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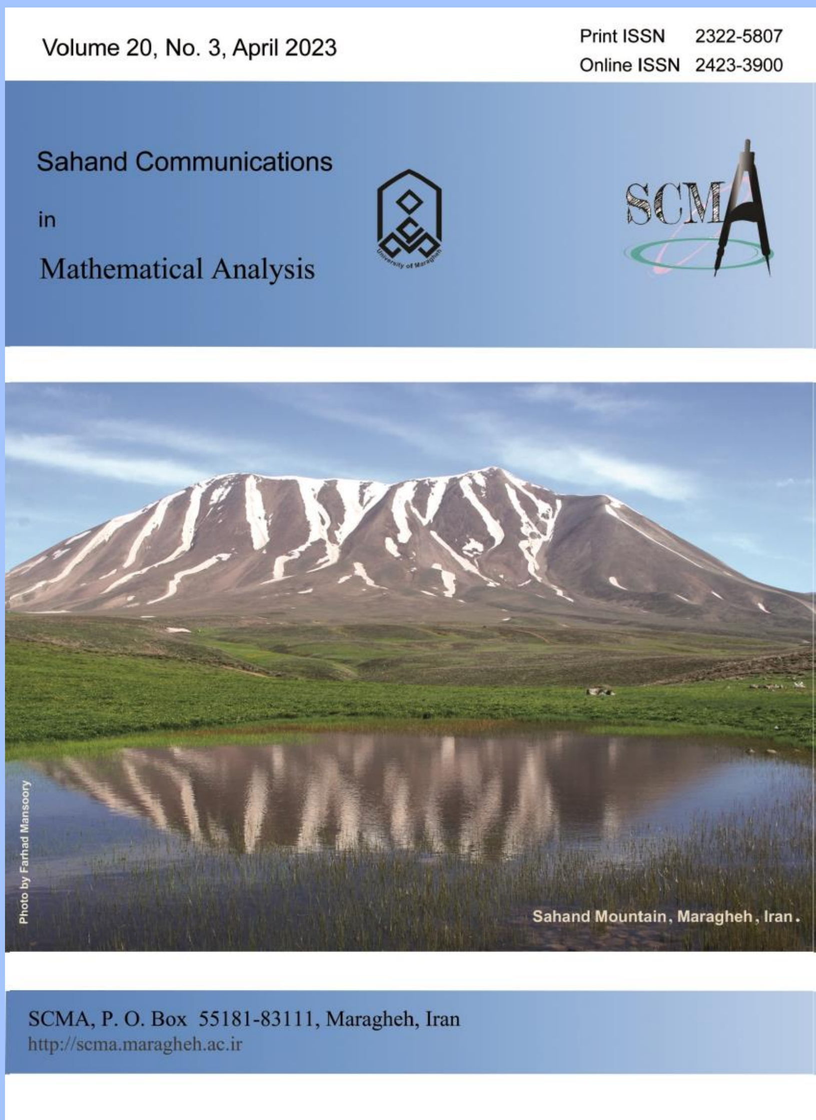

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Fractional Simpson-Type Inequalities for Twice Differentiable Functions

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Abstract. In the literature, several papers are devoted to inequalities of Simpson-type in the case of differentiable convex functions and fractional versions. Moreover, some papers are focused on inequalities of Simpson-type for twice differentiable convex functions. In this research article, we obtain an identity for twice differentiable convex functions. Then, we prove several fractional inequalities of Simpson-type for convex functions.

