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Leila Shahbaz

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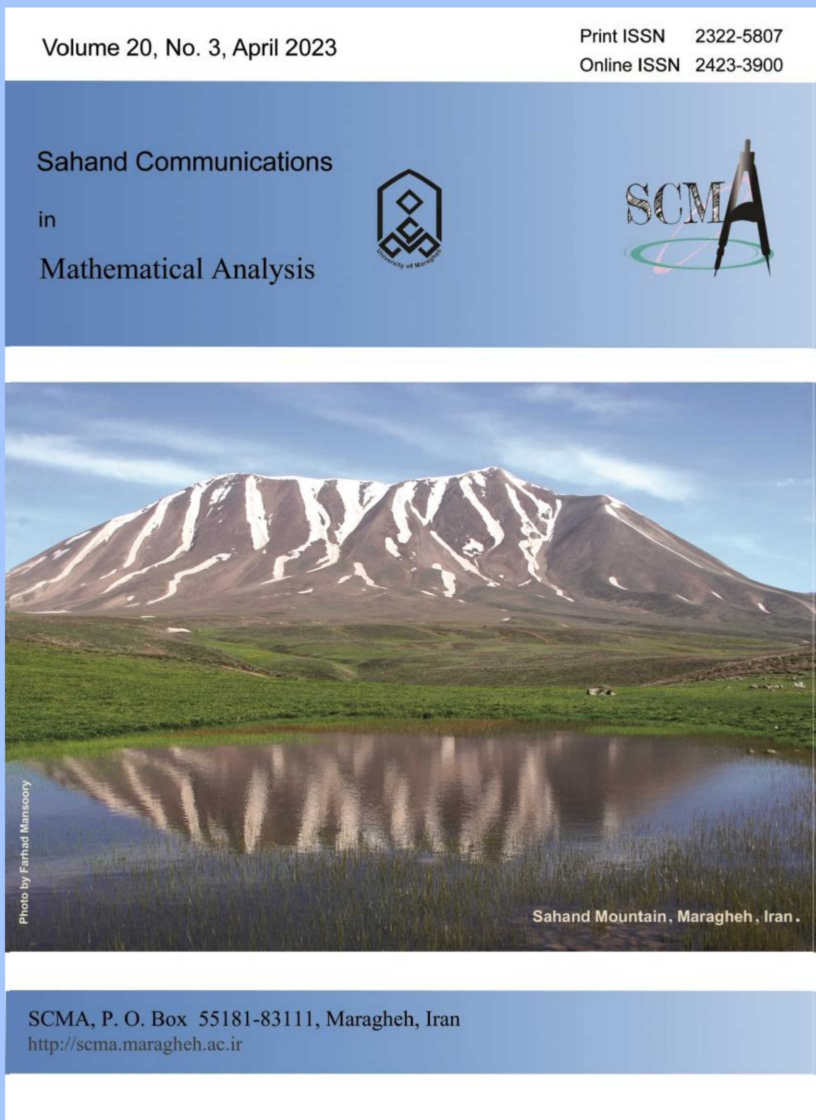

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# The Category of $S$-Fuzzy Posets

Leila Shahbaz

Abstract. In this paper, we define and consider, the category **FPos**-$S$ of all $S$-fuzzy posets and action-preserving monotone maps between them. $S$-fuzzy poset congruences which play an important role in studying the categorical properties of $S$-fuzzy posets are introduced. More precisely, the correspondence between the $S$-fuzzy poset congruences and the fuzzy action and order preserving maps is discussed. We characterize $S$-fuzzy poset congruences on the $S$-fuzzy posets in terms of the fuzzy pseudo orders. Some categorical properties of the category **FPos**-$S$ of all $S$-fuzzy posets is considered. In particular, we characterize products, coproducts, equalizers, coequalizers, pullbacks and pushouts in this category. Also, we consider all forgetful functors between the category **FPos**-$S$ and the categories **FPos** of fuzzy posets, **Pos**-$S$ of $S$-posets, **Pos** of posets, **Act**-$S$ of $S$-acts and **Set** of sets and study the existence of their left and right adjoints. Finally, epimorphisms, monomorphisms and order embeddings in **FPos** and **FPos**-$S$ are studied.

## 1. Introduction and Preliminaries

They have been appeared many kinds of ordered algebras in the literature so far, for example, pogroups, posemigroups, rings and fields equipped with an order and etc. Recently, Fakhruddin in [16, 17] has been studied the category of posets acted on by a pomonoid $S$ (the category of $S$-posets), absolute flatness and amalgams of $S$-posets. After then the properties of $S$-posets have been studied in many papers, for example see [6–11, 25–29]. Historically, fuzzy ordering, which is a generalization of the concept of ordering, has been investigated in the fuzzy context at the beginning by Zadeh, [31] and since then, very researchers,

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* Corresponding author.

were motivated not only by theoretical reasons, but also because of their applicability, to work on this concept and its applications to logic, set theory, algebra, analysis, topology, computer science, control problems, information science, rule based systems and solving relational equations. Recent important results concerning fuzzy orders and their representations are related to De Baets, Bodenhofer at al. [3–5]. In most of these investigations, what is fuzzified is the ordering relation, while the underlying set is crisp. In addition, the authors use T-norms, or more generally, residuated lattices and corresponding operations. Several important algebras are special residuated lattices: Boolean algebras (algebraic counterpart of classical logic), Heyting algebras (intuitionisctic logic), BL-algebras (logic of continuous t-norms), MV-algebras (Lukasiewicz logic), Girard monoids (linear logic) and others (for more information see [20, 22] and references therein). Motivated by the classical approach to $S$-posets, we investigate $S$-fuzzy posets by fuzzification of an ordering relation. The aim of the present paper is to introduce the notion of the actions of a fuzzy ordered semigroup on a fuzzy ordered set ($S$-fuzzy poset) and study some categorical and algebraic ingredients of this category. Similar to the theory of order congruences on fuzzy posets, fuzzy order congruences on $S$-fuzzy posets play an important role in studying the structures of $S$-fuzzy posets. We introduce the concept of pseudo orders on $S$-fuzzy posets to characterize the $S$-fuzzy poset congruences on $S$-fuzzy posets where the pseudo order is a preorder containing the fuzzy order which is compatible with the $S$-action. Some categorical properties of the category **FPos**-$S$ of all $S$-fuzzy posets and action-preserving monotone maps between them is considered. In particular, we describe products, coproducts, equalizers, coequalizers, pullbacks and pushouts in this category and consider epimorphisms, monomorphisms and order embeddings. Also, the existence of the free and cofree objects in the categories **FPos**-$S$ are studied. More precisely, we consider all forgetful functors between the category **FPos**-$S$ and the categories **FPos** of fuzzy posets, **Pos** of posets, **Act**-$S$ of $S$-acts and **Set** of sets, and study the existence of their left and right adjoints. Finally, we characterize epimorphisms and monomorphisms from the perspective of set theory and we give a categorical characterization of order embeddings in the categories **FPos** and **FPos**-$S$.

In rest of this section, we briefly recall the preliminary notions about the actions of a pomonoid on a set and on a poset. For more information, see [10, 24].

Let $S$ be a monoid with 1 as its identity. A (right) $S$-act is a set $A$ equipped with an action $\lambda : A \times S \to A, (\lambda(a, s)$ is denoted by $as)$ such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. An

$S$-map $f : A \to B$ between $S$-acts is an action preserving map, that is $f(as) = f(a)s$ for each $a \in A, s \in S$. The category of all $S$-acts and $S$-maps between them is denoted by **Act**-$S$.

