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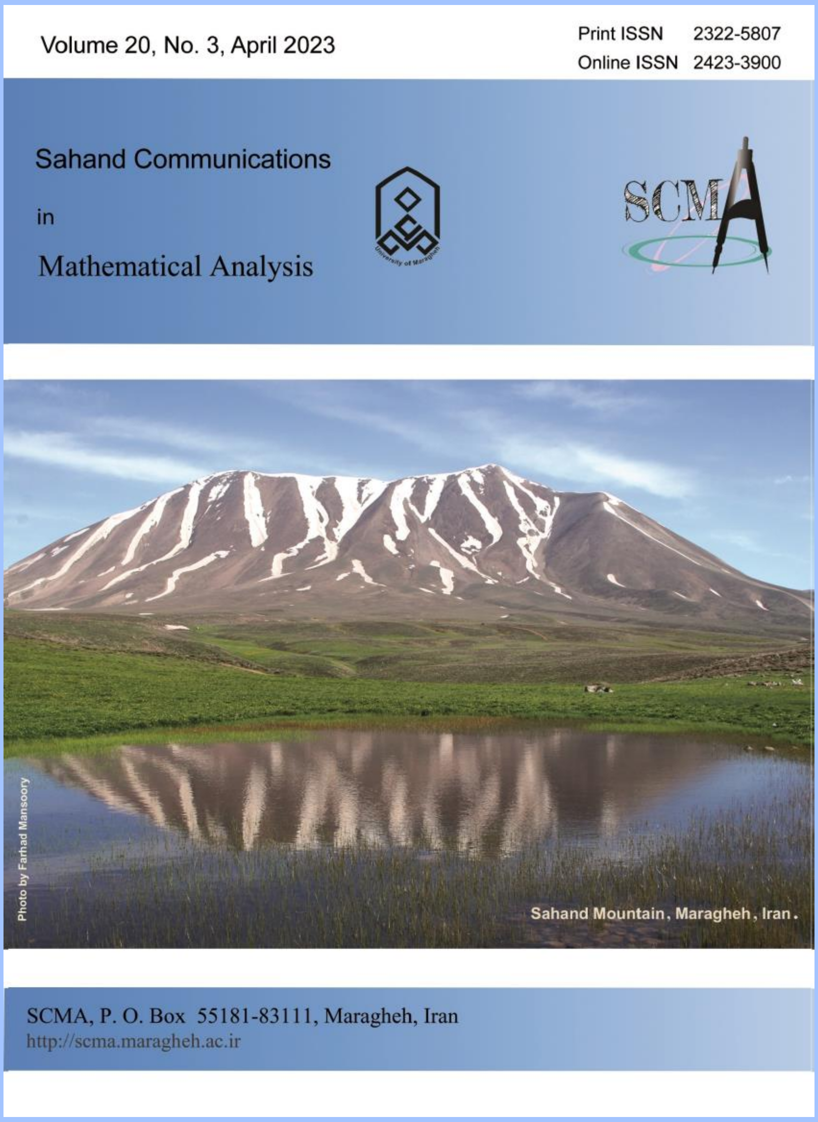

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A New Two-Step Iterative Algorithm and $H(.,.)$-Mixed Mappings for Solving a System of Variational Inclusions

Sumeera Shafi

Abstract. A system of generalized mixed variational inclusion problem (SGMVIP) is considered involving $H(.,.)$-mixed mappings in q-uniformly smooth Banach spaces. By means of proximal-point mapping method, the existence of solution of this system of variational inclusions is given. A new two-step iterative algorithm is proposed for solving SGMVIP. Strong convergence of the proposed algorithm is given.

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1. Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please refer [1, 3–6, 8, 16–18, 20, 21] and the references therein. Zou and Huang [22, 23] introduced and studied $H(.,.)$-accretive mappings, Kazmi *et al.* [9–11] introduced and studied generalized $H(.,.)$-accretive mappings, $H(.,.)-\eta$-proximal-point mappings. In 2011, Li and Huang [14] studied the graph convergence for the $H(.,.)$-accretive mapping and showed the equivalence between graph convergence and proximal-point mapping convergence for the $H(.,.)$-accretive mapping sequence in a Banach space.

