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On Fractional Differential Equations with Riesz-Caputo Derivative and Non-Instantaneous Impulses

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Abstract. This article deals with the existence, uniqueness and Ulam type stability results for a class of boundary value problems for fractional differential equations with Riesz-Caputo fractional derivative. The results are based on Banach contraction principle and Krasnoselskii's fixed point theorem. An illustrative example is given to validate our main results.

1. Introduction

Fractional calculus has recently proven to be a very important tool in modeling numerous phenomena in applications and sciences such as physics, engineering, electrochemistry, geology, stability, controllability, and signal theory, among many others. We refer the reader to [2–4, 11, 14, 15, 17, 21, 26, 29–32] and the references therein for more details.

The stability of functional equations was originally raised by Ulam [38]. Next, by Hyers [19]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [27] demonstrated the existence of unique linear mappings near approximate additive mappings, generalizing Hyers' findings. Several research articles in the literature address the Ulam stabilities of various types of differential and integral equations, see [5–10, 22, 23, 34, 36] and the references therein.

Real-world processes and phenomena can exhibit rapid shifts in state. These modifications have a very brief duration in contrast to the entire longevity of the process, and thus are irrelevant to the evolution

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of the examined process. In such instances, impulsive equations can be employed to construct appropriate mathematical models. Physics, biology, population dynamics, ecology, pharmacokinetics, and other fields all contain such operations. Noninstantaneous impulses are actions that begin at an arbitrary fixed moment and last for a specified time interval. Hernandez and O'Regan [18], studied the existence of solutions to this novel class of abstract differential equations with non-instantaneous impulses. The papers [13, 33–35, 39, 41] can be consulted for fundamental results and recent developments on differential equations with instantaneous and non-instantaneous impulses.

In [25], the authors studied the existence of weak solutions for a class of impulsive nonlinear differential equations with periodic boundary conditions and non-instantaneous impulses. Abbas *et al.* [1] presented some existence results based on Schauder's and Monch's fixed point theorems and the technique of the measure of noncompactness for Cauchy problem of Caputo-Fabrizio fractional differential equations with non-instantaneous impulses. For some applications of non-instantaneous impulses, we recommend the papers [12, 24].

The authors of [14] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha} y(\vartheta) = g(\vartheta, y(\vartheta)), & \vartheta \in \Theta := [0, \varkappa], \\ y(0) = y_0, & y(\varkappa) = y_{\varkappa}, \end{cases}$$

where ${}_0^{RC}D_{\varkappa}^{\alpha}$ is a Riesz-Caputo derivative of order $0 < \alpha \leq 1$, $g : \Theta \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $y_0 \in \mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder fixed point theorem.

The authors of [33] established existence and stability results, with relevant fixed point theorems, to the boundary value problem:

$$\begin{cases} \left({}^{\rho}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2} x\right)(\vartheta) = f\left(\vartheta, x(\vartheta), \left({}^{\rho}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2} x\right)(\vartheta)\right); & \vartheta \in \Omega_i,\ i = 0, \ldots, m, \\ x(\vartheta) = \Psi_i(\vartheta, x(\vartheta)); & \vartheta \in \tilde{\Omega}_i,\ i = 1, \ldots, m, \\ \phi_1 \left({}^{\rho}\mathbb{J}_{a^+}^{1-\zeta_3} x\right)(a^+) + \phi_2 \left({}^{\rho}\mathbb{J}_{m^+}^{1-\zeta_3} x\right)(b) = \phi_3, & \end{cases}$$

where ${}^{\rho}\mathbb{D}_{\varkappa_i^+}^{\zeta_1,\zeta_2}$, ${}^{\rho}\mathbb{J}_{a^+}^{1-\zeta_3}$ are the generalized Hilfer fractional derivative of order $\zeta_1 \in (0, 1)$ and type $\zeta_2 \in [0, 1]$ and generalized fractional integral of order $1 - \zeta_3$, $(\zeta_3 = \zeta_1 + \zeta_2 - \zeta_1\zeta_2)$, respectively, $\phi_1, \phi_2, \phi_3 \in \mathbb{R}$, $\phi_1 \neq 0$, $\Omega_i := (\varkappa_i, \vartheta_{i+1}]; i = 0, \ldots, m$, $\tilde{\Omega}_i := (\vartheta_i, \varkappa_i]; i = 1, \ldots, m$, $a = \vartheta_0 = \varkappa_0 < \vartheta_1 \leq \varkappa_1 < \vartheta_2 \leq \varkappa_2 < \cdots \leq \varkappa_{m-1} < \vartheta_m \leq \varkappa_m < \vartheta_{m+1} = b < \infty$, $x(\vartheta_i^+) = \lim_{\epsilon \to 0^+} x(\vartheta_i + \epsilon)$ and $x(\vartheta_i^-) = \lim_{\epsilon \to 0^-} x(\vartheta_i + \epsilon)$, $f : (a, b] \times \mathbb{R} \times \mathbb{R} \to$

$\mathbb{R}$ is a given function and $\Psi_i : \tilde{\Omega}_i \times \mathbb{R} \to \mathbb{R}$; $i = 1, \ldots, m$ are given continuous functions.

Motivated by the above-mentioned papers, we present some existence, uniqueness and Ulam stability results for the following fractional problem:

$$\left({}^{RC}_{\varkappa_\jmath}D^{\alpha}_{\vartheta_{\jmath+1}}y\right)(\vartheta) = \varphi\left(\vartheta, y\left(\vartheta\right)\right); \quad \vartheta \in \Omega_\jmath,\ \jmath = 0, \ldots, m, \tag{1.1}$$

$$y(\vartheta) = \Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right); \quad \vartheta \in \tilde{\Omega}_\jmath,\ \jmath = 1, \ldots, m, \tag{1.2}$$

$$\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \tag{1.3}$$

where ${}^{RC}_{0}D^{\alpha}_{\varkappa}$ represent the Riesz-Caputo derivative of order $0 < \alpha \leq 1$, $\Theta := [0, \varkappa]$, $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ where $\delta_1 \neq 0$, $\Omega_0 := [0, \vartheta_1]$, $\Omega_\jmath := (\varkappa_\jmath, \vartheta_{\jmath+1}]; \jmath = 1, \ldots, m$, $\tilde{\Omega}_\jmath := (\vartheta_\jmath, \varkappa_\jmath]; \jmath = 1, \ldots, m$, $0 = \vartheta_0 = \varkappa_0 < \vartheta_1 \leq \varkappa_1 < \vartheta_2 \leq \varkappa_2 < \cdots \leq \varkappa_{m-1} < \vartheta_m \leq \varkappa_m < \vartheta_{m+1} = \varkappa < \infty$, $y\left(\vartheta_\jmath^+\right) = \lim\limits_{\epsilon \to 0^+} y(\vartheta_\jmath + \epsilon)$ and $y\left(\vartheta_\jmath^-\right) = \lim\limits_{\epsilon \to 0^-} y(\vartheta_\jmath + \epsilon)$ represent the right and left hand limits of $y(\vartheta)$ at $\vartheta = \vartheta_\jmath$, $\varphi : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $\Psi_\jmath : \tilde{\Omega}_\jmath \times \mathbb{R} \to \mathbb{R}$; $\jmath = 1, \ldots, m$ are given continuous functions.

The following are the primary novelties of the current paper:

- Given the varied conditions we imposed on problem (1.1), our study may be viewed as a partial continuation of the ones in the aforementioned studies.
- The Riesz derivative is a two-sided fractional operator that includes both left and right derivatives, allowing it to capture both past and future memory effects. This capability is useful for fractional modeling on a finite domain in particular.
- By modifying the constants δ_1, δ_2 and δ_3, we can obtain several problems studied in the literature, see for example [14, 15, 17].
- The study of Ulam-Hyers-Rassias stability of a problem with non-instantaneous impulses, delay and anticipation.