Recall that a monoid (semigroup) $S$ is said to be a *pomonoid* (*posemigroup*) if it is also a poset whose partial order $\leq$ is compatible with its binary operation (that is, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$).

A (*right*) *S-poset* is a poset $A$ which is also an $S$-act whose action $\lambda : A \times S \to A$ is order preserving, where $A \times S$ is considered as a poset with componentwise order. An *S-poset map* (or *morphism*) is an action preserving monotone map between $S$-posets. Moreover, regular monomorphisms (equalizers) are exactly order embeddings; that is, (mono)morphisms $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$. The category of all $S$-posets and $S$-poset maps between them is denoted by **Pos**-$S$.

We recall the following from [21]. A residuated lattice (see also [1, 2, 30])
$(L, *, \to, \vee, \wedge, 0, 1)$ is an algebra with four binary operators $*, \to, \vee, \wedge$ on $L$ such that:

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with the greatest element 1 and the least element 0,
(2) $(L, *, 1)$ is a commutative monoid and $*$ is isotonic at both arguments,
(3) $(*, \to)$ is an adjoint pair, i.e. $x * y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in L$.

A residuated lattice is said to be complete if the underlying lattice is complete.

In this paper, $L$ denotes a complete residuated lattice except otherwise specified.

As a generalization of Zadeh's fuzzy set, Goguen introduced $L$-fuzzy set with $L$ being a complete residuated lattice (see [18]).

An $L$-fuzzy set (briefly, fuzzy set) $A$ on $U$ is a map $A : U \to L$ and all the $L$-fuzzy sets on $U$ are denoted by $L^U$.

An $L$-fuzzy relation (briefly, fuzzy relation) $R$ on $U$ is a map $R : U \times U \to L$.

(1) $R$ is reflexive if $R(x, x) = 1$ for all $x \in U$,
(2) $R$ is transitive if $\bigvee_{y \in U} R(x, y) * R(y, z) \leq R(x, z)$ for all $x, z \in U$,
(3) $R$ is symmetric if $R(x, y) = R(y, x)$ for all $x, y \in U$,
(4) $R$ is antisymmetric if for all $x, y \in L, R(x, y) = R(y, x) = 1$ implies $x = y$.

Let $X$ be a set and $e_X : X \times X \to L$ be a fuzzy relation. $e_X$ is called a fuzzy partial order on $X$ if it is reflexive, antisymmetric and transitive. The pair $(X, e_X)$ is called a fuzzy partially ordered set, or briefly fuzzy poset. Let $(X, e_X), (Y, e_Y)$ be two fuzzy posets. A map $f : (X, e_X) \to (Y, e_Y)$ is said to be fuzzy order preserving if $e_X(x, y) \leq e_Y(f(x), f(y))$ for all $x, y \in X$. The category of all fuzzy posets with fuzzy order preserving maps between them is denoted by **FPos**.

Recall that if $e_X$ is a fuzzy partial order on $X$ then the inverse fuzzy relation $e^{-1} \in L^{X \times X}$ defined by $e^{-1}(x, y) = e(y, x)$ of $e$, for all $x, y \in X$, is also a fuzzy partial order on $X$. Moreover, the symmetrization $e^s = e \wedge e^{-1}$ of $e$ is a fuzzy partial order on $X$.

A fuzzy equivalence relation $R$ on a set $X$ is a reflexive, symmetric and transitive fuzzy relation on a set $X$. Fuzzy equivalence relations are introduced by Zadeh (see [32]) as a generalization of equivalence relations. After then much effort have been devoted to study fuzzy equivalence relations in order to measure the degree of indistinguishability or similarity between the objects of a given set (see [13–15]). Also fuzzy equivalence relations have been applied to different contexts such as fuzzy control, approximate reasoning, fuzzy cluster analysis, etc. Various properties of equivalence classes of fuzzy equivalence relations over a complete residuated lattice are investigated in [12]. The set of all fuzzy equivalence relations on a given set $X$ forms a complete lattice. The meet of two fuzzy equivalence relations coincides with the fuzzy sets intersection of the fuzzy relations, but the join of two fuzzy equivalence relations does not coincide with the ordinary fuzzy sets union.

Let $R$ be a fuzzy equivalence relation on a set $X$. $R_a$, for every $a \in X$, is called a fuzzy equivalence class determined by $R$, where $R_a(x) = R(a, x)$ for every $x \in X$. The set $X/R = \{R_a | a \in X\}$ is called the factor set of $X$ with respect to $R$.

**Lemma 1.1** ([12])**.** *Let $R$ be a fuzzy equivalence relation on a set $X$. Then for every $x, y \in X$, the following are true:*

(i) $R(x, y) = \bigvee_{z \in X} R(x, z) * R(z, y)$.
(ii) $R(x, y) = 1$ *if and only if* $R_x = R_y$.

Let $(X, e)$ be a fuzzy poset and $R$ be a fuzzy equivalence relation on $X$. For all $x, y \in X$, define $R_e(x, y) = e \circ R(x, y) = \bigvee_{z \in X} e(x, z) * R(z, y)$. $R_e$ is reflexive but it is not transitive in general. $R_e^T = \bigvee_{n=1}^{\infty} R_e^n$, where $R_e^n = R_e^{n-1} \circ R_e, n \geq 2$, is the transitive closure of $R_e$ (see [19]).

**Remark 1.2.** It is easily checked that the following properties hold:

(i) $e(x, y) \leq R_e^T(x, y)$;
(ii) $R(x, y) \leq R_e^T(x, y) \wedge (R_e^T)^{-1}(x, y)$.

Let $(X, e_X)$ be a fuzzy poset and $R$ be a fuzzy equivalence relation on $X$. $R$ is called a fuzzy order congruence if $R(x, y) = R_e^T(x, y) \wedge (R_e^T)^{-1}(x, y)$, for all $x, y \in X$.

## 2. $S$-Fuzzy Posets

In the following, we introduce the category **FPos**-$S$ of $S$-fuzzy posets and some non trivial examples of $S$-fuzzy posets are given.

**Definition 2.1.** Let $S$ be a semigroup. By a fuzzy ordered semigroup we mean a semigroup $(S, .)$ with the fuzzy partial order $e_S : S \times S \to L$ such that for all $(s_1, s_2), (s'_1, s'_2) \in S \times S$, $e_S(s_1, s_2) \wedge e_S(s'_1, s'_2) \leq e_S(s_1 s'_1, s_2 s'_2)$.

**Definition 2.2.** Let $(S, ., e_S)$ be a fuzzy ordered semigroup and $X$ be a fuzzy poset with the fuzzy partial order $e_X : X \times X \to L$ which is an $S$-act, too. Then we call $X$ an $S$-fuzzy poset if
(1) $e_S(s_1, s_2) \leq e_X(s_1 x, s_2 x)$ for all $(s_1, s_2) \in S \times S$ and $x \in X$;
(2) $e_X(x_1, x_2) \leq e_X(s x_1, s x_2)$ for all $(x_1, x_2) \in X \times X$ and $s \in S$.