1. Introduction

The Inequality of the Simpson-type plays a significant role in many areas of mathematics. The classical inequality of the Simpson-type is described as follows:

Theorem 1.1. *Suppose that $F : [\sigma, \eta] \to \mathbb{R}$ is a four times continuously differentiable mapping on (σ, η), and $\left\|F^{(4)}\right\|_\infty = \sup_{\kappa\in(\sigma,\eta)} \left|F^{(4)}(\kappa)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{F(\sigma)+F(\eta)}{2}+2F\left(\frac{\sigma+\eta}{2}\right)\right]-\frac{1}{\eta-\sigma}\int_\sigma^\eta F(\kappa)d\kappa\right| \leq \frac{1}{2880}\left\|F^{(4)}\right\|_\infty (\eta-\sigma)^4 .$$

Over the years, many authors have considered Simpson-type inequalities across various study categories. In particular, some papers focused on the results of inequalities of Simpson-type for convex functions, since

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the convex theory is an effective and powerful way to solve a large number of problems from different branches of mathematics. For example, it was obtained some inequalities of the Simpson-type in the case of s-convex functions [2]. Afterward, Sarikaya et al. obtained new variants of Simpson-type inequalities based on differentiable convex mapping [26, 27]. In addition to this, some papers were considered the Simpson-type inequalities for various convex classes [6, 13, 21, 23, 24].

In last decades, the fractional calculus has application areas in many various fields such as physics, chemistry and engineering as well as mathematics. Due to its basic properties, definitions and applications in domains of science, fractional calculus has been the center of attraction in pure and applied mathematics, especially the Riemann-Liouville operator. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamical models based on fractional computation. While integer orders are models that are not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches. Because of the importance of fractional calculus mentioned in this paragraph, one can examine distinct fractional integral inequalities extensively. For example, the authors described the Simpson inequalities for differentiable functions to R-L fractional integrals in [4] and [12]. Hence, many papers are devoted to fractional Simpson inequalities in the case of various fractional integral operators [1, 7, 10, 14, 16, 20, 25, 29, 30]. For more papers related to Simpson-type inequalities, one can consult [3, 5, 9, 11, 17–19, 31]. Besides all, Sarikaya et al. proved several Simpson-type inequalities for functions whose second derivatives are convex [28]. This paper aims to extend the results given in [28] for twice differentiable functions to R-L fractional integrals. For this aim, after giving definitions of R-L fractional integral operators, we establish an identity for twice differentiable functions. By utilizing this equality, we show some Simpson type inequalities for mapping whose second derivatives are convex.

Definition 1.2. Let $F \in L_1[\sigma,\eta]$. The Riemann-Liouville (R-L) integrals $\mathcal{J}_{\sigma+}^{\alpha}F$ and $\mathcal{J}_{\eta-}^{\alpha}F$ of order $\alpha>0$ with $\sigma\geq 0$ are defined by

$$\mathcal{J}_{\sigma+}^{\alpha}F(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\kappa}(\kappa-\mu)^{\alpha-1}F(\mu)d\mu,\quad \kappa>\sigma$$

and

$$\mathcal{J}_{\eta-}^{\alpha}F(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\eta}(\mu-\kappa)^{\alpha-1}F(\mu)d\mu,\quad \kappa<\eta,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathcal{J}_{\sigma+}^{0}\digamma(\kappa)$ and $\mathcal{J}_{\eta-}^{0}\digamma(\kappa)$ equal to $\digamma(\kappa)$.

For further explanation and several properties of R-L fractional integrals, please refer to [8, 15, 22].

2. Simpson-Type Inequalities for Twice Differentiable Convex Functions

In this part of paper, we present several fractional inequalities of Simpson type to case of twice differentiable functions.

Lemma 2.1. *Let $\digamma:[\sigma,\eta]\to\mathbb{R}$ be a twice differentiable mapping (σ,η) such that $\digamma''\in L_1([\sigma,\eta])$. Then, the following equality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[\digamma(\sigma)+4\digamma\left(\frac{\sigma+\eta}{2}\right)+\digamma(\eta)\right]\\
&\qquad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}_{\left(\frac{\sigma+\eta}{2}\right)+}^{\alpha}\digamma(\eta)+\mathcal{J}_{\left(\frac{\sigma+\eta}{2}\right)-}^{\alpha}\digamma(\sigma)\right]\\
&\quad=\frac{(\eta-\sigma)^2}{6}\int_0^1 w(\mu)\digamma''(\mu\eta+(1-\mu)\sigma)\,d\mu,
\end{aligned}$$

where

$$w(\mu)=\begin{cases}\mu\left(1-\frac{3\cdot2^{\alpha}}{\alpha+1}\mu^{\alpha}\right), & \mu\in\left[0,\frac{1}{2}\right],\\[2ex] (1-\mu)\left(1-\frac{3\cdot2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right), & \mu\in\left(\frac{1}{2},1\right].\end{cases}$$