The following is how the current paper is arranged. In Section 2, we present certain notations and review some preliminary information on the Riesz-Caputo fractional derivative and auxiliary results. Section 3 presents two solutions to the problem (1.1)-(1.3) based on the Banach contraction principle and Krasnoselskii's fixed point theorem. The Ulam-Hyers-Rassias Stability for our problem is discussed in Section 4. Finally, in the final part, we provide an example to demonstrate the application of our study results.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts which are used throughout this paper.

We denote by $C(\Theta,\mathbb{R})$ the Banach space of all continuous functions from Θ to $\mathbb{R}$, with the norm

$$\|\xi\|_\infty = \sup\{|\xi(\vartheta)| : \vartheta \in \Theta\}.$$

Consider the Banach space

$$\begin{aligned} PC(\Theta,\mathbb{R}) = \Big\{ & y:\Theta\to\mathbb{R} : y|_{\tilde{\Omega}_\jmath} = \Psi_\jmath; \jmath = 1,\dots,m,\ y|_{\Omega_\jmath} \in C(\Omega_\jmath,\mathbb{R}); \\ & \jmath = 0,\dots,m, \text{ and there exist } u\left(\vartheta_\jmath^-\right),\ y\left(\vartheta_\jmath^+\right), y(\varkappa_\jmath^-), \\ & \text{and } y(\varkappa_\jmath^+) \text{ with } y\left(\vartheta_\jmath^-\right) = y(\vartheta_\jmath)\Big\}, \end{aligned}$$

with the norm

$$\|y\|_{PC} = \left\{\sup_{\vartheta\in\Theta} |y(\vartheta)|\right\}.$$

Definition 2.1 ([20])**.** Let $\alpha > 0$. The Riemann-Liouville fractional integral of a function $\varphi \in C(\Theta,\mathbb{R})$ of order α is given by

$${}_0I_\vartheta^\alpha \varphi(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta-\varrho)^{\alpha-1}\varphi(\varrho)d\varrho.$$

Definition 2.2 ([20])**.** Let $\alpha > 0$. The Riesz fractional integral of a function $\varphi \in C(\Theta,\mathbb{R})$ of order α is defined by

$$\begin{aligned} {}_0I_\varkappa^\alpha \varphi(\vartheta) &= \frac{1}{\Gamma(\alpha)} \int_0^\varkappa |\vartheta-\varrho|^{\alpha-1}\varphi(\varrho)d\varrho \\ &= {}_0I_\vartheta^\alpha \varphi(\vartheta) + {}_\vartheta I_\varkappa^\alpha \varphi(\vartheta), \end{aligned}$$

where ${}_0I_\vartheta^\alpha$ is the fractional integral of Riemann-Liouville.

Definition 2.3 ([20])**.** Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}_0$. The Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta,\mathbb{R})$ of order α are given by

$${}_0^CD_\vartheta^\alpha \varphi(\vartheta) = \frac{1}{\Gamma(n+1-\alpha)} \int_0^\vartheta (\vartheta-\varrho)^{n-\alpha}\varphi^{(n+1)}(\varrho)d\varrho.$$

Definition 2.4 ([20])**.** Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}_0$. The Riesz-Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta,\mathbb{R})$ of order α is given by

$$\begin{aligned} {}_0^{RC}D_\varkappa^\alpha \varphi(\vartheta) &= \frac{1}{\Gamma(n+1-\alpha)} \int_0^\varkappa |\vartheta-\varrho|^{n-\alpha}\varphi^{(n+1)}(\varrho)d\varrho \\ &= \frac{1}{2}\left({}_0^CD_\vartheta^\alpha \varphi(\vartheta) + (-1)^{n+1}{}_\vartheta^CD_\varkappa^\alpha \varphi(\vartheta)\right), \end{aligned}$$

where ${}_{0}^{C}D_{\vartheta}^{\alpha}$ is the Caputo derivative. If we take $0<\alpha\leq 1$ and $\varphi\in C(\Theta,\mathbb{R})$, we obtain

$$ {}_{0}^{RC}D_{\varkappa}^{\alpha}\varphi\left(\vartheta\right)=\frac{1}{2}\left({}_{0}^{C}D_{\vartheta}^{\alpha}\varphi\left(\vartheta\right)-{}_{\vartheta}^{C}D_{\varkappa}^{\alpha}\varphi\left(\vartheta\right)\right). $$

Lemma 2.5 ([20])**.** *If $\xi\in C^{n+1}(\Theta,\mathbb{R})$ and $\alpha\in(n,n+1]$, then we have*

$$ {}_{0}I_{\vartheta}^{\alpha}\ {}_{0}^{C}D_{\vartheta}^{\alpha}\xi\left(\vartheta\right)=\xi\left(\vartheta\right)-\sum_{\jmath=0}^{n}\frac{\xi^{(\jmath)}(0)}{\jmath!}\vartheta^{\jmath}, $$

and

$$ {}_{\vartheta}I_{\varkappa}^{\alpha}\ {}_{\vartheta}^{C}D_{\varkappa}^{\alpha}\xi\left(\vartheta\right)=(-1)^{n+1}\left[\xi\left(\vartheta\right)-\sum_{\jmath=0}^{n}\frac{(-1)^{\jmath}\xi^{(\jmath)}(\varkappa)}{\jmath!}(\varkappa-\vartheta)^{\jmath}\right]. $$

Consequently, we may have

$$ {}_{0}I_{\varkappa}^{\alpha}\ {}_{0}^{RC}D_{\varkappa}^{\alpha}\xi\left(\vartheta\right)=\frac{1}{2}\left({}_{0}I_{\vartheta}^{\alpha}\ {}_{0}^{C}D_{\vartheta}^{\alpha}\xi\left(\vartheta\right)+(-1)^{n+1}\ {}_{\vartheta}I_{\varkappa}^{\alpha}\ {}_{\vartheta}^{C}D_{\varkappa}^{\alpha}\xi\left(\vartheta\right)\right). $$

In particular, if $0<\alpha\leq 1$, then we obtain

$$ {}_{0}I_{\varkappa}^{\alpha}\ {}_{0}^{RC}D_{\varkappa}^{\alpha}\xi\left(\vartheta\right)=\xi\left(\vartheta\right)-\frac{1}{2}\left(\xi(0)+\xi\left(\varkappa\right)\right). $$

Lemma 2.6. *Let $\varpi\in C(\Theta,\mathbb{R})$ and $0<\alpha\leq 1$. Then $y\in C(\Theta,\mathbb{R})$ is a solution of*

$$ {}_{0}^{RC}D_{\varkappa}^{\alpha}y\left(\vartheta\right)=\varpi\left(\vartheta\right),\quad \vartheta\in\Theta, \tag{2.1} $$

if and only if y verifies the following integral equation:

$$ y\left(\vartheta\right)=y(0)-\frac{1}{\Gamma(\alpha)}\int_{0}^{\varkappa}\varrho^{\alpha-1}\varpi(\varrho)d\varrho+\frac{1}{\Gamma(\alpha)}\int_{0}^{\varkappa}|\vartheta-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho. \tag{2.2} $$

Proof. From Definition 2.2, Definition 2.4, and Lemma 2.5, we have

$$ {}_{0}I_{\varkappa}^{\alpha}\ {}_{0}^{RC}D_{\varkappa}^{\alpha}y\left(\vartheta\right)=y\left(\vartheta\right)-\frac{1}{2}\left(y(0)+y(\varkappa)\right), $$

which implies that

$$ \begin{aligned} y\left(\vartheta\right)&=\frac{1}{2}\left(y(0)+y(\varkappa)\right)+\ {}_{0}I_{\varkappa}^{\alpha}\varpi\left(\vartheta\right),\\ &=\frac{1}{2}\left(y(0)+y(\varkappa)\right)+\frac{1}{\Gamma(\alpha)}\int_{0}^{\varkappa}|\vartheta-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho\\ &=\frac{1}{2}\left(y(0)+y(\varkappa)\right)+\frac{1}{\Gamma(\alpha)}\int_{0}^{\vartheta}(\vartheta-\varrho)^{\alpha-1}\varpi(\varrho)d\varrho\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\vartheta}^{\varkappa}(\varrho-\vartheta)^{\alpha-1}\varpi(\varrho)d\varrho. \end{aligned} $$