Note that since $S$ and $X$ are fuzzy posets, $S \times X$ is also a fuzzy poset with the fuzzy partial order $e_S \wedge e_X : (S \times X) \times (S \times X) \to L$ given by $(e_S \wedge e_X)((s_1, x_1), (s_2, x_2)) = e_S(s_1, s_2) \wedge e_X(x_1, x_2)$.

**Example 2.3.** Let the residuated lattice $L = \{0, 0.5, 1\}$ and $S = \{1, a, b, c\}$ be a fuzzy ordered monoid with the binary operation and the order given as follows,

| | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |

and $e_S : S \times S \to L$ where

$$e_S = (e_{ij})_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 \\ 1 & 0.5 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.1}$$

Then $A = \{a_1, b_1, a_2, b_2, a_3, b_3\}$ with the action and fuzzy order given as follows is an $S$-fuzzy poset,

| | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $a_1$ | $a_1$ | $b_1$ | $b_1$ | $b_1$ |
| $b_1$ | $b_1$ | $b_1$ | $b_1$ | $b_1$ |
| $a_2$ | $b_2$ | $b_2$ | $b_2$ | $b_2$ |
| $b_2$ | $b_2$ | $b_2$ | $b_2$ | $b_2$ |
| $a_3$ | $a_3$ | $b_3$ | $b_3$ | $b_3$ |
| $b_3$ | $b_3$ | $b_3$ | $b_3$ | $b_3$ |

and $e_A : A \times A \to L$ where

$$e_A = (e_{ij})_A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 1 & 0 & 0 \\ 1 & 0.5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

**Definition 2.4.** (1) Let $S$ be a fuzzy ordered semigroup, $X$ be an $S$-fuzzy poset and $X_1$ be a sub act of $X$. Then $e_{X_1} : X_1 \times X_1 \to L$ defined by $e_{X_1}(x, y) = e_X(x, y)$, for all $x, y \in X_1$, is a fuzzy partial order on $X_1$ and $(X_1, e_{X_1})$ becomes an $S$-fuzzy poset and is called a sub $S$-fuzzy poset of $(X, e_X)$.

(2) Let $(X, e)$ be an $S$-fuzzy poset. Define $\leq = \{(x, y) | e(x, y) = 1\}$. Then $\leq$ is a classical partial order on $X$ and $(X, \leq)$ is called the underlying $S$-fuzzy poset of $(X, e)$.

**Note 2.5.** *Note that, if $\{(X, e^i_X)_{i \in I}\}$ is a family of $S$-fuzzy posets and define $e_X(x, y) = \bigwedge e^i_X(x, y)$ for every $(x, y) \in X \times X$, then it is a fuzzy partial order on $X$ and $(X, e_X)$ becomes an $S$-fuzzy poset.*

**Remark 2.6.** Let $L = \{0, 1\}$ be the truth values. Then the fuzzy orders on a set $X$ are just the classical partial orders on $X$. So the fuzzy posets based on $L$ are classical posets and hence $S$-fuzzy posets are generalizations of $S$-posets. For a classical partial order $\leq$, a fuzzy partial order $e_X : X \times X \to L$ is given by

$$e_X(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 2.7.** Let $(X, e_X)$ and $(Y, e_Y)$ be two $S$-fuzzy posets. A map $f : (X, e_X) \to (Y, e_Y)$ is said to be $S$-fuzzy poset map if it is fuzzy order preserving ($e_X(x, y) \leq e_Y(f(x), f(y))$ for all $x, y \in X$) and action preserving ($f(sx) = sf(x)$ for all $x \in X$ and $s \in S$).

The category of all $S$-fuzzy posets with $S$-fuzzy poset maps between them is denoted by **FPos**-$S$.

**Definition 2.8.** Let $(X, e_X)$ and $(Y, e_Y)$ be two $S$-fuzzy posets. An action preserving map $f : (X, e_X) \to (Y, e_Y)$ is said to be

(1) $S$-fuzzy poset order embedding if $e_X(x, y) = e_Y(f(x), f(y))$ for all $x, y \in X$.
(2) $S$-fuzzy poset order isomorphism if it is a surjective fuzzy order embedding.

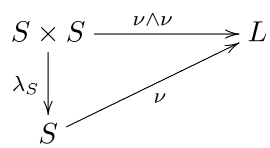
**Example 2.9.** $S$-fuzzy poset order embedding is injective but the converse is not true in general. Take $(\mathbf{1} \sqcup \mathbf{1} = \{0, 1\}, e_{\mathbf{1} \sqcup \mathbf{1}})$ where $e_{\mathbf{1} \sqcup \mathbf{1}}(0, 0) =$

$e_{\mathbf{1}\sqcup\mathbf{1}}(1,1) = 1, e_{\mathbf{1}\sqcup\mathbf{1}}(1,0) = e_{\mathbf{1}\sqcup\mathbf{1}}(0,1) = 0$ and $(\mathbf{2} = \{0,1\}, e_{\mathbf{2}})$ where $e_{\mathbf{2}}(0,0) = e_{\mathbf{2}}(1,1) = 1 = e_{\mathbf{2}}(0,1), e_{\mathbf{2}}(1,0) = 0$ both considered as $S$-fuzzy posets over a one-element fuzzy ordered pomonoid $(S, e_S)$. Let $i : (\mathbf{1} \sqcup \mathbf{1} = \{0,1\}, e_{\mathbf{1}\sqcup\mathbf{1}}) \to (\mathbf{2} = \{0,1\}, e_{\mathbf{2}})$ be the identity $S$-fuzzy poset map. It is clear that $i$ is injective but it is not an order embedding.

Now, recalling the definition of fuzzy $S$-poset from [23] one can easily show that the notion of $S$-fuzzy poset implies the notion of fuzzy $S$-poset.

**Definition 2.10.** A posemigroup $(S, \leq)$ together with a function $\nu : S \to L$ is called a fuzzy posemigroup if

(1) $x \leq y$ implies $\nu(x) \leq \nu(y)$, for all $x, y \in S$;
(2) $\nu(s) \wedge \nu(r) \leq \nu(rs)$, for every $r, s \in S$, that the following diagram is a fuzzy triangle:




which $\lambda_S((s,r)) = sr$, for every $s, r \in S$.

**Remark 2.11.** If $(S, e_S)$ is a fuzzy ordered semigroup then it is a fuzzy posemigroup. For, define $\leq = \{(x,y) \in S \times S | e_S(x,y) = 1\}$. Then $\leq$ is a classical partial order on $S$. For, if $s_1 \leq s_2, s'_1 \leq s'_2$ then $e_S(s_1, s_2) = 1, e_S(s'_1, s'_2) = 1$. Hence $e_S(s_1, s_2) \wedge e_S(s'_1, s'_2) \leq e_S(s_1 s'_1, s_2 s'_2)$ which means that $e_S(s_1 s'_1, s_2 s'_2) = 1$ and so $s_1 s'_1 \leq s_2 s'_2$. Thus $(S, \leq)$ is a posemigroup. Define $\nu : S \to L$ given by $\nu(s) = e_S(s,s)$ and let $s_1 \leq s_2$. Then $\nu(s_1) = e_S(s_1, s_1) = 1 \leq \nu(s_2) = e_S(s_2, s_2) = 1$. Also, $\nu(s) \wedge \nu(r) = e_S(s,s) \wedge e_S(r,r) \leq \nu(rs) = e_S(rs, rs) = 1$. Therefore, each fuzzy ordered semigroup is a fuzzy posemigroup.

