1}\int_0^1 t^n\Big(\tfrac{\sqrt{t}}{2\sqrt{1-t}}\left|\mathcal{C}^{(n)}(x)\right|+m\tfrac{\sqrt{1-t}}{2\sqrt{t}}\left|\mathcal{C}^{(n)}\left(\tfrac{\bar{e}}{m}\right)\right|\\
&\quad-cmt(1-t)\left(x-\tfrac{\bar{e}}{m}\right)^2\Big)dt\Bigg)\\
&=\tfrac{1}{2\times n!}\Bigg((x-\ell)^{n+1}\Bigg(\left|\mathcal{C}^{(n)}(\ell)\right|\int_0^1 t^{\frac12}(1-t)^{n-\frac12}dt\\
&\quad+m\left|\mathcal{C}^{(n)}\left(\tfrac{x}{m}\right)\right|\int_0^1 t^{-\frac12}(1-t)^{n+\frac12}dt\\
&\quad-2\left(\ell-\tfrac{x}{m}\right)^2cm\int_0^1 t(1-t)^{n+1}dt\Bigg)\\
&\quad+(\bar{e}-x)^{n+1}\Bigg(\left|\mathcal{C}^{(n)}(x)\right|\int_0^1 t^{n+\frac12}(1-t)^{-\frac12}dt\\
&\quad+m\left|\mathcal{C}^{(n)}\left(\tfrac{\bar{e}}{m}\right)\right|\int_0^1 t^{n-\frac12}(1-t)^{\frac12}dt\\
&\quad-2\left(x-\tfrac{\bar{e}}{m}\right)^2cm\int_0^1 t^{n+1}(1-t)dt\Bigg)\Bigg)\\
&=\tfrac{1}{2(n!)}\Bigg((x-\ell)^{n+1}\Big(B\left(\tfrac32,n+\tfrac12\right)\left|\mathcal{C}^{(n)}(\ell)\right|\\
&\quad+mB\left(\tfrac12,n+\tfrac32\right)\left|\mathcal{C}^{(n)}\left(\tfrac{x}{m}\right)\right|-2cmB(2,n+2)\left(\ell-\tfrac{x}{m}\right)^2\Big)\\
&\quad+(\bar{e}-x)^{n+1}\Big(B\left(n+\tfrac32,\tfrac12\right)\left|\mathcal{C}^{(n)}(x)\right|\\
&\quad+mB\left(n+\tfrac12,\tfrac32\right)\left|\mathcal{C}^{(n)}\left(\tfrac{\bar{e}}{m}\right)\right|-2cmB(n+2,2)\left(x-\tfrac{\bar{e}}{m}\right)^2\Big)\Bigg)\\
&=\tfrac{\Gamma\left(n+\frac12\right)}{4(n+1)\times(n!)^2}\sqrt{\pi}
\end{aligned}
$$

$$\times\left((x-\ell)^{n+1}\left|\mathcal{C}^{(n)}(\ell)\right|+(2n+1)\,m\,(x-\ell)^{n+1}\left|\mathcal{C}^{(n)}\left(\tfrac{x}{m}\right)\right|\right.$$
$$\left.+\,(2n+1)(\bar{e}-x)^{n+1}\left|\mathcal{C}^{(n)}(x)\right|+m\,(\bar{e}-x)^{n+1}\left|\mathcal{C}^{(n)}\left(\tfrac{\bar{e}}{m}\right)\right|\right)$$
$$-\,\tfrac{c(n+1)}{(n+3)!m}\left((x-\ell)^{n+1}(x-m\ell)^2+(\bar{e}-x)^{n+1}(\bar{e}-mx)^2\right),$$

which is the desired result. □

Corollary 3.3. *In Theorem 3.2, if we assume that* $\left|\mathcal{C}^{(n)}(x)\right|\leq M$ *for all* $x\in[\ell,\bar{e}]\,,m=1$ *and* $c\to 0$*, we obtain*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\tfrac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq\tfrac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{\pi}}{(n!)^2}\left(\tfrac{(x-\ell)^{n+1}+(\bar{e}-x)^{n+1}}{2}\right)M.$$

Remark 3.4. Corollary 3.3 will be reduced to Theorem 2 from [30], if take $n=1$.