Proof. With the help of the integration by parts, it follows

(2.1)

$$\begin{aligned}
\mathcal{K}_1&=\int_0^{\frac{1}{2}}\mu\left(1-\frac{3\cdot2^{\alpha}}{\alpha+1}\mu^{\alpha}\right)\digamma''(\mu\eta+(1-\mu)\sigma)\,d\mu\\
&=\mu\left(\mu-\frac{3\cdot2^{\alpha}}{\alpha+1}\mu^{\alpha}\right)\left.\frac{\digamma'(\mu\eta+(1-\mu)\sigma)}{\eta-\sigma}\right|_0^{\frac{1}{2}}\\
&\quad+\frac{1}{\eta-\sigma}\int_0^{\frac{1}{2}}\left(1-3\cdot2^{\alpha}\mu^{\alpha}\right)\digamma'(\mu\eta+(1-\mu)\sigma)\,d\mu\\
&=\frac{1}{\eta-\sigma}\left[\frac{1}{2}-\frac{3}{2(\alpha+1)}\right]\digamma'\left(\frac{\sigma+\eta}{2}\right)\\
&\quad-\frac{1}{\eta-\sigma}\left.\left[\frac{1-3\cdot2^{\alpha}\mu^{\alpha}}{\eta-\sigma}\digamma(\mu\eta+(1-\mu)\sigma)\right]\right|_0^{\frac{1}{2}}
\end{aligned}$$

$$+\frac{3\cdot 2^{\alpha}\alpha}{\eta-\sigma}\int_0^{\frac{1}{2}}\mu^{\alpha-1}F'\left(\mu\eta+(1-\mu)\sigma\right)d\mu$$
$$=\frac{1}{\eta-\sigma}\left[\frac{1}{2}-\frac{3}{2(\alpha+1)}\right]F'\left(\frac{\sigma+\eta}{2}\right)+\frac{2}{(\eta-\sigma)^2}F\left(\frac{\sigma+\eta}{2}\right)$$
$$+\frac{1}{(\eta-\sigma)^2}F(\sigma)-\frac{2^{\alpha}3\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha+2}}\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma).$$

Similarly, we obtain

(2.2)
$$\mathcal{K}_2=\int_{\frac{1}{2}}^{1}(1-\mu)\left(1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right)F''\left(\mu\eta+(1-\mu)\sigma\right)d\mu$$
$$=-\frac{1}{\eta-\sigma}\left[\frac{1}{2}-\frac{3}{2(\alpha+1)}\right]F'\left(\frac{\sigma+\eta}{2}\right)+\frac{2}{(\eta-\sigma)^2}F\left(\frac{\sigma+\eta}{2}\right)$$
$$+\frac{1}{(\eta-\sigma)^2}F(\eta)-\frac{2^{\alpha}3\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha+2}}\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta).$$

Equations (2.1) and (2.2) yield the following equality:

$$\frac{(\eta-\sigma)^2}{6}(\mathcal{K}_1+\mathcal{K}_2)=\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\eta}{2}\right)+F(\eta)\right]$$
$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right].$$

This is the end of the proof of Lemma 2.1. □

Remark 2.2. If we choose $\alpha=1$ in Lemma 2.1, then Lemma 2.1 reduces to [28, Lemma 2.1].