**Proposition 2.12.** *Each $S$-fuzzy poset is a fuzzy $S$-poset.*

*Proof.* If $A$ is an $S$-fuzzy poset, similar to the above remark, one can define $\leq_S, \nu_S, \leq_A$ and $\nu_A$ and then show that $A$ is a fuzzy $S$-poset. □

## 3. Representations of Fuzzy Ordered Semigroups and Monoids by Transformations of a Fuzzy Poset

As $S$-posets correspond to representations of pomonoids or posemigroups by transformations on posets, $S$-fuzzy posets are representations of fuzzy ordered semigroups and monoids by transformations of a fuzzy poset.

The set $\mathcal{T}(A)$ of all transformations of a fuzzy poset $(A, e_A)$ forms a fuzzy ordered monoid under the composition of mappings and the order

$e : \mathcal{T}(A) \times \mathcal{T}(A) \to L$ given by $e(f,g) = \bigwedge_{a \in A} e_A(f(a), g(a))$. If we write the mappings on the left, the fuzzy ordered monoid $\mathcal{T}(A)$ will be denoted by $\mathcal{T}^l(A)$ and if we write the mappings on the right, the fuzzy ordered monoid $\mathcal{T}(A)$ will be denoted by $\mathcal{T}^r(A)$.

**Definition 3.1.** Let $(S, e_S)$ be a fuzzy ordered semigroup and $(A, e_A)$ be a fuzzy poset. A fuzzy ordered semigroup homomorphism $\psi : (S, e_S) \to (\mathcal{T}^r(A), e)$ is called a representation of $(S, e_S)$ by transformations of $(A, e_A)$.

If $(S, e_S)$ is a fuzzy ordered monoid and $\psi$ is a fuzzy ordered monoid homomorphism then $\psi$ is called a unitary representation of $(S, e_S)$ by transformations of $(A, e_A)$.

**Proposition 3.2.** *Every unitary representation of a fuzzy ordered monoid $(S, e_S)$ by transformations in $\mathcal{T}^r(A)$ of a fuzzy poset $(A, e_A)$ turns $(A, e_A)$ into a right $S$-fuzzy poset. Conversely, for every right unitary $S$-fuzzy poset $(A, e_A)$ there is an associated representation of $(S, e_S)$ by transformations in $\mathcal{T}^r(A)$.*

*Every unitary representation of a fuzzy ordered monoid $(S, e_S)$ by transformations in $\mathcal{T}^l(A)$ of a fuzzy poset $(A, e_A)$ turns $(A, e_A)$ into a left $S$-fuzzy poset. Conversely, for every left $S$-fuzzy poset $(A, e_A)$ there is an associated representation of $(S, e_S)$ by transformations in $\mathcal{T}^l(A)$.*

## 4. Fuzzy Order Congruences on $S$-Fuzzy Posets

The aim of this section is to study $S$-fuzzy poset congruences on an $S$-fuzzy poset $A$ and to characterize the $S$-fuzzy poset congruences by the concept of fuzzy pseudo orders on $A$. Some homomorphism theorems of $S$-fuzzy posets are given. Finally, some examples of the non-trivial $S$-fuzzy poset congruence on an $S$-fuzzy poset is given.

**Definition 4.1.** Let $(X, e_X)$ be an $S$-fuzzy poset and $R$ be a fuzzy equivalence relation on $X$. $R$ is called an $S$-fuzzy poset congruence if

(i) $R(x,y) = R_e^T(x,y) \wedge (R_e^T)^{-1}(x,y)$,
(ii) $R_e^T(x,y) \leq R_e^T(sx, sy)$ for all $x, y \in X$ and $s \in S$.

A suitable fuzzy partial order on the fuzzy quotient set $X/R$ is defined by $e_{X/R}(R_x, R_y) = R_e^T(x,y)$ for all $x, y \in X$ and a suitable action on the fuzzy quotient set $X/R$ is defined by $sR_x(y) = R(sx, y)$ for all $s \in S$ and $x, y \in X$.

It is easily seen that $X/R$ with the above action is an $S$-act. Also, $e_{X/R} : X/R \times X/R \to L$ is a map. In the following it is proved that it is a fuzzy partial order on $X/R$. First, $e_{X/R}(R_x, R_x) = R_e^T(x,x) = 1$, so $e_{X/R}$ is reflexive. Also, $e_{X/R}(R_x, R_y) = e_{X/R}(R_y, R_x) = 1$ implies $R_e^T(x,y) = R_e^T(y,x) = 1$, i.e. $R(x,y) = 1$. Then by Lemma 1.1, $R_x =$

$R_y$. Thus $e_{X/R}$ is antisymmetric. Finally, $e_{X/R}(R_x, R_y) * e_{X/R}(R_y, R_z) = R_e^T(x,y) * R_e^T(y,z) \leq R_e^T(x,z) = e_{X/R}(R_x, R_z)$ which follows that $e_{X/R}$ is transitive. Now,

$$e_S(s_1, s_2) \leq e_S(s_1x, s_2x) \leq R_e^T(s_1x, s_2x) = e_{X/R}(s_1R_x, s_2R_x)$$

for all $s_1, s_2 \in S$ and $x \in X$. Also, $e_{X/R}(R_x, R_y) = R_e^T(x,y) \leq R_e^T(sx, sy) = e_{X/R}(R_{sx}, R_{sy}) = e_{X/R}(sR_x, sR_y)$. Therefore, $e_{X/R}$ is a fuzzy partial order on the $S$-fuzzy poset $X/R$.

**Proposition 4.2.** *Let $(X, e_X)$ be an $S$-fuzzy poset and $R$ be an $S$-fuzzy poset congruence on $(X, e_X)$. Then the fuzzy quotient map $\pi_R : (X, e_X) \to (X/R, e_{X/R})$ given by $\pi_R(x) = R_x$ is an $S$-fuzzy poset map.*

*Proof.* By Remark 1.2, $e(x,y) \leq R_e^T(x,y) = e_{X/R}(R_x, R_y)$. Also,

$$\pi_R(sx) = R_{sx} = sR_x = s\pi_R(x).$$

□

**Definition 4.3.** Let $(X, e_X)$ be an $S$-fuzzy poset and $H \in L^{X \times X}$ be an arbitrary reflexive fuzzy relation which is compatible with the $S$-action. Then $\nu(H) = H_e^T \wedge (H_e^T)^{-1}$, is the least $S$-fuzzy poset congruence that contains $H$. We will call $\nu(H)$, the $S$-fuzzy poset congruence generated by $H$.

We now give an example to illustrate that each fuzzy equivalence relation is not an $S$-fuzzy poset congruence and an example to illustrate that there exist $S$-fuzzy poset congruences.

**Example 4.4.** Let $(S, e_S), (A, e_A)$ be defined as in Example 2.3. We define
$R : A \times A \to L$ as follows:

$$R = (R_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{4.1}$$

We want to show that each fuzzy equivalence relation is not an $S$-fuzzy poset congruence.

Through direct computing, one gets

$$R_e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 0.5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{4.2}$$

It is seen that $R_e$ is reflexive, but it is not transitive. Then

$$R_e^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{4.3}$$

and so

$$(R_e^T) \wedge (R_e^T)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \neq R, \tag{4.4}$$

which shows that $R$ is not an $S$-fuzzy poset congruence.