Corollary 3.5. *In Corollary 3.3, if* $n=2$*, we obtain*

$$\left|\tfrac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\tfrac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|\leq\tfrac{3\pi}{32}\left(\tfrac{(x-\ell)^3+(\bar{e}-x)^3}{\bar{e}-\ell}\right)M.$$

Corollary 3.6. *In Theorem 3.2, if* $m=1$*, we obtain*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\tfrac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq\tfrac{\Gamma\left(n+\frac{1}{2}\right)}{4(n+1)\times(n!)^2}\sqrt{\pi}\left((x-\ell)^{n+1}\left|\mathcal{C}^{(n)}(\ell)\right|+(\bar{e}-x)^{n+1}\left|\mathcal{C}^{(n)}(\bar{e})\right|\right.$$
$$\left.+\,(2n+1)\left((x-\ell)^{n+1}+(\bar{e}-x)^{n+1}\right)\left|\mathcal{C}^{(n)}(x)\right|\right)$$
$$-\,\tfrac{c(n+1)}{(n+3)!}\left((x-\ell)^{n+3}+(\bar{e}-x)^{n+3}\right).$$

Corollary 3.7. *By letting* c *tend to* 0*, Corollary 3.6 gives*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\tfrac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq\tfrac{\Gamma\left(n+\frac{1}{2}\right)}{4(n+1)\times(n!)^2}\sqrt{\pi}\left((x-\ell)^{n+1}\left|\mathcal{C}^{(n)}(\ell)\right|+(\bar{e}-x)^{n+1}\left|\mathcal{C}^{(n)}(\bar{e})\right|\right.$$
$$\left.+\,(2n+1)\left((x-\ell)^{n+1}+(\bar{e}-x)^{n+1}\right)\left|\mathcal{C}^{(n)}(x)\right|\right).$$

Corollary 3.8. *In Theorem 3.2, if* $n=1$*, we obtain*

$$\left|\tfrac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)\right|$$

$$\leq \tfrac{\pi}{16}\left(\tfrac{(x-\ell)^2}{\bar{e}-\ell}\left|\mathcal{C}'(\ell)\right|+3\left(\tfrac{(\bar{e}-x)^2}{\bar{e}-\ell}\left|\mathcal{C}'(x)\right|+m\tfrac{(x-\ell)^2}{\bar{e}-\ell}\left|\mathcal{C}'\left(\tfrac{x}{m}\right)\right|\right)\right.$$
$$\left.+\, m\tfrac{(\bar{e}-x)^2}{\bar{e}-\ell}\left|\mathcal{C}'\left(\tfrac{\bar{e}}{m}\right)\right|\right)-\tfrac{c}{12m}\left(\tfrac{(x-\ell)^2(x-m\ell)^2+(\bar{e}-x)^2(\bar{e}-mx)^2}{\bar{e}-\ell}\right).$$

Corollary 3.9. *By letting c tend to 0, Corollary 3.8 gives*

$$\left|\tfrac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)\right|$$
$$\leq \tfrac{\pi}{16}\left(\tfrac{(x-\ell)^2|\mathcal{C}'(\ell)|+m(\bar{e}-x)^2\left|\mathcal{C}'\left(\frac{b}{m}\right)\right|}{\bar{e}-\ell}+3\left(\tfrac{(\bar{e}-x)^2|\mathcal{C}'(x)|+m(x-\ell)^2\left|\mathcal{C}'\left(\frac{x}{m}\right)\right|}{\bar{e}-\ell}\right)\right).$$

Corollary 3.10. *In Theorem 3.2, if $n=2$, we obtain*

$$\left|\tfrac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\tfrac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|$$
$$\leq \tfrac{\pi}{64}\left(\tfrac{(x-\ell)^3}{\bar{e}-\ell}\left|\mathcal{C}''(\ell)\right|+5\left(\tfrac{(\bar{e}-x)^3}{\bar{e}-\ell}\left|\mathcal{C}''(x)\right|+m\tfrac{(x-\ell)^3}{\bar{e}-\ell}\left|\mathcal{C}''\left(\tfrac{x}{m}\right)\right|\right)\right.$$
$$\left.+\, m\tfrac{(\bar{e}-x)^3}{\bar{e}-\ell}\left|\mathcal{C}''\left(\tfrac{\bar{e}}{m}\right)\right|\right)-\tfrac{c}{40m}\left(\tfrac{(x-\ell)^3(x-m\ell)^2+(\bar{e}-x)^3(\bar{e}-mx)^2}{\bar{e}-\ell}\right).$$