Theorem 2.3. *Let the conditions of Lemma 2.1 hold true and the function $|F''|$ is convex on $[\sigma,\eta]$. Then, we get*

$$\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\eta}{2}\right)+F(\eta)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\right|$$
$$\leq\frac{(\eta-\sigma)^2}{6}\mathcal{A}(\alpha)\left[\left|F''(\sigma)\right|+\left|F''(\eta)\right|\right],$$

where

(2.3)
$$\mathcal{A}(\alpha)=\tfrac{1}{4(\alpha+2)}\left(\alpha\left(\tfrac{\alpha+1}{3}\right)^{\frac{2}{\alpha}}+\tfrac{3}{\alpha+1}\right)-\tfrac{1}{8}.$$

Proof. By taking modulus in Lemma 2.1, we have

(2.4)

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\eta}{2}\right)+F(\eta)\right]\right. \\
&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\right| \\
&\quad\leq\frac{(\eta-\sigma)^2}{6}\int_0^1|w(\mu)|\left|F''(\mu\eta+(1-\mu)\sigma)\right|d\mu \\
&\quad=\frac{(\eta-\sigma)^2}{6}\left[\int_0^{\frac{1}{2}}\mu\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|\left|F''(\mu\eta+(1-\mu)\sigma)\right|d\mu\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1(1-\mu)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|\left|F''(\mu\eta+(1-\mu)\sigma)\right|d\mu\right].
\end{aligned}$$

With the help of the convexity of $|F''|$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\eta}{2}\right)+F(\eta)\right]\right. \\
&\qquad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\right| \\
&\quad\leq\frac{(\eta-\sigma)^2}{6}\left[\int_0^{\frac{1}{2}}\mu\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|\left[\mu\left|F''(\eta)\right|+(1-\mu)\left|F''(\sigma)\right|\right]d\mu\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1(1-\mu)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|\left[\mu\left|F''(\eta)\right|+(1-\mu)\left|F''(\sigma)\right|\right]d\mu\right] \\
&\quad=\frac{(\eta-\sigma)^2}{6}\left\{\left[\int_0^{\frac{1}{2}}\mu^2\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|d\mu\right.\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^1\mu(1-\mu)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|d\mu\right]\left|F''(\eta)\right| \\
&\qquad+\left[\int_0^{\frac{1}{2}}\mu(1-\mu)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|d\mu\right. \\
&\qquad\left.\left.+\int_{\frac{1}{2}}^1(1-\mu)^2\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|d\mu\right]\left|F''(\sigma)\right|\right\} \\
&\quad=\frac{(\eta-\sigma)^2}{6}\mathcal{A}(\alpha)\left[\left|F''(\sigma)\right|+\left|F''(\eta)\right|\right].
\end{aligned}$$

This completes the proof of Theorem 2.3. ∎

Remark 2.4. If we choose $\alpha = 1$ in Theorem 2.3, then Theorem 2.3 reduces to [28, Theorem 2.2].

Theorem 2.5. *If the assumptions of Lemma 2.1 are valid and the function $|F''|^q$, $q > 1$ is convex on $[\sigma, \eta]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\sigma) + 4F\left(\frac{\sigma+\eta}{2}\right) + F(\eta)\right]\right. \\
&\qquad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta) + \mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\right| \\
&\quad \leq \frac{(\eta-\sigma)^2}{6}\Theta(\alpha, p) \\
&\qquad \times \left[\left(\frac{|F''(\eta)|^q + 3|F''(\sigma)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|F''(\eta)|^q + |F''(\sigma)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Here, $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$\Theta(\alpha, p) = \left(\int_0^{\frac{1}{2}} \mu^p \left|1 - \frac{3 \cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|^p d\mu\right)^{\frac{1}{p}}.$$

Proof. By applying Hölder inequality in inequality (2.4), we get

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(\sigma) + 4F\left(\frac{\sigma+\eta}{2}\right) + F(\eta)\right]\right. \\
&\qquad \left. - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta) + \mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\right| \\
&\quad \leq \frac{(\eta-\sigma)^2}{6}\left[\left(\int_0^{\frac{1}{2}} \mu^p \left|1 - \frac{3 \cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|^p d\mu\right)^{\frac{1}{p}}\right. \\
&\qquad \times \left(\int_0^{\frac{1}{2}} \left|F''(\mu\eta + (1-\mu)\sigma)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_{\frac{1}{2}}^1 (1-\mu)^p \left|1 - \frac{3 \cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|^p d\mu\right)^{\frac{1}{p}} \\
&\qquad \left. \times \left(\int_{\frac{1}{2}}^1 \left|F''(\mu\eta + (1-\mu)\sigma)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