For $\vartheta = 0$, we have

$$y(\varkappa) = y(0) - \frac{2}{\Gamma(\alpha)} \int_0^{\varkappa} \varrho^{\alpha-1} \varpi(\varrho) d\varrho.$$

Then, the final solution is given by:

$$y(\vartheta) = y(0) - \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \varrho^{\alpha-1} \varpi(\varrho) d\varrho + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\vartheta - \varrho|^{\alpha-1} \varpi(\varrho) d\varrho.$$

Conversely, we can easily show by Lemma 2.5 that if ξ verifies equation (2.2), then it satisfied the equation (2.1). □

2.1. Some Fixed Point Theorems.

Theorem 2.7 (Banach's fixed point theorem [37]). *Let E be a Banach space and $\mathcal{H} : E \longrightarrow E$ be a contraction, i.e. there exists $\jmath \in [0,1)$ such that*

$$\|\mathcal{H}(\xi_1) - \mathcal{H}(\xi_2)\| \leq \jmath \|\xi_1 - \xi_2\|, \quad \forall \xi_1, \xi_2 \in E.$$

Then $\mathcal{H}$ has a unique fixed point.

Theorem 2.8 (Krasnoselskii's fixed point theorem [16]). *Let D be a closed, convex, and nonempty subset of a Banach space E, and A, B the operators such that*

1) *$Ax + By \in D$ for all $x, y \in D$;*
2) *A is compact and continuous;*
3) *B is a contraction mapping.*

Then there exists $z \in D$ such that $z = Az + Bz$.

3. Main Results

Definition 3.1. By a solution of problem (1.1)-(1.3), we mean a function $y \in PC(\Theta, \mathbb{R})$ that verifies the equations (1.1), (1.2) and the boundary condition (1.3).

Theorem 3.2. *The function* y *verifies* (1.1)-(1.3) *if and only if it verifies*
(3.1)

$$y(\vartheta)=\begin{cases}\dfrac{\delta_3}{\delta_1}-\dfrac{\delta_2\Psi_m\left(\varkappa_m,y\left(t_m^-\right)\right)}{\delta_1}\\ \quad+\dfrac{\delta_2}{\Gamma(\alpha)\delta_1}\displaystyle\int_{\varkappa_m}^{\varkappa}\varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ \quad-\dfrac{\delta_2}{\Gamma(\alpha)\delta_1}\displaystyle\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ \quad-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\vartheta_1}\varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ \quad+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^{\vartheta_1}|\vartheta-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho, & \vartheta\in\Omega_0,\\ \\ \Psi_\jmath\left(\varkappa_\jmath,y\left(\vartheta_\jmath^-\right)\right)\\ \quad-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}\varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ \quad+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho, & \vartheta\in\Omega_\jmath;\jmath=1,\dots,m,\\ \\ \Psi_\jmath\left(\vartheta,y\left(\vartheta_\jmath^-\right)\right), & \vartheta\in\tilde{\Omega}_\jmath;\jmath=1,\dots,m.\end{cases}$$

Proof. Assume y satisfies (1.1)-(1.3). If $\vartheta\in\Omega_0$, then

$${}_0^{RC}D_{\vartheta_1}^{\alpha}y(\vartheta)=\varphi\left(\vartheta,y(\vartheta)\right).$$

By Lemma 2.6, we get

$$\begin{aligned}y(\vartheta)=y(0)&-\frac{1}{\Gamma(\alpha)}\int_0^{\vartheta_1}\varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{\vartheta_1}|\vartheta-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho.\end{aligned}$$

If $\vartheta\in\tilde{\Omega}_1$, then we have $y(\vartheta)=\Psi_1(\vartheta,y(\vartheta_1^-))$.
If $\vartheta\in\Omega_1$, then Lemma 2.6 implies

$$\begin{aligned}y(\vartheta)&=y(\varkappa_1)-\frac{1}{\Gamma(\alpha)}\int_{\varkappa_1}^{\vartheta_2}\varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\varkappa_1}^{\vartheta_2}|\vartheta-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ &=\Psi_1\left(\varkappa_1,y\left(\vartheta_1^-\right)\right)-\frac{1}{\Gamma(\alpha)}\int_{\varkappa_1}^{\vartheta_2}\varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\varkappa_1}^{\vartheta_2}|\vartheta-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho.\end{aligned}$$

If $\vartheta \in \tilde{\Omega}_2$, then we have $y(\vartheta) = \Psi_2(\vartheta, y(\vartheta_2^-))$.
If $\vartheta \in \Omega_2$, then Lemma 2.6 implies

$$\begin{aligned} y(\vartheta) = y(\varkappa_2) &- \frac{1}{\Gamma(\alpha)} \int_{\varkappa_2}^{\vartheta_3} \varrho^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ &+ \frac{1}{\Gamma(\alpha)} \int_{\varkappa_2}^{\vartheta_3} |\vartheta - \varrho|^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ &= \Psi_2\left(\varkappa_2, y\left(\vartheta_2^-\right)\right) - \frac{1}{\Gamma(\alpha)} \int_{\varkappa_2}^{\vartheta_3} \varrho^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ &+ \frac{1}{\Gamma(\alpha)} \int_{\varkappa_2}^{\vartheta_3} |\vartheta - \varrho|^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho. \end{aligned}$$

Repeating the process in this way, for $\vartheta \in \Theta$ we can obtain

(3.2)

$$y(\vartheta) = \begin{cases} y(0) - \frac{1}{\Gamma(\alpha)} \int_0^{\vartheta_1} \varrho^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^{\vartheta_1} |\vartheta - \varrho|^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho, & \vartheta \in \Omega_0, \\ \Psi_\jmath\left(\varkappa_\jmath, y\left(\vartheta_\jmath^-\right)\right) \\ \quad - \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho, & \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ \Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right), & \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m. \end{cases}$$

Taking $\vartheta = \varkappa$ in (3.2), we obtain

$$\begin{aligned} y(\varkappa) = \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right) &- \frac{1}{\Gamma(\alpha)} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ &+ \frac{1}{\Gamma(\alpha)} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho. \end{aligned}$$

Using the condition (1.3), we get

$$\begin{aligned} y(0) = \frac{\delta_3}{\delta_1} - \frac{\delta_2 \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right)}{\delta_1} &+ \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho \\ &- \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi(\varrho, y(\varrho))\, d\varrho. \end{aligned}$$

Substituting the value of $y(0)$ in (3.2), we obtain (3.1).

Reciprocally, for $\vartheta \in \Omega_0$, taking $\vartheta = 0$, we get

$$y(0) = \frac{\delta_3}{\delta_1} - \frac{\delta_2 \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right)}{\delta_1} + \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho$$
$$- \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho,$$

and for $\vartheta \in \Omega_m$, taking $\vartheta = \varkappa$, we get

$$y(\varkappa) = \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right) - \frac{1}{\Gamma(\alpha)} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho.$$

Thus, we can obtain $\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3$, which implies that (1.3) is verified. Next, apply ${}^{RC}_{\varkappa_\jmath}D^{\alpha}_{\vartheta_{\jmath+1}}(\cdot)$ on both sides of (3.1), where $\jmath = 0, \ldots, m$. Then, by Lemma 2.6 we get the equation (1.1). Also, it is clear that y verifies (1.2). □

We are now in a position to prove the existence result of the problem (1.1)-(1.3) based on the Banach's contraction principle.

Let us assume the following assumptions:

($Ax1$) The function $\varphi : \Theta \times \mathbb{R} \to \mathbb{R}$ is continuous.