**Example 4.5.** Let the residuated lattice $L = \{0, 0.5, 1\}$, $S = \{1, a, b, c\}$ be a fuzzy ordered monoid with the binary operation and the order given as follows,

| | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |

and $e_S : S \times S \to L$ where

$$e_S = (e_{ij})_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

Let $X = \{x_1, x_2, x_3, x_4\}$ be an $S$-fuzzy poset with the fuzzy order and action given as follows.

| | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $x_1$ | $x_1$ | $x_3$ | $x_3$ | $x_3$ |
| $x_2$ | $x_2$ | $x_3$ | $x_3$ | $x_3$ |
| $x_3$ | $x_3$ | $x_4$ | $x_4$ | $x_4$ |
| $x_4$ | $x_4$ | $x_4$ | $x_4$ | $x_4$ |

and $e_X : X \times X \to L$ where

$$e_X = (e_{ij})_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 \\ 1 & 0.5 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.6}$$

Define $R : X \times X \to L$ as follows:

$$R = (R_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0.5 \\ 0 & 0.5 & 1 & 1 \\ 0 & 0.5 & 1 & 1 \end{pmatrix}. \tag{4.7}$$

We want to show that $R$ is an $S$-fuzzy poset congruence.

Through direct computing, we get

$$R_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 0.5 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{4.8}$$

It is seen that $R_e$ is reflexive, but it is not transitive. Then

$$R_e^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \tag{4.9}$$

and so

$$(R_e^T) \wedge (R_e^T)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0.5 \\ 0 & 0.5 & 1 & 1 \\ 0 & 0.5 & 1 & 1 \end{pmatrix} = R. \tag{4.10}$$

So $R$ is an $S$-fuzzy poset congruence on an $S$-fuzzy poset $(X, e_X)$. The corresponding fuzzy quotient set is $X/R = \{R_{x_1}, R_{x_2}, R_{x_3} = R_{x_4}\}$ and the fuzzy partial order on $X/R$ is defined by

$$e_{X/R} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.11}$$

**Definition 4.6.** A fuzzy pseudo order $\sigma$ on an $S$-fuzzy poset $(X, e_X)$ is a reflexive and transitive order relation on $(X, e_X)$ which is compatible with the $S$-action ($\sigma(x, y) \leq \sigma(sx, sy)$ for all $x, y \in X$ and $s \in S$).

The following lemma is the characterization of the fuzzy pseudo orders that contain the fuzzy partial order. This is needed to obtain the relationship between the $S$-fuzzy poset congruences and the fuzzy pseudo order that contains the fuzzy partial order.

**Lemma 4.7.** *Let $(X, e_X)$ be an $S$-fuzzy poset and $\sigma \in L^{X\times X}$. The following are equivalent:*

(1) *$\sigma$ is a fuzzy pseudo order on $X$.*
(2) *There exist an $S$-fuzzy poset $(Y, e_Y)$ and an $S$-fuzzy poset map $\varphi : (X, e_X) \to (Y, e_Y)$ such that $\sigma = H_\varphi$, where $H_\varphi(x, y) = \bigwedge_{a\in Y} e_Y(\varphi(y), a) \to e_Y(\varphi(x), a)$ is called the directed kernel of $\varphi$.*

*Proof.* (1) $\Rightarrow$ (2) Let $\sigma$ be a fuzzy pseudo order containing the fuzzy partial order. Define $R = \sigma \wedge \sigma^{-1}$. Then by Lemma 3.4 of [21], $R \leq R_e^T \wedge (R_e^T)^{-1} \leq \sigma \wedge \sigma^{-1} = R$. Also, since $\sigma$ is action preserving, $R$ is action preserving, too, which means that $R$ is an $S$-fuzzy poset congruence. Also, by Lemma 3.4 of [21], $H_{\pi_R} = \sigma$ where $\pi_R : (X, e_X) \to (X/R, e_\sigma)$ is the quotient $S$-fuzzy poset map and $e_Y(f(x), f(y)) = H_\varphi(x, y)$, for all $x, y \in X$. $H_\varphi$ contains the fuzzy partial order and it is reflexive and transitive. It remains to prove that $H_\varphi$ is compatible with the $S$-action. For, $H_\varphi(x, y) = e_Y(f(x), f(y)) \leq e_Y(sf(x), sf(y)) \leq e_Y(f(sx), f(sy)) = H_\varphi(sx, sy)$, for all $x, y \in X, s \in S$, as desired. □

**Theorem 4.8.** *Let $(X, e_X)$ be an $S$-fuzzy poset and $R$ be a fuzzy equivalence relation on $(X, e_X)$. Then the following are equivalent:*

(1) *$R$ is an $S$-fuzzy poset congruence on $(X, e_X)$.*
(2) *There exists a fuzzy pseudo order $\sigma$ that contains the fuzzy partial order such that $R = \sigma \wedge \sigma^{-1}$.*
(3) *There exist an $S$-fuzzy poset $(Y, e_Y)$ and an $S$-fuzzy poset map $\varphi : (X, e_X) \to (Y, e_Y)$ with $R = H_\varphi \wedge (H_\varphi)^{-1}$.*

*Proof.* (1) $\Leftrightarrow$ (3) is clear by Lemma 4.7.

(1) $\Rightarrow$ (2) By the proof of Lemma 4.7, $H_{\pi_R}$ is the fuzzy pseudo order which is needed.

(2) $\Rightarrow$ (1) Let $R = \sigma \wedge \sigma^{-1}$ where $\sigma$ is a fuzzy pseudo order that contains the fuzzy partial order. Then for all $x, y \in X, e(x, y) \leq \sigma(x, y)$ and $R(x, y) \leq \sigma(x, y)$, and so $R \leq R_e^T \wedge (R_e^T)^{-1} \leq \sigma \wedge \sigma^{-1} = R$. Therefore, $R$ is an $S$-fuzzy poset congruence. □

For a given $S$-fuzzy poset congruence $R$, the quotient set $X/R$ may support several different compatible fuzzy orders, it is necessary to specify which fuzzy order is being considered on the quotient set $X/R$. A fuzzy pseudo order $\rho \geq e$ on an $S$-fuzzy poset $(X, e_X)$ can generate an $S$-fuzzy poset congruence $\bar{\rho} = \rho \wedge \rho^{-1}$. The induced fuzzy partial order $e_{X/\bar{\rho}}$

on the quotient $S$-fuzzy poset $X/\bar{\rho}$ is defined by $e_{X/\bar{\rho}}(\bar{\rho}_x, \bar{\rho}_y) = \rho(x, y)$, which is denoted by $e_\rho$.

Note that if $R$ is an $S$-fuzzy poset congruence on an $S$-fuzzy poset $(X, e_X)$, then the fuzzy pseudo order that contains the fuzzy partial order $e_X$ and $R$ is not unique.

**Corollary 4.9.** *Let $(X, e_X)$ be an $S$-fuzzy poset and $R$ be an $S$-fuzzy poset congruence on $(X, e_X)$. Then $R_e^T$ is the least fuzzy pseudo order that contains $e$ and $R$.*

In the following the isomorphism theorems of $S$-fuzzy posets based on pseudo orders are given. Congruences have the essential role in isomorphism theorems in any category, but in the case of $S$-fuzzy posets, pseudo orders play the role congruences which are general than the congruences.