Corollary 3.11. *By letting c tend to 0, Corollary 3.10 gives*

$$\left|\tfrac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\tfrac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|$$
$$\leq \tfrac{\pi}{64}\left(\tfrac{(x-\ell)^3|\mathcal{C}''(\ell)|+m(\bar{e}-x)^3\left|\mathcal{C}''\left(\frac{\bar{e}}{m}\right)\right|}{\bar{e}-\ell}+5\left(\tfrac{(\bar{e}-x)^3|\mathcal{C}''(x)|+m(x-\ell)^3\left|\mathcal{C}''\left(\frac{x}{m}\right)\right|}{\bar{e}-\ell}\right)\right).$$

Theorem 3.12. *Let $\mathcal{C}:[\ell,\bar{e}^*]\to\mathbb{R}$ be n-times differentiable mapping such that $\left|\mathcal{C}^{(n)}\right|^q\in L^1[\ell,\bar{e}^*]$ where $0\leq\ell<\bar{e}\leq\frac{\bar{e}}{m}\leq\bar{e}^*$. If $\left|\mathcal{C}^{(n)}\right|^q$ is strongly m-MT-convex, where $q,p>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $n\in\mathbb{N}$ with $n\geq 1$, then the following inequality holds*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\tfrac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq \tfrac{(x-\ell)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\tfrac{\pi}{4}\left|\mathcal{C}^{(n)}(\ell)\right|^q+m\tfrac{\pi}{4}\left|\mathcal{C}^{(n)}\left(\tfrac{x}{m}\right)\right|^q-\tfrac{c}{6m}(x-m\ell)^2\right)^{\frac{1}{q}}$$
$$+\tfrac{(\bar{e}-x)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\tfrac{\pi}{4}\left|\mathcal{C}^{(n)}(x)\right|^q+m\tfrac{\pi}{4}\left|\mathcal{C}^{(n)}\left(\tfrac{\bar{e}}{m}\right)\right|^q-\tfrac{c}{6m}(\bar{e}-mx)^2\right)^{\frac{1}{q}}.$$

Proof. From lemma 2.7, properties of modulus and Hölder's inequality, we have

(3.1)

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\tfrac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$

$$\leq \frac{1}{n!}\left((x-\ell)^{n+1}\int_0^1 (1-t)^n \left|\mathcal{C}^{(n)}\left(t\ell+(1-t)\,x\right)\right| dt\right.$$
$$\left.+\,(\bar{e}-x)^{n+1}\int_0^1 t^n \left|\mathcal{C}^{(n)}\left(tx+(1-t)\,\bar{e}\right)\right| dt\right)$$
$$\leq \frac{(x-\ell)^{n+1}}{n!}\left(\int_0^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathcal{C}^{(n)}\left(t\ell+(1-t)\,x\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\,\frac{(\bar{e}-x)^{n+1}}{n!}\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathcal{C}^{(n)}\left(tx+(1-t)\,\bar{e}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{(x-\ell)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|\mathcal{C}^{(n)}\left(t\ell+(1-t)\,x\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\,\frac{(\bar{e}-x)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\int_0^1 \left|\mathcal{C}^{(n)}\left(tx+(1-t)\,\bar{e}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|\mathcal{C}^{(n)}\right|^q$ the strongly m-MT-convex function, we have

(3.2)
$$\int_0^1 \left|\mathcal{C}^{(n)}\left(t\ell+(1-t)\,x\right)\right|^q dt$$
$$\leq \int_0^1 \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\left|\mathcal{C}^{(n)}\left(\ell\right)\right|^q + m\frac{\sqrt{1-t}}{2\sqrt{t}}\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q\right.$$
$$\left.-cmt(1-t)\left(\ell-\frac{x}{m}\right)^2\right) dt$$
$$= \frac{B\left(\frac{3}{2},\frac{1}{2}\right)}{2}\left|\mathcal{C}^{(n)}\left(\ell\right)\right|^q + \frac{B\left(\frac{1}{2},\frac{3}{2}\right)}{2}m\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q - cB\left(2,2\right)m\left(\ell-\frac{x}{m}\right)^2$$
$$= \frac{\pi}{4}\left|\mathcal{C}^{(n)}\left(\ell\right)\right|^q + m\frac{\pi}{4}\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q - \frac{cm}{6}\left(\ell-\frac{x}{m}\right)^2.$$