Motivated and inspired by the above works and by the ongoing research in this direction[2, 7, 12, 13], we introduce and study a system of

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generalized mixed variational inclusion problem involving $H(.,.)$-mixed mappings, a natural generalization of accretive (monotone) mappings in q-uniformly smooth Banach spaces. Using proximal-point mapping method, we suggest a new two-step iterative algorithm for solving the system. Furthermore, we prove that the sequences generated by the algorithm converge strongly to a solution of the system.

2. Proximal-Point Mapping and Formulation of Problem

We need the following definitions and results from the literature.

Let X be a real Banach space equipped with norm $\|.\|$ and $X^\star$ be the topological dual space of X. Let $\langle .,. \rangle$ be the dual pair between X and $X^\star$ and 2^X be the power set of X.

Definition 2.1 ([19])**.** For $q > 1$, a mapping $J_q : X \to 2^{X^\star}$ is said to be generalized duality mapping, if it is defined by

$$J_q(x) = \left\{ f \in X^\star : \langle x, f \rangle = \|x\|^q, \|x\|^{q-1} = \|f\| \right\}, \quad \forall x \in X.$$

In particular, J_2 is the usual normalized duality mapping on X, given as

$$J_q(x) = \|x\|^{q-2} J_2(x), \quad \forall x (\neq 0) \in X.$$

Note that if $X \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2 ([19])**.** A Banach space X is said to be smooth if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$, defined by

$$\rho_X(\sigma) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = \sigma \right\}.$$

Definition 2.3 ([19])**.** A Banach space X is said to be

(i) uniformly smooth if $\lim_{\sigma \to 0} \dfrac{\rho_X(\sigma)}{\sigma} = 0$,

(ii) q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that $\rho_X(\sigma) \leq c\sigma^q$, $\sigma \in [0, \infty)$.

Note that if X is uniformly smooth, J_q becomes single-valued.

Lemma 2.4 ([19])**.** *Let $q > 1$ be a real number and let X be a smooth Banach space. Then the following statements are equivalent:*

(i) *X is q-uniformly smooth.*

(ii) *There is a constant $c_q > 0$ such that for every $x, y \in X$, the following inequality holds*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q \|y\|^q.$$

Lemma 2.5 ([15]). *Let* $\{a^n\}, \{b^n\}$ *and* $\{c^n\}$ *be sequences of non-negative real numbers that satisfy:*

$$a^{n+1} \leq (1-d^n)a^n + b^n + c^n, \quad \forall n \geq 0,$$

where $d^n \in (0,1)$, $\sum\limits_{n=0}^{\infty} d^n = +\infty$, $\lim\limits_{n\to\infty} b^n = 0$ *and* $\sum\limits_{n=0}^{\infty} c^n < \infty$. *Then* $\sum\limits_{n=0}^{\infty} a^n = 0$.

Lemma 2.6 ([7]). *A mapping* $f : X \to X$ *is said to be*

(i) δ*-strongly accretive with* $\delta > 0$*, if*

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq \delta\|x-y\|^q, \quad \forall x, y \in X.$$

(ii) μ*-cocoercive with* $\mu > 0$*, if*

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq \mu\|f(x)-f(y)\|^q, \quad \forall x, y \in X.$$

(iii) γ*-relaxed cocoercive with* $\gamma > 0$*, if*

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq -\gamma\|f(x)-f(y)\|^q, \quad \forall x, y \in X.$$

(iv) β*-Lipschitz continuous with* $\beta > 0$*, if*

$$\|f(x) - f(y)\| \leq \beta\|x-y\|, \quad \forall x, y \in X.$$

(v) α*-expansive with* $\alpha > 0$*, if*

$$\|f(x) - f(y)\| \geq \alpha\|x-y\|, \quad \forall x, y \in X.$$

if $\alpha = 1$*, then it is expansive.*

Definition 2.7. Let $H : X \times X \to X$ and $A, B : X \to X$ be single-valued mapings. Then,

(i) $H(A, .)$ is said to be μ-cocoercive with respect to A if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), J_q(x-y)\rangle \geq \mu\|Ax - Ay\|^q, \quad \forall x, y, u \in X.$$

(ii) $H(., B)$ is said to be γ-relaxed accretive with respect to B if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x-y)\rangle \geq -\gamma\|x-y\|^q, \quad \forall x, y, u \in X.$$