**Theorem 4.10** (Homomorphism theorem)**.** *Let $(X, e_X)$ and $(Y, e_Y)$ be two $S$-fuzzy posets and $f : (X, e_X) \to (Y, e_Y)$ be an $S$-fuzzy poset map. Then there exists a unique $S$-fuzzy poset order embedding $g : (X/K_f, e_{H_f}) \to (Y, e_Y)$ such that $g \circ \pi_{K_f} = f$, where $K_f = H_f \wedge (H_f)^{-1}$.*

*Proof.* For $R = K_f$, $(X/R, e_{X/R})$ is the quotient $S$-fuzzy poset, where $e_{X/R}(R_x, R_y) = H_f(x, y)$. Defining $g : (X/R, e_{X/R}) \to (Y, e_Y)$ by $g(R_x) = f(x)$, it is easily shown that $g$ is a well-defined map. Also, $e_{X/R}(R_x, R_y) = H_f(x, y) = e_Y(f(x), f(y)) = e_Y(g(R_x), g(R_y))$, which shows that $g$ is an order embedding. The uniqueness of $g$ with this property is obvious. □

The following theorem is a generalization of the homomorphism theorem.

**Theorem 4.11** (Decomposition Theorem)**.** *Let $(X, e_X)$, $(Y, e_Y)$ and $(Z, e_Z)$ be $S$-fuzzy posets. Let $f : (X, e_X) \to (Y, e_Y)$ be a surjective $S$-fuzzy poset map and $g : (X, e_X) \to (Z, e_Z)$ be an $S$-fuzzy poset map with $H_f \leq H_g$. Then there exists a unique $S$-fuzzy poset map $h : (Y, e_Y) \to (Z, e_Z)$ such that $h \circ f = g$. Moreover, $h$ is a fuzzy order embedding if and only if $H_f = H_g$ and $h$ is surjective if and only if $g$ is surjective.*

## 5. Limits and Colimits

In this section, some of the limits and colimits in the category of $S$-fuzzy posets are considered.

**Limits.** Let $\{(X_i, e_{X_i})_{i\in I}\}$ be a family of $S$-fuzzy posets. Define $e_X(x, y) = \bigwedge e_{X_i}(x_i, y_i)$ for $X = \prod_{i\in I} X_i$ and every $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}) \in \prod_{i\in I} X_i \times \prod_{i\in I} X_i$, then it is a fuzzy partial order on the product of $\{(X_i, e_{X_i})_{i\in I}\}$ and $(X, e_X)$ becomes an $S$-fuzzy poset.

The terminal $S$-fuzzy poset is the singleton $S$-fuzzy poset and the initial $S$-fuzzy poset is empty.

The equalizer of a pair $f, g : (X, e_X) \to (Y, e_Y)$ of $S$-fuzzy poset maps is given by $(E = \{x \in X | f(x) = g(x)\}, e_E)$ with the action and order inherited from $(X, e_X)$.

The pullback of $S$-fuzzy poset maps $f : (X, e_X) \to (Z, e_Z)$ and $g : (Y, e_Y) \to (Z, e_Z)$ is the sub $S$-fuzzy poset $(P = \{(x, y) \in X \times Y | f(x) = g(y)\}, (e_X \wedge e_Y)|_P)$ of $(X \times Y, e_X \wedge e_Y)$ together with the restricted projection maps.

**Colimits.** $S$-fuzzy poset congruences play an essential role in studying the structure of $S$-fuzzy posets. Now, using $S$-fuzzy poset congruences, we construct some colimits of the category of $S$-fuzzy posets such as pushouts and coequalizers.

Let $\{(X_i, e_{X_i})_{i \in I}\}$ be a family of $S$-fuzzy posets. Define

$$e_X(x, y) = \begin{cases} e_{X_i}(x, y) & \text{if } x, y \in X_i \\ 0 & \text{otherwise} \end{cases}$$

for $X = \coprod_{i \in I} X_i$ and every $(x, y) \in \coprod_{i \in I} X_i \times \coprod_{i \in I} X_i$, then it is a fuzzy partial order on the coproduct of $\{(X_i)_{i \in I}\}$ and $(X, e_X)$ becomes an $S$-fuzzy poset.

Let $(X, e_X)$ and $(Y, e_Y)$ be $S$-fuzzy posets and $f, g : (X, e_X) \to (Y, e_Y)$ be $S$-fuzzy poset maps. The coequalizer of $f$ and $g$ is the quotient $S$-fuzzy poset $(Y/\nu(K), e_{\nu(K)})$ where $\nu(K)$ is an $S$-fuzzy poset congruence generated by the fuzzy relation $K$ on $(Y, e_Y)$ defined by

$$K(x, y) = \begin{cases} 1 & \text{if } x = y \text{ or } x = f(a), y = g(a) \text{ or } x = g(a), y = f(a) \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in X$ and $x, y \in Y$.

Let $f : (X, e_X) \to (Y, e_Y)$ and $g : (X, e_X) \to (Z, e_Z)$ be $S$-fuzzy poset maps. Then the pushout of the pair $f$ and $g$ is the quotient $S$-fuzzy poset, $((Y \sqcup Z)/\nu(K), e_{\nu(K)})$, where $Y \sqcup Z$ is the coproduct of $(Y, e_Y)$ and $(Z, e_Z)$ and $\nu(K)$ is the $S$-fuzzy poset congruence generated by the following fuzzy relation on $Y \sqcup Z$,

$$K(y, z) = \begin{cases} 1 & \text{if } y = z \text{ or } y = (1, f(x)), z = (2, g(x)) \text{ or } y = (2, g(x)), z = (1, f(x)) \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$ and $y, z \in Y \sqcup Z$.

## 6. Adjoint Relations

In this section, we consider all forgetful functors between the category **FPos**-$S$ and the categories **FPos** of fuzzy posets, **Pos**-$S$ of $S$-posets,

**Pos** of posets, **Act**-$S$ of $S$-acts and **Set** of sets and study the existence of free and cofree objects.

### 6.1. Adjoint Relations of $S$-Fuzzy Posets with Fuzzy Posets.

**Definition 6.1.** By a free $S$-fuzzy poset on a fuzzy poset $(P, e_P)$ we mean an $S$-fuzzy poset $(F, e_F)$ together with a fuzzy order preserving map $\tau : (P, e_P) \to (F, e_F)$ with the universal property that given any $S$-fuzzy poset $(A, e_A)$ and a fuzzy order preserving map $f : (P, e_P) \to (A, e_A)$ there exists a unique $S$-fuzzy poset map $\bar{f} : (F, e_F) \to (A, e_A)$ such that $\bar{f} \circ \tau = f$.

Note that this notion generalizes the notion of a free $S$-poset on a poset, defined in [10] and also it gives an adjoint pair between the categories **FPos**-$S$ and **FPos**.