Similarly, we have

$$\int_0^1 \left|\mathcal{C}^{(n)}\left(tx+(1-t)\,\bar{e}\right)\right|^q dt \tag{3.3}$$
$$\leq \frac{\pi}{4}\left|\mathcal{C}^{(n)}\left(x\right)\right|^q + m\frac{\pi}{4}\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^q - \frac{cm}{6}\left(x-\frac{\bar{e}}{m}\right)^2.$$

Combining (3.1)-(3.3) we get the desired result. □

Corollary 3.13. *In Theorem 3.12, if we assume that* $\left|\mathcal{C}^{(n)}\left(x\right)\right| \leq M$ *for all* $x \in [\ell, \bar{e}]$, $m = 1$ *and* $c \to 0$, *we obtain*

$$\left|\int_\ell^{\bar{e}} \mathcal{C}\left(u\right)du - \sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}\left(x\right)\right|$$

$$\leq \frac{(x-\ell)^{n+1}+(\bar{e}-x)^{n+1}}{n!(np+1)^{\frac{1}{p}}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} M.$$

Remark 3.14. Corollary 3.13 will be reduced to Theorem 3 from [30], if $n=1$.

Corollary 3.15. *In Corollary 3.13, if $n=2$, we obtain*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\frac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right| \leq \frac{(x-\ell)^3+(\bar{e}-x)^3}{2(\bar{e}-\ell)(2p+1)^{\frac{1}{p}}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} M.$$

Corollary 3.16. *In Theorem 3.12, if we take $m=1$, we obtain*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq \frac{\pi^{\frac{1}{q}}(x-\ell)^{n+1}}{2^{\frac{1}{q}}n!(np+1)^{\frac{1}{p}}}\left(\frac{\left|\mathcal{C}^{(n)}(\ell)\right|^q+\left|\mathcal{C}^{(n)}(x)\right|^q}{2}-\frac{c}{3\pi}(x-\ell)^2\right)^{\frac{1}{q}}$$
$$+\frac{\pi^{\frac{1}{q}}(\bar{e}-x)^{n+1}}{2^{\frac{1}{q}}n!(np+1)^{\frac{1}{p}}}\left(\frac{\left|\mathcal{C}^{(n)}(x)\right|^q+\left|\mathcal{C}^{(n)}(\bar{e})\right|^q}{2}-\frac{c}{3\pi}(\bar{e}-x)^2\right)^{\frac{1}{q}}.$$

Corollary 3.17. *By letting c tend to 0, Corollary 3.16 gives*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq \frac{\pi^{\frac{1}{q}}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}}\left(\frac{(x-\ell)^{n+1}}{n!}\left(\frac{\left|\mathcal{C}^{(n)}(\ell)\right|^q+\left|\mathcal{C}^{(n)}(x)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^{n+1}}{n!}\left(\frac{\left|\mathcal{C}^{(n)}(x)\right|^q+\left|\mathcal{C}^{(n)}(\bar{e})\right|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 3.18. *In Theorem 3.12, if we take $n=1$, we obtain*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)\right|$$
$$\leq \frac{\pi^{\frac{1}{q}}}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{(x-\ell)^2}{\bar{e}-\ell}\left(\frac{|\mathcal{C}'(\ell)|^q+m\left|\mathcal{C}'\left(\frac{x}{m}\right)\right|^q}{2}-\frac{c}{3\pi m}(x-m\ell)^2\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^2}{\bar{e}-\ell}\left(\frac{|\mathcal{C}'(x)|^q+m\left|\mathcal{C}'\left(\frac{\bar{e}}{m}\right)\right|^q}{2}-\frac{c}{3\pi m}(\bar{e}-mx)^2\right)^{\frac{1}{q}}\right).$$

Corollary 3.19. *By letting c tend to 0, Corollary 3.18 gives*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)\right|$$

$$\leq \frac{\pi^{\frac{1}{q}}}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{(x-\ell)^2}{\bar{e}-\ell}\left(\frac{|\mathcal{C}'(\ell)|^q+m\left|\mathcal{C}'\left(\frac{x}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^2}{\bar{e}-\ell}\left(\frac{|\mathcal{C}'(x)|^q+m\left|\mathcal{C}'\left(\frac{\bar{e}}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 3.20. *In Theorem 3.12, if we take $n=2$, we obtain*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\frac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|$$
$$\leq \frac{\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left(\frac{(x-\ell)^3}{\bar{e}-\ell}\left(\frac{|\mathcal{C}''(\ell)|^q+m\left|\mathcal{C}''\left(\frac{x}{m}\right)\right|^q}{2}-\frac{c}{3\pi m}(x-m\ell)^2\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^3}{\bar{e}-\ell}\left(\frac{|\mathcal{C}''(x)|^q+m\left|\mathcal{C}''\left(\frac{\bar{e}}{m}\right)\right|^q}{2}-\frac{c}{3\pi m}(\bar{e}-mx)^2\right)^{\frac{1}{q}}\right).$$