With the help of the convexity of $|F''|^q$, we obtain

$$\left|\frac{1}{6}\left[F(\sigma) + 4F\left(\frac{\sigma+\eta}{2}\right) + F(\eta)\right]\right.$$

$$
\begin{aligned}
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\Bigg| \\
&\leq \frac{(\eta-\sigma)^2}{6}\left(\int_0^{\frac{1}{2}}\mu^p\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|^p d\mu\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left[\mu\left|F''(\eta)\right|^q+(1-\mu)\left|F''(\sigma)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left[\mu\left|F''(\eta)\right|^q+(1-\mu)\left|F''(\sigma)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right] \\
&=\frac{(\eta-\sigma)^2}{6}\Theta(\alpha,p) \\
&\quad\times\left[\left(\frac{|F''(\eta)|^q+3|F''(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\eta)|^q+|F''(\sigma)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This finishes the proof of Theorem 2.5. □

Corollary 2.6. *Let us consider $\alpha = 1$ in Theorem 2.5. Then, the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\eta}{2}\right)+F(\eta)\right]-\frac{1}{(\eta-\sigma)}\int_{\sigma}^{\eta}F(\kappa)\,d\kappa\right| \\
&\quad\leq\frac{(\eta-\sigma)^2}{6}\left(\int_0^{\frac{1}{2}}\mu^p\,|1-3\mu|^p\,d\mu\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{|F''(\eta)|^q+3|F''(\sigma)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\eta)|^q+|F''(\sigma)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Theorem 2.7. *Let the assumptions of Lemma 2.1 hold and the function $|F''|^q$, $q \geq 1$ is convex on $[\sigma,\eta]$. Then, we get*

$$
\begin{aligned}
&\Bigg|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\eta}{2}\right)+F(\eta)\right] \\
&\quad-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta-\sigma)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F(\eta)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F(\sigma)\right]\Bigg| \\
&\quad\leq\frac{(\eta-\sigma)^2}{6}\left(\mathcal{A}(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\mathcal{B}(\alpha)\left|F''(\eta)\right|^q+\mathcal{C}(\alpha)\left|F''(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\mathcal{C}(\alpha)\left|F''(\eta)\right|^q+\mathcal{B}(\alpha)\left|F''(\sigma)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $\mathcal{A}(\alpha)$ *is defined as in (2.3) and*

$$\begin{cases} \mathcal{B}(\alpha)=\frac{1}{4(\alpha+3)}\left[\frac{\alpha}{3}\left(\frac{\alpha+1}{3}\right)^{\frac{3}{\alpha}}+\frac{3}{2(\alpha+1)}\right]-\frac{1}{24}, \\ \\ \mathcal{C}\left(\alpha\right)=\mathcal{A}\left(\alpha\right)-\mathcal{B}\left(\alpha\right)=\frac{1}{4(\alpha+2)}\left(\alpha\left(\frac{\alpha+1}{3}\right)^{\frac{2}{\alpha}}+\frac{3}{\alpha+1}\right) \\ \\ \qquad -\frac{1}{4(\alpha+3)}\left[\frac{\alpha}{3}\left(\frac{\alpha+1}{3}\right)^{\frac{3}{\alpha}}+\frac{3}{2(\alpha+1)}\right]-\frac{1}{12}. \end{cases}$$