**Theorem 6.2.** *For a given fuzzy poset $(P, e_P)$ and a fuzzy ordered monoid $(S, e_S)$, the free $S$-fuzzy poset on $(P, e_P)$ is given by $P \times S$, with the order $e_P \wedge e_S : (P \times S) \times (P \times S) \to L$ given by $e_P \wedge e_S((x_1, s_1), (x_2, s_2)) = e_P(x_1, x_2) \wedge e_S(s_1, s_2)$ and the action $(x, s)t = (x, st)$ for all $x_1, x_2, x \in P$ and $s_1, s_2, s, t \in S$.*

*Proof.* With the order and the action defined above, $(P \times S, e_P \wedge e_S)$ is clearly an $S$-fuzzy poset and the map $\tau : (P, e_P) \to (P \times S, e_P \wedge e_S)$ given by $\tau(x) = (x, 1)$ is a universal fuzzy order preserving map. For, if $f : (P, e_P) \to (A, e_A)$ is any fuzzy order preserving map to an $S$-fuzzy poset $(A, e_A)$ then the map $\bar{f} : (P \times S, e_P \wedge e_S) \to (A, e_A)$ defined by $\bar{f}(x, s) = f(x)s$ is the unique $S$-fuzzy poset map with $\bar{f} \circ \tau = f$. □

**Corollary 6.3.** *The free functor $F_1$ : **FPos** $\to$ **FPos** $- S$ given by $F_1(P, e_P) = (P \times S, e_P \wedge e_S)$ is a left adjoint to the forgetful functor $U_1$ : **FPos** $- S \to$ **FPos**.*

**Definition 6.4.** By a cofree $S$-fuzzy poset on a fuzzy poset $(P, e_P)$ we mean an $S$-fuzzy poset $(C, e_C)$ together with a fuzzy order preserving map $\sigma : (C, e_C) \to (P, e_P)$ with the universal property that given any $S$-fuzzy poset $(A, e_A)$ and a fuzzy order preserving map $\alpha : (A, e_A) \to (P, e_P)$ there exists a unique $S$-fuzzy poset map $\bar{\alpha} : (A, e_A) \to (C, e_C)$ such that $\sigma \circ \bar{\alpha} = \alpha$.

**Theorem 6.5.** *For a given fuzzy poset $(P, e_P)$ and a fuzzy ordered monoid $(S, e_S)$, the cofree $S$-fuzzy poset on $(P, e_P)$ is the set $((P, e_P)^{(S, e_S)}, e)$, of all fuzzy order preserving maps from $(S, e_S)$ to $(P, e_P)$, with the order $e : (P, e_P)^{(S, e_S)} \times (P, e_P)^{(S, e_S)} \to L$ given by $e(f, g) = \bigwedge_{s \in S} e_P(f(s), g(s))$ and the action $(fs)(t) = f(st)$ for all $f, g \in (P, e_P)^{(S, e_S)}$ and $s, t \in S$.*

*Proof.* One can check that with the order and the action defined above, $((P, e_P)^{(S,e_S)}, e)$ is an $S$-fuzzy poset. The cofree map $\sigma : (P, e_P)^{(S,e_S)} \to (P, e_P)$ given by $\sigma(f) = f(1)$ is monotone, since

$$\bigwedge_{s\in S} e_P(f(s), g(s)) = e(f, g) \leq e_P(\sigma(f), \sigma(g)) = e_P(f(1), g(1)).$$

Also, for a given fuzzy order preserving map $\alpha : (A, e_A) \to (P, e_P)$ from an $S$-fuzzy poset $(A, e_A)$, there exists a unique $S$-fuzzy poset map $\bar{\alpha} : (A, e_A) \to (P, e_P)$ given by $\bar{\alpha}(a)(s) = \alpha(as)$ such that $\sigma \circ \bar{\alpha} = \alpha$. $\bar{\alpha}$ is fuzzy order preserving, since $e_A(x, y) \leq e_A(xs, ys) \leq e_P(\alpha(xs), \alpha(ys))$ for all $x, y \in A, s \in S$ and hence $e_A(x, y) \leq \bigwedge_{s\in S} e_P(\alpha(xs), \alpha(ys)) = e(\bar{\alpha}(x), \bar{\alpha}(y))$. □

**Corollary 6.6.** *The (cofree) functor* $K_1 : \mathbf{FPos} \to \mathbf{FPos} - S$ *given by* $K_1(P, e_P) = ((P, e_P)^{(S,e_S)}, e)$ *is a right adjoint to the forgetful functor* $U_1 : \mathbf{FPos} - S \to \mathbf{FPos}$.

6.2. **Adjoint relations of ($S$-)fuzzy posets with ($S$-)posets.** Define $H : \mathbf{FPos} - S \to \mathbf{Pos} - S$ given by $H(A, e_A) = (A, \leq)$, where $(A, \leq)$ is the underlying $S$-fuzzy poset of $(A, e_A)$. Then $H$ is a functor from $\mathbf{FPos} - S$ to $\mathbf{Pos} - S$. Also, define $K : \mathbf{Pos} - S \to \mathbf{FPos} - S$ given by $K(A, \leq) = (A, e_A)$, where

$$e_A(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Then $K$ is a functor from $\mathbf{Pos} - S$ to $\mathbf{FPos} - S$. It is shown that $K \dashv H$ and $H \dashv K$. Similar functors give an adjoint pair between the category of fuzzy posets and the category of posets.

6.3. **Adjoint relations of $S$-fuzzy posets with $S$-acts.** The following result is immediately obvious.

**Theorem 6.7.** *The (free) functor* $F_2 : \mathbf{Act} - S \to \mathbf{FPos} - S$ *given by* $F_2(A) = (A, e_A)$*, where* $e_A : A \times A \to L$ *is given by*

$$e_A(a, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise,} \end{cases}$$

*is a left adjoint to the forgetful functor* $U_2 : \mathbf{FPos} - S \to \mathbf{Act} - S$.

**Remark 6.8.** The forgetful functor $U_2 : \mathbf{FPos} - S \to \mathbf{Act} - S$ does not have a right adjoint. Because, if $K' : \mathbf{Act} - S \longrightarrow \mathbf{FPos} - S$ is a right adjoint of $U_2$ then $H \circ K' : \mathbf{Act} - S \xrightarrow{K'} \mathbf{FPos} - S \xrightarrow{H} \mathbf{Pos} - S$ would be a right adjoint of the forgetful functor $U : \mathbf{Pos} - S \longrightarrow \mathbf{Act} - S$. But by the note after Theorem 17 of [10], the forgetful functor $U : \mathbf{Pos} - S \longrightarrow \mathbf{Act} - S$ does not have a right adjoint.

6.4. **Adjoint relations of $S$-fuzzy posets with sets.**

**Definition 6.9.** By a free fuzzy poset on a set $X$ we mean a fuzzy poset $(X, e_X)$, where $e_X : X \times X \to L$ is given by

$$e_X(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

together with the identity map $i : X \to (X, e_X)$ with the universal property that given any fuzzy poset $(P, e_P)$ and a map $f : X \to P$ there exists a unique fuzzy poset map $\bar{f} : (X, e_X) \to (P, e_P)$ such that $\bar{f} \circ i = f$.