($Ax2$) There exists a constant $\psi_1 > 0$ where

$$|\varphi(\vartheta, \xi) - \varphi(\vartheta, \bar{\xi})| \leq \psi_1 |\xi - \bar{\xi}|,$$

for any $\xi, \bar{\xi} \in \mathbb{R}$ and $\vartheta \in \Omega_\jmath$; $\jmath = 0, \ldots, m$.

($Ax3$) The functions $\Psi_\jmath$ are continuous and there exist constants $\wp_\jmath > 0$ such that

$$|\Psi_\jmath(\vartheta, \xi) - \Psi_\jmath(\vartheta, \bar{\xi})| \leq \wp_\jmath |\xi - \bar{\xi}|,$$

for any $\xi, \bar{\xi} \in \mathbb{R}$, $\jmath = 1, \ldots, m$.

Set

$$\wp^* = \max_{\jmath = 1, \ldots, m} \{\wp_\jmath\}.$$

Theorem 3.3. *Assume that the assumptions* ($Ax1$)-($Ax3$) *hold. If*

$$\beta := \wp^* \left(1 + \frac{|\delta_2|}{|\delta_1|}\right) + \frac{2(\delta_1 + \delta_2)\psi_1 \varkappa^\alpha}{\delta_1 \Gamma(\alpha + 1)} < 1, \tag{3.3}$$

then the implicit fractional problem (1.1)-(1.3) *has a unique solution on* Θ.

Proof. Let us transform the problem (1.1)-(1.3) into a fixed point problem by defining the operator $\aleph : PC(\Theta,\mathbb{R}) \longrightarrow PC(\Theta,\mathbb{R})$ by:

(3.4)

$$\aleph y(\vartheta) = \begin{cases} \frac{\delta_3}{\delta_1} - \frac{\delta_2 \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right)}{\delta_1} \\ \quad + \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad - \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad - \frac{1}{\Gamma(\alpha)} \int_0^{\vartheta_1} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^{\vartheta_1} |\vartheta - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho, & \vartheta \in \Omega_0, \\ \\ \Psi_\jmath\left(\varkappa_\jmath, y\left(\vartheta_\jmath^-\right)\right) \\ \quad - \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad + \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho, & \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ \\ \Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right), & \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m. \end{cases}$$

Obviously, the fixed points of the operator $\varkappa$ are solutions of the problem (1.1)-(1.3).
Let $y, z \in PC(\Theta,\mathbb{R})$. Then for $\vartheta \in \Omega_0$ we have

$$\begin{aligned} |\aleph y(\vartheta) - \aleph z(\vartheta)| \leq \; & \frac{|\delta_2|}{|\delta_1|} \left| \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right) - \Psi_m\left(\varkappa_m, z\left(t_m^-\right)\right) \right| \\ & + \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} |\varphi\left(\varrho, y(\varrho)\right) - \varphi(\varrho, z(\varrho))| d\varrho \\ & + \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} |\varphi\left(\varrho, y(\varrho)\right) - \varphi(\varrho, z(\varrho))| d\varrho \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{\vartheta_1} \varrho^{\alpha-1} |\varphi\left(\varrho, y(\varrho)\right) - \varphi(\varrho, z(\varrho))| d\varrho \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{\vartheta_1} |\vartheta - \varrho|^{\alpha-1} |\varphi\left(\varrho, y(\varrho)\right) - \varphi(\varrho, z(\varrho))| d\varrho. \end{aligned}$$

Then, by $(Ax2)$ we find that

$$|\aleph y(\vartheta) - \aleph z(\vartheta)| \leq \frac{|\delta_2| \wp_\jmath}{|\delta_1|} \|y - z\|_{PC}$$

$$
\begin{aligned}
&+\frac{\delta_2\psi_1}{\Gamma(\alpha)\delta_1}\|y-z\|_{PC}\int_{\varkappa_m}^{\varkappa}\varrho^{\alpha-1}d\varrho\\
&+\frac{\delta_2\psi_1}{\Gamma(\alpha)\delta_1}\|y-z\|_{PC}\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\alpha-1}d\varrho\\
&+\frac{\psi_1}{\Gamma(\alpha)}\|y-z\|_{PC}\int_{0}^{\vartheta_1}\varrho^{\alpha-1}d\varrho\\
&+\frac{\psi_1}{\Gamma(\alpha)}\|y-z\|_{PC}\int_{0}^{\vartheta_1}|\vartheta-\varrho|^{\alpha-1}d\varrho\\
\leq&\left[\frac{2(\delta_1+\delta_2)\psi_1\varkappa^{\alpha}}{\delta_1\Gamma(\alpha+1)}+\frac{|\delta_2|\wp^*}{|\delta_1|}\right]\|y-z\|_{PC}.
\end{aligned}
$$

For $\vartheta\in\Omega_\jmath;\jmath=1,\dots,m$, we have

$$
\begin{aligned}
|\aleph y(\vartheta)-\aleph z(\vartheta)|\leq&\left|\Psi_\jmath\left(\varkappa_\jmath,y\left(\vartheta_\jmath^-\right)\right)-\Psi_\jmath\left(\varkappa_\jmath,z\left(\vartheta_\jmath^-\right)\right)\right|\\
&+\frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}\varrho^{\alpha-1}|\varphi(\varrho,y(\varrho))-\varphi(\varrho,z(\varrho))|d\varrho\\
&+\frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\alpha-1}|\varphi(\varrho,y(\varrho))-\varphi(\varrho,z(\varrho))|d\varrho.
\end{aligned}
$$

Then by $(Ax2)$ and $(Ax3)$, we find that

$$
\begin{aligned}
|\aleph y(\vartheta)-\aleph z(\vartheta)|\leq&\wp_\jmath\|y-z\|_{PC}\\
&+\frac{\psi_1}{\Gamma(\alpha)}\|y-z\|_{PC}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}\varrho^{\alpha-1}d\varrho\\
&+\frac{\psi_1}{\Gamma(\alpha)}\|y-z\|_{PC}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\alpha-1}d\varrho\\
\leq&\left[\wp^*+\frac{2\psi_1\varkappa^{\alpha}}{\Gamma(\alpha+1)}\right]\|y-z\|_{PC}.
\end{aligned}
$$

For $\vartheta\in\tilde{\Omega}_\jmath;\jmath=1,\dots,m$, we have

$$
\begin{aligned}
|\aleph y(\vartheta)-\aleph z(\vartheta)|\leq&\left|\Psi_\jmath\left(\vartheta,y\left(\vartheta_\jmath^-\right)\right)-\Psi_\jmath(\vartheta,z\left(\vartheta_\jmath^-\right))\right|\\
\leq&\wp^*\|y-z\|_{PC}.
\end{aligned}
$$

Thus, we can conclude that

$$\|\aleph y-\aleph z\|_{PC}\leq\beta\|y-z\|_{PC}.$$

Consequently, by the Banach's contraction principle, the operator $\aleph$ has a unique fixed point which is solution of the problem (1.1)-(1.3). □

Our second result is based on Krasnoselskii's fixed point theorem.

Remark 3.4. Let us put

$$\lambda_2(\vartheta) = |\varphi(\vartheta, 0)|, \qquad \Phi_2^k(\vartheta) = |\Psi_\jmath(\vartheta, 0)|, \quad \psi_1 = \tilde{\lambda}_1, \ \wp^* = \Phi_1,$$

then the hypothesis $(Ax2)$ implies that

$$|\varphi(\vartheta, \xi)| \leq \tilde{\lambda}_1 |\xi| + \lambda_2(\vartheta),$$

and the hypothesis $(Ax3)$ implies that

$$|\Psi_\jmath(\vartheta, \xi)| \leq \Phi_1 |\xi| + \Phi_2,$$

for $\vartheta \in \Theta$, $\xi \in \mathbb{R}$ and $\lambda_2, \Phi_2^k \in C(\Theta, \mathbb{R}_+)$, with

$$\tilde{\lambda}_2 = \sup_{\vartheta \in \Theta} \lambda_2(\vartheta), \qquad \Phi_2 = \sup_{\vartheta \in \Theta} \Phi_2^k(\vartheta).$$

Theorem 3.5. *Assume* (Ax1)-(Ax3) *hold. If*

$$\tilde{\beta} := \left[\Phi_1 \left(1 + \frac{2|\delta_2|}{|\delta_1|} \right) + \frac{2\delta_2 \tilde{\lambda}_1 \varkappa^\alpha}{\delta_1 \Gamma(\alpha+1)} \right] < 1, \tag{3.5}$$

then the problem (1.1)-(1.3) *has at least one solution in* $PC(\Theta, \mathbb{R})$.