Corollary 3.21. *By letting c tend to 0, Corollary 3.20 gives*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\frac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|$$
$$\leq \frac{\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left(\frac{(x-\ell)^3}{\bar{e}-\ell}\left(\frac{|\mathcal{C}''(\ell)|^q+m\left|\mathcal{C}''\left(\frac{x}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^3}{\bar{e}-\ell}\left(\frac{|\mathcal{C}''(x)|^q+m\left|\mathcal{C}''\left(\frac{\bar{e}}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right).$$

Theorem 3.22. *Let $\mathcal{C}:[\ell,\bar{e}^*]\to\mathbb{R}$ be n-times differentiable mapping such that $\left|\mathcal{C}^{(n)}\right|^q\in L^1[\ell,\bar{e}^*]$ where $0\leq\ell<\bar{e}\leq\frac{\bar{e}}{m}\leq\bar{e}^*$. If $\left|\mathcal{C}^{(n)}\right|^q$ is strongly m-MT-convex, where $q\geq 1$ and $n\in\mathbb{N}$ with $n\geq 1$, then the following inequality holds*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$
$$\leq \frac{(x-\ell)^{n+1}}{(n+1)!}\left(\frac{(n+1)}{2\times(n+1)!}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(\ell)\right|^q\right.$$
$$\left.+\frac{(2n+1)\sqrt{\pi}}{2}m\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q-\frac{(n+1)!2c}{\Gamma\left(n+\frac{1}{2}\right)(n+2)(n+3)m}(x-m\ell)^2\right)^{\frac{1}{q}}$$
$$+\frac{(\bar{e}-x)^{n+1}}{(n+1)!}\left(\frac{(n+1)}{2\times(n+1)!}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{(2n+1)\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(x)\right|^q\right.$$
$$\left.+\frac{\sqrt{\pi}}{2}m\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^q-\frac{(n+1)!2c}{\Gamma\left(n+\frac{1}{2}\right)(n+2)(n+3)m}(\bar{e}-mx)^2\right)^{\frac{1}{q}}.$$

where $\Gamma(.)$ *is the Euler gamma function.*

Proof. From lemma 2.7, properties of modulus, power mean and the strong m-MT-convexity of $\left|\mathcal{C}^{(n)}\right|^q$, we have

$$\left|\int_\ell^{\bar{e}} \mathcal{C}(u)\,du - \sum_{k=0}^{n-1} \frac{(\bar{e}-x)^{k+1}+(-1)^k(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$

$$\leq \frac{(x-\ell)^{n+1}}{n!}\left(\int_0^1 (1-t)^n\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^n\left|\mathcal{C}^{(n)}(t\ell+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(\bar{e}-x)^{n+1}}{n!}\left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^n\left|\mathcal{C}^{(n)}(tx+(1-t)\bar{e})\right|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-\ell)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 (1-t)^n\left|\mathcal{C}^{(n)}(t\ell+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(\bar{e}-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1 t^n\left|\mathcal{C}^{(n)}(tx+(1-t)\bar{e})\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{(x-\ell)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1\left(\frac{(1-t)^n\sqrt{t}}{2\sqrt{1-t}}\left|\mathcal{C}^{(n)}(\ell)\right|^q + m\frac{(1-t)^n\sqrt{1-t}}{2\sqrt{t}}\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q\right.\right.$$

$$\left.\left.-\ cmt(1-t)^{n+1}\left(\ell-\frac{x}{m}\right)^2\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{(\bar{e}-x)^{n+1}}{n!(n+1)^{1-\frac{1}{q}}}\left(\int_0^1\left(\frac{t^n\sqrt{t}}{2\sqrt{1-t}}\left|\mathcal{C}^{(n)}(x)\right|^q + m\frac{t^n\sqrt{1-t}}{2\sqrt{t}}\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^q\right.\right.$$