**Lemma 6.10.** *The free functor $F' : \mathbf{Set} \longrightarrow \mathbf{FPos}$ given by $F'(X) = (X, e_X)$, where $e_X : X \times X \to L$ is given by*

$$e_X(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

*is a left adjoint to the forgetful functor $U' : \mathbf{FPos} \longrightarrow \mathbf{Set}$.*

By composing the two free functors, one from **Set** to **FPos** and the other from **FPos** to **FPos**-$S$ one gets:

**Theorem 6.11.** *The free functor $F'_2 : \mathbf{Set} \longrightarrow \mathbf{FPos} - S$ given by $F'_2(X) = F_1 \circ F'(X) = F_1(X, e_X)$ is a left adjoint to the forgetful functor $U'_2 : \mathbf{FPos} - S \longrightarrow \mathbf{Set}$. More precisely, the free $S$-fuzzy poset on a set $X$, is $(X \times S, e_X \wedge e_S)$ where the fuzzy order $e_X : X \times X \to L$ is given by*

$$e_X(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$$

*and the action is given by $(x, s)t = (x, st)$ for $x \in X, s, t \in S$.*

**Remark 6.12.** The forgetful functor $U'_2 : \mathbf{FPos} - S \longrightarrow \mathbf{Set}$ does not have a right adjoint. For, let $H' : \mathbf{Set} \longrightarrow \mathbf{FPos} - S$ be a right adjoint for $U'_2$. Then $H \circ H' : \mathbf{Set} \longrightarrow \mathbf{FPos} - S \longrightarrow \mathbf{Pos} - S$ is a right adjoint for a forgetful functor $U : \mathbf{Pos} - S \longrightarrow \mathbf{Set}$ which is a contradiction by Remark 16 of [10].

## 7. Epimorphisms and Monomorphisms

In this section, we give a set theoretic characterization of the epimorphisms and monomorphisms in **FPos** and **FPos**-$S$ and a categorical characterization of order embeddings. Note that an epimorphism in **FPos** is a morphism that is right cancelable under composition.

**Lemma 7.1.** *Epimorphisms in* **FPos** *are exactly surjective fuzzy poset maps.*

*Proof.* Assume that $f : (P, e_P) \to (Q, e_Q)$ is an epimorphism in **FPos** but it is not surjective. Take $a \in Q$ with $a \notin \mathrm{Im} f$. Define a new fuzzy poset $(R, e_R)$ as follows:
$R = (Q - \{a\}) \cup \{b, c\}$ with the order $e_R : R \times R \to L$ given by

$$e_R(c, x) = \begin{cases} 1 & \text{if } e_Q(a, x) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$e_R(x, c) = \begin{cases} 1 & \text{if } e_Q(a, x) \neq 1 \\ e_Q(x, a) & \text{if } e_Q(a, x) = 1, \end{cases}$$

$$e_R(y, b) = \begin{cases} 1 & \text{if } e_Q(y, a) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$e_R(b, y) = \begin{cases} 1 & \text{if } e_Q(y, a) \neq 1 \\ e_Q(a, y) & \text{if } e_Q(y, a) = 1, \end{cases}$$

and $e_R(x, y) = e_Q(x, y)$ for all $x, y \in Q$ and $x, y \neq a$. Define $g : (Q, e_Q) \to (R, e_R)$ given by $g(x) = x$ for all $x \neq a$ and $g(a) = b$ and $h : (Q, e_Q) \to (R, e_R)$ given by $h(x) = x$ for all $x \neq a$ and $h(a) = c$. It is clear that $g \circ f = h \circ f$ but $g \neq h$ which contradicts the assumption that $f$ is an epimorphism. The converse is obvious. □

**Theorem 7.2.** *Epimorphisms in* **FPos**-$S$ *are exactly surjective* $S$*-fuzzy poset maps.*

*Proof.* Using the adjunction given in Corollary 6.3 and the fact that left adjoints preserve colimits, and in particular epimorphisms, one gets that epimorphisms in **FPos**-$S$ are exactly $S$-fuzzy poset maps which are epimorphisms in **FPos**. Then the result follows by the fact that epimorphisms in **FPos** are exactly surjective morphisms by Lemma 7.1. □

Note that a monomorphism in **FPos**-$S$ is a morphism that is left cancelable under composition.

**Lemma 7.3.** *Monomorphisms in* **FPos** *are exactly injective fuzzy poset maps.*

*Proof.* Let $f : (P, e_P) \to (Q, e_Q)$ be a fuzzy poset monomorphism and suppose that $f$ is not injective. i.e. $f(a) = f(b)$ for distinct elements $a, b \in P$. Take $S = (\{a, b\}, e_S)$ with the discrete order, $e_S : S \times S \to L$ given by $e_S(a, a) = e_S(b, b) = 1, e_S(a, b) = e_S(b, a) = 0$ and define $g : (S, e_S) \to (P, e_P)$ given by $g(a) = g(b) = a$ and $h : (S, e_S) \to (P, e_P)$ given by $h(a) = h(b) = b$. Then $f \circ g = f \circ h$ but $g \neq h$ which contradicts that $f$ is a monomorphism. Hence $f$ is injective. The converse is clear. □

**Theorem 7.4.** *Monomorphisms in* **FPos**-$S$ *are exactly injective* $S$*-fuzzy poset maps.*

*Proof.* Using the adjunction given in Corollary 6.6 and the fact that right adjoints preserve limits, and in particular monomorphisms, one gets that monomorphisms in **FPos**-$S$ are exactly $S$-fuzzy poset maps which are monomorphisms in **FPos**. Then the result follows by the fact that monomorphisms in **FPos** are exactly injective fuzzy poset maps by Lemma 7.3. □

**Theorem 7.5.** *A morphism* $f : (P, e_P) \to (Q, e_Q)$ *in* **FPos** (**FPos**-$S$) *is an order embedding if and only if it is a regular monomorphism.*

*Proof.* Let $f : (P, e_P) \to (Q, e_Q)$ be an order embedding. Then $f$ is a monomorphism and it is enough to show that it is regular. For, let $g : (T, e_T) \to (Q, e_Q)$ be any fuzzy poset map which equalizes every pair of fuzzy poset maps from $(Q, e_Q)$ equalized by $f$ and $g(T) \nsubseteq f(P)$. Then, by the proof of Lemma 7.3, there exists a fuzzy poset $(S, e_S)$ and $u, v : (Q, e_Q) \to (S, e_S)$ which differ only at one point $a$ in $g(T)$, not in $f(P)$. Hence, $u \circ f = v \circ f$ whereas $u \circ g \neq v \circ g$, which is a contradiction. Therefore, $g(T) \subseteq f(P)$ and $f^{-1} \circ g$, where $f^{-1}$ is the inverse of $f$ on $f(P)$, provides the factorization.

For the converse, let $f : (P, e_P) \to (Q, e_Q)$ be a regular monomorphism and let $a, b \in P$. Also, let $(T, e_T)$ be the sub fuzzy poset of $(Q, e_Q)$ determined by $\{f(a), f(b)\}$, with the fuzzy order $e_T : T \times T \to L$ given by $e_T(f(a), f(a)) = e_T(f(b), f(b)) = e_T(f(a), f(b)) = e_Q(f(a), f(b)), e_T(f(b), f(a)) = e_Q(f(b), f(a))$ and $g : (T, e_T) \to (Q, e_Q)$ be the natural embedding. It is clear that $g$ equalizes any pair $u, v : (Q, e_Q) \to (S, e_S)$ equalized by $f$. Hence there exists a fuzzy poset map $h : (T, e_T) \to (P, e_P)$ such that $g = f \circ h$. Since $g(f(a)) = f(a)$, one gets $h(f(a)) = a$ and since $g(f(b)) = f(b)$, one gets $h(f(b)) = b$ and hence $e_Q(f(a), f(b)) = e_T(f(a), f(b)) \leq e_P(h(f(a)), h(f(b))) = e_P(a, b)$. Therefore, $f$ is an order embedding. □

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Department of Mathematics, University of Maragheh, Maragheh, 55181-83111, Iran.

*Email address*: `leilashahbaz@yahoo.com, l_shahbaz@maragheh.ac.ir`