Proof. Consider the set

$$\Upsilon_\sigma = \left\{ \xi \in PC(\Theta, \mathbb{R}) : \|\xi\|_{PC} \leq \sigma \right\},$$

where

$$\sigma \geq 2 \max \Bigg\{ \Phi_1 \sigma + \Phi_2, \left[\frac{|\delta_3| + |\delta_2| \Phi_2}{|\delta_1|} + \frac{2|\delta_2| \varkappa^\alpha \tilde{\lambda}_2}{\Gamma(\alpha+1)|\delta_1|} \right] + \sigma \left[\frac{|\delta_2| \Phi_1}{|\delta_1|} + \frac{2|\delta_2| \varkappa^\alpha \tilde{\lambda}_1}{\Gamma(\alpha+1)|\delta_1|} \right] \Bigg\}.$$

We define the operators $\aleph_1$ and $\aleph_2$ on Υ_σ by

$$\aleph_1 x(\vartheta) = \begin{cases} \dfrac{\delta_3}{\delta_1} - \dfrac{\delta_2 \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right)}{\delta_1} \\ \quad + \dfrac{\delta_2}{\Gamma(\alpha)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad - \dfrac{\delta_2}{\Gamma(\alpha)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho, & \vartheta \in \Omega_0, \\ \Psi_\jmath\left(\varkappa_\jmath, y\left(\vartheta_\jmath^-\right)\right), & \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ 0, & \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m. \end{cases} \tag{3.6}$$

and

(3.7)

$$\aleph_2 x(\vartheta)=\begin{cases} -\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho \\ \quad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta-\varrho|^{\alpha-1}\varphi\left(\varrho,y(\varrho)\right)d\varrho, & \vartheta\in\Omega_\jmath;\jmath=0,\ldots,m, \\ \Psi_\jmath\left(\vartheta,y\left(\vartheta_\jmath^-\right)\right), & \vartheta\in\tilde{\Omega}_\jmath;\jmath=1,\ldots,m. \end{cases}$$

Then, we can write the following operator equation

$$\aleph y(\vartheta)=\aleph_1 y(\vartheta)+\aleph_2 y(\vartheta),\quad y\in PC(\Theta,\mathbb{R}).$$

Step 1: We prove that $\aleph_1 x+\aleph_2 y\in\Upsilon_\sigma$ for any $x,y\in\Upsilon_\sigma$.
For $\vartheta\in\Omega_0$, by (3.6) and Remark 3.4, we obtain

$$\begin{aligned} |(\aleph_1 x)(\vartheta)| &\le \frac{|\delta_3|}{|\delta_1|}+\frac{|\delta_2\Psi_m\left(\varkappa_m,x\left(t_m^-\right)\right)|}{|\delta_1|} \\ &\quad+\frac{|\delta_2|}{\Gamma(\alpha)|\delta_1|}\int_{\varkappa_m}^{\varkappa}\varrho^{\alpha-1}|\varphi\left(\varrho,x(\varrho)\right)|d\varrho \\ &\quad+\frac{|\delta_2|}{\Gamma(\alpha)|\delta_1|}\int_{\varkappa_m}^{\varkappa}|\varkappa-\varrho|^{\alpha-1}|\varphi\left(\varrho,x(\varrho)\right)|d\varrho \\ &\le \frac{|\delta_3|}{|\delta_1|}+\frac{|\delta_2|(\Phi_1\sigma+\Phi_2)}{|\delta_1|}+\frac{2|\delta_2|\varkappa^\alpha\left(\tilde{\lambda}_1\sigma+\tilde{\lambda}_2\right)}{\Gamma(\alpha+1)|\delta_1|} \\ &\le \left[\frac{|\delta_3|+|\delta_2|\Phi_2}{|\delta_1|}+\frac{2|\delta_2|\varkappa^\alpha\tilde{\lambda}_2}{\Gamma(\alpha+1)|\delta_1|}\right] \\ &\quad+\sigma\left[\frac{|\delta_2|\Phi_1}{|\delta_1|}+\frac{2|\delta_2|\varkappa^\alpha\tilde{\lambda}_1}{\Gamma(\alpha+1)|\delta_1|}\right], \end{aligned}$$

and for $\vartheta\in\Omega_\jmath;\jmath=1,\ldots,m$, we have

$$\begin{aligned} |(\aleph_1 x)(\vartheta)| &\le |\Psi_\jmath\left(\varkappa_\jmath,x\left(\vartheta_\jmath^-\right)\right)| \\ &\le \Phi_1\sigma+\Phi_2, \end{aligned}$$

then for each $\vartheta\in\Theta$ we get

$$\|\aleph_1 x\|_{PC}\le\max\left\{\Phi_1\sigma+\Phi_2,\left[\frac{|\delta_3|+|\delta_2|\Phi_2}{|\delta_1|}+\frac{2|\delta_2|\varkappa^\alpha\tilde{\lambda}_2}{\Gamma(\alpha+1)|\delta_1|}\right]\right.$$
$$\left.+\sigma\left[\frac{|\delta_2|\Phi_1}{|\delta_1|}+\frac{2|\delta_2|\varkappa^\alpha\tilde{\lambda}_1}{\Gamma(\alpha+1)|\delta_1|}\right]\right\}. \tag{3.8}$$

For $\vartheta \in \Omega_\jmath; \jmath = 0, \dots, m$, by (3.7) and Remark 3.4, we obtain

$$\begin{aligned}|(\aleph_2 y)(\vartheta)| &\leq \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\alpha-1} |\varphi(\varrho, y(\varrho))| d\varrho \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\alpha-1} |\varphi(\varrho, y(\varrho))| d\varrho \\ &\leq \frac{2|\delta_2| \varkappa^\alpha \left(\tilde{\lambda}_1 \sigma + \tilde{\lambda}_2\right)}{\Gamma(\alpha+1)|\delta_1|},\end{aligned}$$

and for $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \dots, m$, we have

$$\begin{aligned}|(\aleph_2 y)(\vartheta)| &\leq \left|\Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right)\right| \\ &\leq \Phi_1 \sigma + \Phi_2,\end{aligned}$$

then for each $\vartheta \in \Theta$ we get

$$\|\aleph_2 y\|_{PC} \leq \max\left\{\Phi_1 \sigma + \Phi_2, \frac{2|\delta_2| \varkappa^\alpha \left(\tilde{\lambda}_1 \sigma + \tilde{\lambda}_2\right)}{\Gamma(\alpha+1)|\delta_1|}\right\}. \tag{3.9}$$

From (3.8) and (3.9), for each $\vartheta \in \Theta$ we have,

$$\begin{aligned}\|\aleph_1 x + \aleph_2 y\|_{PC} &\leq \|\aleph_1 x\|_{PC} + \|\aleph_2 y\|_{PC} \\ &\leq \sigma,\end{aligned}$$

thus $\aleph_1 x + \aleph_2 y \in \Upsilon_\sigma$.