Proof. Let us consider the power-mean inequality in (2.4). Then, it follows

(2.5)

$$\begin{aligned} &\left|\frac{1}{6}\left[F\left(\sigma\right)+4F\left(\frac{\sigma+\eta}{2}\right)+F\left(\eta\right)\right]\right. \\ &\qquad\left.-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\eta-\sigma\right)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)+}F\left(\eta\right)+\mathcal{J}^{\alpha}_{\left(\frac{\sigma+\eta}{2}\right)-}F\left(\sigma\right)\right]\right| \\ &\quad\leq\frac{\left(\eta-\sigma\right)^{2}}{6}\left[\left(\int_{0}^{\frac{1}{2}}\mu\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|d\mu\right)^{1-\frac{1}{q}}\right. \\ &\qquad\times\left(\int_{0}^{\frac{1}{2}}\mu\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|\left|F''\left(\mu\eta+\left(1-\mu\right)\sigma\right)\right|^{q}d\mu\right)^{\frac{1}{q}} \\ &\qquad+\left(\int_{\frac{1}{2}}^{1}\left(1-\mu\right)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\left(1-\mu\right)^{\alpha}\right|d\mu\right)^{1-\frac{1}{q}} \\ &\qquad\left.\times\left(\int_{\frac{1}{2}}^{1}\left(1-\mu\right)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\left(1-\mu\right)^{\alpha}\right|\left|F''\left(\mu\eta+\left(1-\mu\right)\sigma\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Since $|F''|^{q}$ is convex, we obtain

$$\begin{aligned} &\int_{0}^{\frac{1}{2}}\mu\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|\left|F''\left(\mu\eta+\left(1-\mu\right)\sigma\right)\right|^{q}d\mu \\ &\quad\leq\int_{0}^{\frac{1}{2}}\mu\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|\left[\mu\left|F''\left(\eta\right)\right|^{q}+\left(1-\mu\right)\left|F''\left(\sigma\right)\right|^{q}\right]d\mu \\ &\quad=\left|F''\left(\eta\right)\right|^{q}\int_{0}^{\frac{1}{2}}\mu^{2}\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|d\mu \\ &\qquad+\left|F''\left(\sigma\right)\right|^{q}\int_{0}^{\frac{1}{2}}\mu\left(1-\mu\right)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}\mu^{\alpha}\right|d\mu \\ &\quad=\mathcal{B}\left(\alpha\right)\left|F''\left(\eta\right)\right|^{q}+\mathcal{C}\left(\alpha\right)\left|F''\left(\sigma\right)\right|^{q}, \end{aligned} \tag{2.6}$$

and similarly

$$\begin{aligned}
(2.7)\qquad & \int_{\frac{1}{2}}^{1}(1-\mu)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|\left|F''(\mu\eta+(1-\mu)\sigma)\right|^{q}d\mu \\
& \leq \left|F''(\eta)\right|^{q}\int_{\frac{1}{2}}^{1}\mu(1-\mu)\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|d\mu \\
& \quad+\left|F''(\sigma)\right|^{q}\int_{\frac{1}{2}}^{1}(1-\mu)^{2}\left|1-\frac{3\cdot 2^{\alpha}}{\alpha+1}(1-\mu)^{\alpha}\right|d\mu \\
& =\mathcal{C}(\alpha)\left|F''(\eta)\right|^{q}+\mathcal{B}(\alpha)\left|F''(\sigma)\right|^{q}.
\end{aligned}$$

If we substitute the inequalities (2.6) and (2.7) in (2.5), then we establish the required result. □

Remark 2.8. If it is assigned $\alpha = 1$ in Theorem 2.7, then Theorem 2.7 becomes to [28, Theorem 2.5].

3. Conclusion and Future Work

In this investigation, we prove an identity for twice differentiable convex functions. Moreover we obtain the fractional version of Simpson-type inequalities in the case of twice differentiable functions. we also show that our obtained results generalize the inequalities established by Sarikaya et al. in [28].

In the future papers of authors, they can generalize our obtained results by utilizing another type of convex function classes or other kinds of fractional integrals. More precisely, one can generalize our results by using various versions of convex function classes specially Harmonic convex functions, fractional calculus etc. In future works, the ideas and strategies for our results about Simpson-type inequalities by Riemann-Liouville fractional fractional integrals may open new avenues for further research in this field. More precisely, one can obtain Simpson-type inequalities for convex functions by using quantum calculus. Furthermore, interested readers can apply these resulting inequalities to different types of fractional integrals such as k-Riemann- Liouville fractional integrals, conformable fractional integrals, Hadamard fractional integrals, Katugampola fractional integrals, etc.

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