$$\left.\left.-\ cmt(1-t)^{n+1}\left(x-\frac{\bar{e}}{m}\right)^2\right)dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-\ell)^{n+1}}{2^{\frac{1}{q}}(n!)(n+1)^{1-\frac{1}{q}}}\left(B\left(\frac{3}{2},n+\frac{1}{2}\right)\left|\mathcal{C}^{(n)}(\ell)\right|^q + B\left(\frac{1}{2},n+\frac{3}{2}\right)m\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q\right.$$

$$\left.-\ 2cB(2,n+2)\,m\left(\ell-\frac{x}{m}\right)^2\right)^{\frac{1}{q}}$$

$$+\frac{(\bar{e}-x)^{n+1}}{2^{\frac{1}{q}}(n!)(n+1)^{1-\frac{1}{q}}}\left(B\left(n+\frac{3}{2},\frac{1}{2}\right)\left|\mathcal{C}^{(n)}(x)\right|^q + mB\left(n+\frac{1}{2},\frac{3}{2}\right)\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^q\right.$$

$$\left.-\ 2cB(2,n+2)\,m\left(x-\frac{\bar{e}}{m}\right)^2\right)^{\frac{1}{q}}$$

$$=\frac{(n+1)^{\frac{1}{q}}(x-\ell)^{n+1}}{((n+1)!)^{1+\frac{1}{q}}}\left(\frac{1}{2}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(\ell)\right|^q\right.$$

$$\left.+\frac{(2n+1)\sqrt{\pi}}{2}m\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q - \frac{(n+1)!2c}{\Gamma\left(n+\frac{1}{2}\right)(n+2)(n+3)m}(x-m\ell)^2\right)^{\frac{1}{q}}$$

$$+\frac{(n+1)^{\frac{1}{q}}(\bar{e}-x)^{n+1}}{((n+1)!)^{1+\frac{1}{q}}}\left(\frac{1}{2}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{(2n+1)\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(x)\right|^{q}\right.$$

$$\left.+\frac{\sqrt{\pi}}{2}m\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^{q}-\frac{(n+1)!2c}{\Gamma\left(n+\frac{1}{2}\right)(n+2)(n+3)m}(\bar{e}-mx)^{2}\right)^{\frac{1}{q}}.$$

The proof is completed. □

Corollary 3.23. *In Theorem 3.22, if we assume that* $\left|\mathcal{C}^{(n)}(x)\right|\leq M$ *for all* $x\in[\ell,\bar{e}]\,, m=1$ *and* $c\to 0$*, we obtain*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^{k}(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$

$$\leq\frac{(n+1)^{\frac{2}{q}}}{((n+1)!)^{1+\frac{1}{q}}}\left(\frac{\sqrt{\pi}}{2}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left((x-\ell)^{n+1}+(b-x)^{n+1}\right)M.$$

Remark 3.24. Corollary 3.23 will be reduced to Theorem 4 from [30], if $n=1$.

Corollary 3.25. *In Corollary 3.23, if* $n=2$*, we obtain*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\frac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|\leq\left(\frac{9}{16}\pi\right)^{\frac{1}{q}}\left(\frac{(x-\ell)^{3}+(\bar{e}-x)^{3}}{6(b-a)}\right)M.$$

Corollary 3.26. *In Theorem 3.22, if we take* $m=1$*, we obtain*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^{k}(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$

$$\leq\frac{(x-\ell)^{n+1}}{(n+1)!}\left(\frac{(n+1)}{2\times(n+1)!}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(\ell)\right|^{q}\right.$$

$$\left.+\frac{(2n+1)\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(x)\right|^{q}-\frac{(n+1)!2c}{\Gamma\left(n+\frac{1}{2}\right)(n+2)(n+3)}(x-\ell)^{2}\right)^{\frac{1}{q}}$$

$$+\frac{(\bar{e}-x)^{n+1}}{(n+1)!}\left(\frac{(n+1)}{2\times(n+1)!}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{(2n+1)\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(x)\right|^{q}\right.$$

$$\left.+\frac{\sqrt{\pi}}{2}\left|\mathcal{C}^{(n)}(\bar{e})\right|^{q}-\frac{(n+1)!2c}{\Gamma\left(n+\frac{1}{2}\right)(n+2)(n+3)}(\bar{e}-x)^{2}\right)^{\frac{1}{q}}.$$