Step 2: $\aleph_1$ is a contraction.
Let $y, z \in PC(\Theta, \mathbb{R})$. Then for $\vartheta \in \Omega_0$ we have

$$\begin{aligned}|\aleph_1 y(\vartheta) - \aleph_1 z(\vartheta)| &\leq \frac{|\delta_2|}{|\delta_1|} \left|\Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right) - \Psi_m\left(\varkappa_m, z\left(t_m^-\right)\right)\right| \\ &\quad + \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} |\varphi(\varrho, y(\varrho)) - \varphi(\varrho, z(\varrho))| d\varrho \\ &\quad + \frac{\delta_2}{\Gamma(\alpha)\delta_1} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} |\varphi(\varrho, y(\varrho)) - \varphi(\varrho, z(\varrho))| d\varrho.\end{aligned}$$

Then, by $(Ax2)$ and Remark 3.4, we find that

$$\begin{aligned}|\aleph_1 y(\vartheta) - \aleph_1 z(\vartheta)| &\leq \frac{|\delta_2| \wp^*}{|\delta_1|} \|y - z\|_{PC} \\ &\quad + \frac{|\delta_2| \psi_1}{\Gamma(\alpha)|\delta_1|} \|y - z\|_{PC} \int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} d\varrho \\ &\quad + \frac{|\delta_2| \psi_1}{\Gamma(\alpha)|\delta_1|} \|y - z\|_{PC} \int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} d\varrho\end{aligned}$$

$$\leq \left[\frac{2|\delta_2|\psi_1\varkappa^\alpha}{|\delta_1|\Gamma(\alpha+1)} + \frac{|\delta_2|\wp^*}{|\delta_1|}\right]\|y-z\|_{PC}.$$

For $\vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m$, we have

$$\begin{aligned}|(\aleph_1 y)(\vartheta) - (\aleph_1 z)(\vartheta)| &\leq |\Psi_\jmath\left(\varkappa_\jmath, y\left(\vartheta_\jmath^-\right)\right) - \Psi_\jmath\left(\varkappa_\jmath, z\left(\vartheta_\jmath^-\right)\right)| \\ &\leq \Phi_1\|y-z\|_{PC}.\end{aligned}$$

Then, for each $\vartheta \in \Theta$, we have

$$\|\aleph_1 y - \aleph_1 z\|_{PC} \leq \tilde{\beta}\|y-z\|_{PC}.$$

Then by (3.5), the operator $\aleph_1$ is a contraction.

Step 3: $\aleph_2$ is continuous and compact. Let $\{y_n\}$ be a sequence where $y_n \to y$ in $PC(\Theta, \mathbb{R})$.
For $\vartheta \in \Omega_\jmath; \jmath = 0, \ldots, m$, we have,

$$\begin{aligned}&|(\aleph_2 y_n)(\vartheta) - (\aleph_2 y)(\vartheta)| \\ &\quad\leq \frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\alpha-1}|\varphi(\varrho, y_n(\varrho)) - \varphi(\varrho, y(\varrho))|d\varrho \\ &\qquad + \frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta-\varrho|^{\alpha-1}|\varphi(\varrho, y_n(\varrho)) - \varphi(\varrho, y(\varrho))|d\varrho.\end{aligned}$$

For each $\vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$, we have,

$$|(\aleph_2 y_n)(\vartheta) - (\aleph_2 y)(\vartheta)| \leq \left|\Psi_\jmath(\vartheta, y_n\left(\vartheta_\jmath^-\right)) - \Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right)\right|.$$

Since $y_n \to y$ and since φ and $\Psi_\jmath$ are continuous, then we may obtain

$$\|\aleph_2 y_n - \aleph_2 y\|_{PC} \to 0 \quad \text{as} \quad n \to \infty.$$

Then $\aleph_2$ is continuous. Now we demonstrate that $\aleph_2$ is uniformly bounded on Υ_σ. Let $y \in \Upsilon_\sigma$. Thus, for $\vartheta \in \Theta$,

$$\|\aleph_2 y\|_{PC} \leq \max\left\{\Phi_1\sigma + \Phi_2, \frac{2|\delta_2|\varkappa^\alpha\left(\tilde{\lambda}_1\sigma + \tilde{\lambda}_2\right)}{\Gamma(\alpha+1)|\delta_1|}\right\}.$$

Consequently, $\aleph_2$ is uniformly bounded on Υ_σ. We take $y \in \Upsilon_\sigma$ and $0 < \gamma_1 < \gamma_2 \leq \varkappa$. Then for $\gamma_1, \gamma_2 \in \Omega_\jmath; \jmath = 0, \ldots, m$,

$$\begin{aligned}&|(\aleph_2 y)(\gamma_1) - (\aleph_2 y)(\gamma_2)| \\ &\quad\leq \left|\frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\gamma_1-\varrho|^{\alpha-1}\varphi(\varrho, y(\varrho))\,d\varrho \right.\\ &\qquad \left. - \frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\gamma_2-\varrho|^{\alpha-1}\varphi(\varrho, y(\varrho))\,d\varrho\right|\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \left| |\gamma_1 - \varrho|^{\alpha-1} - |\gamma_2 - \varrho|^{\alpha-1} \right| |\varphi(\varrho, y(\varrho))| d\varrho$$
$$\leq \frac{\tilde{\lambda}_1 \sigma + \tilde{\lambda}_2}{\Gamma(\alpha)} \int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \left| |\gamma_1 - \varrho|^{\alpha-1} - |\gamma_2 - \varrho|^{\alpha-1} \right| d\varrho,$$

note that

$$|(\aleph_2 y)(\gamma_1) - (\aleph_2 y)(\gamma_2)| \to 0 \quad \text{as} \quad \gamma_1 \to \gamma_2.$$

And for $\gamma_1, \gamma_2 \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m$,

$$|(\aleph_2 y)(\gamma_1) - (\aleph_2 y)(\gamma_2)| \leq \left| \Psi_\jmath \left(\gamma_1, y\left(\vartheta_\jmath^-\right)\right) - \Psi_\jmath \left(\gamma_2, y\left(\vartheta_\jmath^-\right)\right) \right|,$$

note that since $\Psi_\jmath$ are continuous

$$|(\aleph_2 y)(\gamma_1) - (\aleph_2 y)(\gamma_2)| \to 0 \quad \text{as} \quad \gamma_1 \to \gamma_2.$$

Thus, $\aleph_2 \Upsilon_\sigma$ is equicontinuous on Θ, which implies that $\aleph_2 \Upsilon_\sigma$ is relatively compact. By Arzela-Ascoli Theorem, $\aleph_2$ is compact. By Theorem 2.8, we conclude that $\Im$ admit, at least a fixed point which is a solution to the problem (1.1)-(1.3). □

4. Ulam-Hyers-Rassias stability

Now, we consider the Ulam stability for problem (1.1)-(1.3). For this, we take inspiration from the following papers [28, 34, 40] and the references therein. Let $y \in PC(\Theta, \mathbb{R})$, $\epsilon > 0$, $\zeta > 0$ and $\xi : \Theta \longrightarrow [0, \infty)$ be a continuous function. We consider the following inequalities:

$$\begin{cases} \left| \left({}^{RC}_{\varkappa_\jmath} D^\alpha_{\vartheta_{\jmath+1}} y \right)(\vartheta) - \varphi(\vartheta, y(\vartheta)) \right| \leq \epsilon, & \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \left| y(\vartheta) - \Psi_\jmath \left(\vartheta, y\left(\vartheta_\jmath^-\right)\right) \right| \leq \epsilon, & \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m. \end{cases} \tag{4.1}$$

$$\begin{cases} \left| \left({}^{RC}_{\varkappa_\jmath} D^\alpha_{\vartheta_{\jmath+1}} y \right)(\vartheta) - \varphi(\vartheta, y(\vartheta)) \right| \leq \xi(\vartheta), & \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \left| y(\vartheta) - \Psi_\jmath \left(\vartheta, y\left(\vartheta_\jmath^-\right)\right) \right| \leq \zeta, & \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m. \end{cases} \tag{4.2}$$

and

$$\begin{cases} \left| \left({}^{RC}_{\varkappa_\jmath} D^\alpha_{\vartheta_{\jmath+1}} y \right)(\vartheta) - \varphi(\vartheta, y(\vartheta)) \right| \leq \epsilon\xi(\vartheta), & \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m, \\ \left| y(\vartheta) - \Psi_\jmath \left(\vartheta, y\left(\vartheta_\jmath^-\right)\right) \right| \leq \epsilon\zeta, & \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m. \end{cases} \tag{4.3}$$

Definition 4.1. Problem (1.1)-(1.3) is Ulam-Hyers (U-H) stable if there exists a real number $a_\varphi > 0$ such that for each $\epsilon > 0$ and for each solution

$x \in PC(\Theta, \mathbb{R})$ of inequality (4.1) there exists a solution $y \in PC(\Theta, \mathbb{R})$ of (1.1)-(1.3) with

$$|x(\vartheta) - y(\vartheta)| \leq \epsilon a_{\varphi}, \quad \vartheta \in \Theta.$$

Definition 4.2. Problem (1.1)-(1.3) is generalized Ulam-Hyers (G.U-H) stable if there exists $K_{\varphi} : C([0,\infty),[0,\infty))$ with $K_{\varphi}(0) = 0$ such that for each $\epsilon > 0$ and for each solution $x \in PC(\Theta, \mathbb{R})$ of inequality (4.1) there exists a solution $y \in PC(\Theta, \mathbb{R})$ of (1.1)-(1.3) with

$$|x(\vartheta) - y(\vartheta)| \leq K_{\varphi}(\epsilon), \quad \vartheta \in \Theta.$$

Definition 4.3. Problem (1.1)-(1.3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to (ξ, ζ) if there exists a real number $a_{\varphi,\xi} > 0$ such that for each $\epsilon > 0$ and for each solution $x \in PC(\Theta, \mathbb{R})$ of inequality (4.3) there exists a solution $y \in PC(\Theta, \mathbb{R})$ of (1.1)-(1.3) with

$$|x(\vartheta) - y(\vartheta)| \leq \epsilon a_{\varphi,\xi}(\xi(\vartheta) + \zeta), \quad \vartheta \in \Theta.$$

Definition 4.4. Problem (1.1)-(1.3) is generalized Ulam-Hyers-Rassias (G.U-H-R) stable with respect to (ξ, ζ) if there exists a real number $a_{\varphi,\xi} > 0$ such that for each solution $x \in PC(\Theta, \mathbb{R})$ of inequality (4.2) there exists a solution $y \in PC(\Theta, \mathbb{R})$ of (1.1)-(1.3) with

$$|x(\vartheta) - y(\vartheta)| \leq a_{\varphi,\xi}(\xi(\vartheta) + \zeta), \quad \vartheta \in \Theta.$$

Remark 4.5. It is clear that :

(i) Definition 4.1 $\Longrightarrow$ Definition 4.2
(ii) Definition 4.3 $\Longrightarrow$ Definition 4.4
(iii) Definition 4.3 for $\xi(.) = \zeta = 1 \Longrightarrow$ Definition 4.1

Remark 4.6. A function $x \in PC(\Theta, \mathbb{R})$ is a solution of inequality (4.3) if and only if there exist $\upsilon \in PC(\Theta, \mathbb{R})$ and a sequence $\upsilon_{\jmath}, \jmath = 0, \dots, m$ such that

(i) $|\upsilon(\vartheta)| \leq \epsilon\xi(\vartheta)$, $\vartheta \in \Omega_{\jmath}, \jmath = 0, \dots, m$; and $|\upsilon_{\jmath}| \leq \epsilon\zeta$, $\vartheta \in \tilde{\Omega}_{\jmath}, \jmath = 1, \dots, m$,
(ii) $\left({}^{RC}_{\varkappa_{\jmath}}D^{\alpha}_{\vartheta_{\jmath+1}} y\right)(\vartheta) = \varphi(\vartheta, y(\vartheta)) + \upsilon(\vartheta)$, $\vartheta \in \Omega_{\jmath}, \jmath = 0, \dots, m$,
(iii) $y(\vartheta) = \Psi_{\jmath}\left(\vartheta, y\left(\vartheta_{\jmath}^{-}\right)\right) + \upsilon_{\jmath}$, $\vartheta \in \tilde{\Omega}_{\jmath}, \jmath = 1, \dots, m$.

Theorem 4.7. *Assume that in addition to* (Ax1)-(Ax3) *and (3.3), the following hypothesis holds.*

(*Ax4*) *There exist a nondecreasing function* $\xi : \Theta \longrightarrow [0, \infty)$ *and* $\kappa_{\xi} > 0$ *where for* $\vartheta \in \Omega_{\jmath}; \jmath = 0, \dots, m$, *we have*

$$({}_0I^{\alpha}_{\varkappa}\xi)(\vartheta) \leq \kappa_{\xi}\xi(\vartheta).$$

Then the problem (1.1)-(1.3) *is U-H-R stable with respect to* (ξ, ζ).

Proof. Let $x \in PC(\Theta, \mathbb{R})$ be a solution of inequality (4.3), and let us suppose that y is the unique solution of the problem

$$\begin{cases} \left({}^{RC}_{\varkappa_\jmath} D^{\alpha}_{\vartheta_{\jmath+1}} y \right)(\vartheta) = \varphi\left(\vartheta, y\left(\vartheta\right)\right); & \vartheta \in \Omega_\jmath,\ \jmath = 0, \ldots, m, \\ y(\vartheta) = \Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right); & \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m, \\ \delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, & \\ y(\varkappa_\jmath) = x(\varkappa_\jmath); & \jmath = 0, \ldots, m, \\ y(\vartheta_\jmath) = x(\vartheta_\jmath); & \jmath = 1, \ldots, m+1. \end{cases}$$

By Theorem 3.2, we obtain for each $\vartheta \in \Theta$

$$y(\vartheta) = \begin{cases} \dfrac{\delta_3}{\delta_1} - \dfrac{\delta_2 \Psi_m\left(\varkappa_m, y\left(t_m^-\right)\right)}{\delta_1} \\ \quad + \dfrac{\delta_2}{\Gamma(\alpha)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad - \dfrac{\delta_2}{\Gamma(\alpha)\delta_1} \displaystyle\int_{\varkappa_m}^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{\vartheta_1} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{\vartheta_1} |\vartheta - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho, & \vartheta \in \Omega_0, \\ \Psi_\jmath\left(\varkappa_\jmath, y\left(\vartheta_\jmath^-\right)\right) \\ \quad - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} \varrho^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}} |\vartheta - \varrho|^{\alpha-1} \varphi\left(\varrho, y(\varrho)\right) d\varrho, & \vartheta \in \Omega_\jmath; \jmath = 1, \ldots, m, \\ \Psi_\jmath\left(\vartheta, y\left(\vartheta_\jmath^-\right)\right), & \vartheta \in \tilde{\Omega}_\jmath; \jmath = 1, \ldots, m. \end{cases}$$

Since x is a solution of the inequality (4.3), by Remark 4.6, we have

$$\begin{cases} \left({}^{RC}_{\varkappa_\jmath} D^{\alpha}_{\vartheta_{\jmath+1}} x \right)(\vartheta) = \varphi\left(\vartheta, x\left(\vartheta\right)\right) + \upsilon(\vartheta), & \vartheta \in \Omega_\jmath, \jmath = 0, \ldots, m; \\ x(\vartheta) = \Psi_\jmath\left(\vartheta, x\left(\vartheta_\jmath^-\right)\right) + \upsilon_\jmath, & \vartheta \in \tilde{\Omega}_\jmath, \jmath = 1, \ldots, m. \end{cases} \tag{4.4}$$

Clearly, the solution of (4.4) is given by

$$x(\vartheta)=\begin{cases} y(\varkappa_\jmath)-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}\varrho^{\alpha-1}\left(\varphi\left(\varrho,x(\varrho)\right)+\upsilon(\varrho)\right)d\varrho \\ \quad +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\alpha-1}(\varphi\left(\varrho,x(\varrho)\right)+\upsilon(\varrho))d\varrho, \\ \qquad\qquad\qquad\qquad \text{if } \ \vartheta\in\Omega_\jmath,\jmath=1,\ldots,m, \\ \Psi_\jmath\left(\vartheta,y\left(\vartheta_\jmath^-\right)\right)+\upsilon_\jmath, \qquad \text{if } \vartheta\in\tilde{\Omega}_\jmath,\jmath=1,\ldots,m. \end{cases}$$