Corollary 3.27. *By letting* c *tend to* 0*, Corollary 3.26 gives*

$$\left|\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\sum_{k=0}^{n-1}\frac{(\bar{e}-x)^{k+1}+(-1)^{k}(x-\ell)^{k+1}}{(k+1)!}\mathcal{C}^{(k)}(x)\right|$$

$$\leq\left(\frac{(n+1)^{2}\sqrt{\pi}}{2(n+1)!}\Gamma\left(n+\frac{1}{2}\right)\right)^{\frac{1}{q}}\left(\frac{(x-\ell)^{n+1}}{(n+1)!}\left(\frac{\left|\mathcal{C}^{(n)}(\ell)\right|^{q}+(2n+1)\left|\mathcal{C}^{(n)}(x)\right|^{q}}{2(n+1)}\right)^{\frac{1}{q}}\right.$$

$$+ \frac{(\bar{e}-x)^{n+1}}{(n+1)!}\left(\frac{(2n+1)\left|\mathcal{C}^{(n)}(x)\right|^q+\left|\mathcal{C}^{(n)}(\bar{e})\right|^q}{2(n+1)}\right)^{\frac{1}{q}}\Bigg).$$

Corollary 3.28. *In Theorem 3.22, if we take $n=1$, we obtain*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)\right|$$
$$\leq \frac{(x-\ell)^2}{2(\bar{e}-\ell)}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{|\mathcal{C}'(\ell)|^q+3m\left|\mathcal{C}'\left(\frac{x}{m}\right)\right|^q}{4}-\frac{c}{3\pi m}(x-m\ell)^2\right)^{\frac{1}{q}}$$
$$+\frac{(\bar{e}-x)^2}{2(\bar{e}-\ell)}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{3|\mathcal{C}'(x)|^q+m\left|\mathcal{C}'\left(\frac{\bar{e}}{m}\right)\right|^q}{4}-\frac{c}{3\pi m}(\bar{e}-mx)^2\right)^{\frac{1}{q}}.$$

Corollary 3.29. *By letting c tend to 0, Corollary 3.28 gives*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)\right|\leq\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{(x-\ell)^2}{2(\bar{e}-\ell)}\left(\frac{|\mathcal{C}'(\ell)|^q+3m\left|\mathcal{C}'\left(\frac{x}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^2}{2(\bar{e}-\ell)}\left(\frac{3|\mathcal{C}'(x)|^q+m\left|\mathcal{C}'\left(\frac{\bar{e}}{m}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right).$$

Corollary 3.30. *In Theorem 3.22, if we take $n=2$, we obtain*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\frac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|$$
$$\leq\frac{(x-\ell)^3}{6(b-\ell)}\left(\frac{9\pi}{16}\right)^{\frac{1}{q}}\left(\frac{\left|\mathcal{C}^{(n)}(\ell)\right|^q+5m\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q}{6}-\frac{4c}{15\pi m}(x-m\ell)^2\right)^{\frac{1}{q}}$$
$$+\frac{(\bar{e}-x)^3}{6(\bar{e}-\ell)}\left(\frac{9\pi}{16}\right)^{\frac{1}{q}}\left(\frac{5\left|\mathcal{C}^{(n)}(x)\right|^q+m\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^q}{6}-\frac{4c}{15\pi m}(\bar{e}-mx)^2\right)^{\frac{1}{q}}.$$

Corollary 3.31. *By letting c tend to 0, Corollary 3.30 gives*

$$\left|\frac{1}{\bar{e}-\ell}\int_{\ell}^{\bar{e}}\mathcal{C}(u)\,du-\mathcal{C}(x)+\left(\frac{\ell+\bar{e}}{2}-x\right)\mathcal{C}'(x)\right|$$
$$\leq\left(\frac{9\pi}{16}\right)^{\frac{1}{q}}\left(\frac{(x-\ell)^3}{6(\bar{e}-\ell)}\left(\frac{\left|\mathcal{C}^{(n)}(\ell)\right|^q+5m\left|\mathcal{C}^{(n)}\left(\frac{x}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\bar{e}-x)^3}{6(\bar{e}-\ell)}\left(\frac{5\left|\mathcal{C}^{(n)}(x)\right|^q+m\left|\mathcal{C}^{(n)}\left(\frac{\bar{e}}{m}\right)\right|^q}{6}\right)^{\frac{1}{q}}\right).$$

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