Hence, for each $\vartheta\in\Omega_\jmath$,$\jmath=0,\ldots,m$, we have

$$\begin{aligned} |x\left(\vartheta\right)-y\left(\vartheta\right)| &\leq \frac{1}{\Gamma(\alpha)}\int_{\varkappa_\jmath}^{\vartheta_{\jmath+1}}|\vartheta-\varrho|^{\alpha-1}|\varphi\left(\varrho,x(\varrho)\right)-\varphi\left(\varrho,y(\varrho)\right)|d\varrho \\ &\quad +\left({}_0I_\varkappa^\alpha|\upsilon(\tau)|\right) \\ &\leq \epsilon\kappa_\xi\xi(\vartheta)+\frac{\psi_1}{\Gamma(\alpha)}\|x-y\|_{PC}\int_0^\varkappa|\vartheta-\varrho|^{\alpha-1}d\varrho \\ &\leq \epsilon\kappa_\xi\xi(\vartheta)+\frac{\psi_1\varkappa^\alpha}{\Gamma(\alpha)}\|x-y\|_{PC}. \end{aligned}$$

And for each $\vartheta\in\tilde{\Omega}_\jmath$,$\jmath=1,\ldots,m$, we have

$$\begin{aligned} |x\left(\vartheta\right)-y\left(\vartheta\right)| &\leq \left|\Psi_\jmath\left(\vartheta,x\left(\vartheta_\jmath^-\right)\right)-\Psi_\jmath\left(\vartheta,y\left(\vartheta_\jmath^-\right)\right)\right|+|\upsilon_\jmath| \\ &\leq \wp^*\left|x\left(\vartheta\right)-y\left(\vartheta\right)\right|+\epsilon\zeta \\ &\leq \wp^*\|x-y\|_{PC}+\epsilon\zeta. \end{aligned}$$

Thus

$$\|x-y\|_{PC}\leq\left[\epsilon\kappa_\xi\xi(\vartheta)+\epsilon\zeta\right]+\left[\wp^*+\frac{\psi_1\varkappa^\alpha}{\Gamma(\alpha)}\right]\|x-y\|_{PC}.$$

Then for each $\vartheta\in\Theta$, we have

$$\|x-y\|_{PC}\leq a_\xi\epsilon(\zeta+\xi(\vartheta)),$$

where

$$a_\xi=\frac{1+\kappa_\xi}{1-\left[\wp^*+\dfrac{\psi_1\varkappa^\alpha}{\Gamma(\alpha)}\right]}.$$

Hence, the problem (1.1)-(1.3) is U-H-R stable with respect to (ξ,τ). □

Remark 4.8. If the conditions (*Ax1*)-(*Ax3*) and (3.3) are satisfied, then by Theorem 4.7 and Remark 4.5, it is clear the problem (1.1)-(1.3) is U-H-R stable and G.U-H-R stable. And if $\xi(\cdot)=\zeta=1$, then problem (1.1)-(1.3) is also G.U-H stable and U-H stable.

5. An Example

Consider the following impulsive problem which is an example of our problem (1.1)-(1.3).

$$\left({}^{RC}_{\varkappa_\jmath}D^{\frac{1}{2}}_{\vartheta_{\jmath+1}}y\right)(\vartheta)=\varphi\left(\vartheta,y\left(\vartheta\right)\right);\quad \vartheta\in\Omega_0\cup\Omega_1, \tag{5.1}$$

$$y(\vartheta)=\frac{|y\left(\vartheta_\jmath^-\right)|+3e^{\vartheta}}{233e^{\vartheta}+133|y\left(\vartheta_\jmath^-\right)|},\quad \vartheta\in\tilde{\Omega}_1, \tag{5.2}$$

$$y(0)+y(\varkappa)=0, \tag{5.3}$$

where $\Omega_0=(0,e]$, $\Omega_1=(3,\pi]$, $\tilde{\Omega}_1=(e,3]$, $\varkappa_0=0$, $\vartheta_1=e$ and $\varkappa_1=3$, with $\alpha=\frac{1}{2}$, $\jmath\in\{0,1\}$, $\delta_1=\delta_2=1$ and $\delta_3=0$.
Set

$$\varphi\left(\vartheta,y\left(\vartheta\right)\right)=\frac{|\cos(\vartheta)|\vartheta+|\sin(\vartheta)|}{322e^{\vartheta+2}(1+|y|)},\quad \vartheta\in\Omega_0\cup\Omega_1,\ y\in\mathbb{R}.$$

Clearly, the function φ is continuous. Hence the condition (*Ax1*) is satisfied.

For each $x,\bar{x}\in\mathbb{R}$ and $\vartheta\in\Omega_0\cup\Omega_1$, we have

$$\begin{aligned}|\varphi(\vartheta,x)-\varphi(\vartheta,\bar{x})|&\le\frac{|\cos(\vartheta)|\vartheta+|\sin(\vartheta)|}{322e^{\vartheta+2}}|x-\bar{x}|\\&\le\frac{1+\sqrt{\pi}}{322e^2}|x-\bar{x}|.\end{aligned}$$

Hence condition (*Ax2*) is satisfied with $\psi_1=\dfrac{1+\sqrt{\pi}}{322e^2}$. And let

$$\Psi_1\left(\vartheta,x\left(\vartheta_1^-\right)\right)=\frac{x}{233e^{\vartheta}+133x},\quad x\in[0,\infty),$$

and $x,y\in[0,\infty)$. Then, we have

$$\begin{aligned}\left|\Psi_1\left(\vartheta,x\left(\vartheta_1^-\right)\right)-\Psi_1\left(\vartheta,x\left(\vartheta_1^-\right)\right)\right|&=\left|\frac{x}{233e^{\vartheta}+133x}-\frac{y}{233e^{\vartheta}+133y}\right|\\&=\frac{233e^{\vartheta}|x-y|}{(233e^{\vartheta}+133x)(233e^{\vartheta}+133y)}\\&\le\frac{1}{233}|x-y|,\end{aligned}$$

and so the condition (*Ax3*) is satisfied with $\wp^*=\dfrac{1}{233}$.

Also, the condition (3.3) of Theorem 3.3 is satisfied, for

$$\beta=\wp^*\left(1+\frac{|\delta_2|}{|\delta_1|}\right)+\frac{2(\delta_1+\delta_2)\psi_1\varkappa^{\alpha}}{\delta_1\Gamma(\alpha+1)}$$

$$
\begin{aligned}
&= \frac{2}{233} + \frac{8 + 8\sqrt{\pi}}{322e^2} \\
&\approx 0.0179056991156396 \\
&< 1.
\end{aligned}
$$

Then the problem (5.1)-(5.3) has a unique solution in $PC([0,\pi],\mathbb{R})$. Hypothesis ($Ax4$) is satisfied with $\zeta = 1$, $\xi(\vartheta) = \sqrt{\pi}$ and $\kappa_\xi = 4$. Indeed, for each $\vartheta \in \Omega_0 \cup \Omega_1$, we get

$$
\begin{aligned}
{}_0I_\pi^\alpha \sqrt{\pi} &= \frac{1}{\Gamma(\alpha)} \int_0^\pi |\vartheta - \varrho|^{\alpha-1} \sqrt{\pi} d\varrho \\
&\leq \frac{\sqrt{\pi}}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta - \varrho)^{\alpha-1} d\varrho + \frac{\sqrt{\pi}}{\Gamma(\alpha)} \int_\vartheta^\pi (\varrho - \vartheta)^{\alpha-1} d\varrho \\
&\leq 4\sqrt{\pi}.
\end{aligned}
$$

Consequently, Theorem 4.7 implies that the problem (5.1)-(5.3) is U-H-R stable.

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