(iii) $H(A, .)$ is said to be r_1-Lipschitz continuous with respect to A if there exists a constant $r_1 > 0$ such that

$$\|H(Ax, .) - H(Ay, .)\| \leq r_1\|x-y\|, \quad \forall x, y \in X.$$

(iv) $H(., B)$ is said to be r_2-Lipschitz continuous with respect to B if there exists a constant $r_2 > 0$ such that

$$\|H(., Bx) - H(., By)\| \leq r_2\|x-y\|, \quad \forall x, y \in X.$$

Example 2.8. Consider a 2-uniformly smooth Banach space $X = \mathbb{R}^2$ with the usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax = \begin{pmatrix} 2sx_1 - 2sx_2 \\ \\ -2sx_1 + 3sx_2 \end{pmatrix}, \qquad By = \begin{pmatrix} -2sy_1 + 2sy_2 \\ \\ -2sy_1 - 2sy_2 \end{pmatrix}$$

for all scalers $s \in \mathbb{R}$ and for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

Suppose that $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $H(Ax, By) = Ax + By$. Then $H(A, B)$ is $\dfrac{1}{5s}$-cocoercive with respect to A and $2s$-relaxed accretive with respect to B, and $\sqrt{13}s$-Lipschitz continuous with respect to A and $\sqrt{8}s$-Lipschitz continuous with respect to B.

Indeed, for any $u \in X$,

$$\begin{aligned}
&\langle H(Ax, u) - H(Ay, u), x - y\rangle \\
&\quad = \langle Ax - Ay, x - y\rangle \\
&\quad = \langle (2sx_1 - 2sx_2, -2sx_1 + 3sx_2) \\
&\qquad -(2sy_1 - 2sy_2, -2sy_1 + 3sy_2), (x_1 - y_1, x_2 - y_2)\rangle \\
&\quad = \langle 2s(x_1 - y_1) - 2s(x_2 - y_2), -2s(x_1 - y_1) \\
&\qquad +3s(x_2 - y_2), (x_1 - y_1, x_2 - y_2)\rangle \\
&\quad = 2s(x_1 - y_1)^2 - 4s(x_1 - y_1)(x_2 - y_2) + 3s(x_2 - y_2)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|Ax - Ay\|^2 \\
&\quad = \langle Ax - Ay, Ax - Ay\rangle \\
&\quad = \langle (2sx_1 - 2sx_2, -2sx_1 + 3sx_2) - (2sy_1 - 2sy_2, -2sy_1 + 3sy_2), \\
&\qquad ((2sx_1 - 2sx_2, -2sx_1 + 3sx_2) - (2sy_1 - 2sy_2, -2sy_1 + 3sy_2))\rangle \\
&\quad = \langle 2s(x_1 - y_1) - 2s(x_2 - y_2), -2s(x_1 - y_1) + 3s(x_2 - y_2), \\
&\qquad (2s(x_1 - y_1) - 2s(x_2 - y_2), -2s(x_1 - y_1) + 3s(x_2 - y_2))\rangle \\
&\quad = 4s^2(x_1 - y_1)^2 - 4s^2(x_1 - y_1)(x_2 - y_2) - 4s^2(x_1 - y_1)(x_2 - y_2) \\
&\qquad + 4s^2(x_2 - y_2)^2 + 4s^2(x_1 - y_1)^2 - 6s^2(x_1 - y_1)(x_2 - y_2) \\
&\qquad - 6s^2(x_1 - y_1)(x_2 - y_2) + 9s^2(x_2 - y_2)^2 \\
&\quad = 8s^2(x_1 - y_1)^2 - 20s^2(x_1 - y_1)(x_2 - y_2) + 13s^2(x_2 - y_2)^2 \\
&\quad \le 10s^2(x_1 - y_1)^2 - 20s^2(x_1 - y_1)(x_2 - y_2) + 15s^2(x_2 - y_2)^2 \\
&\quad = 5s\left[2s(x_1 - y_1)^2 - 4s(x_1 - y_1)(x_2 - y_2) + 3s(x_2 - y_2)^2\right] \\
&\quad = 5s\left[\langle H(Ax, u) - H(Ay, u), x - y\rangle\right],
\end{aligned}$$

which implies that

$$\langle H(Ax,u) - H(Ay,u), x - y\rangle \geq \frac{1}{5s}\|Ax - Ay\|^2,$$

that is, $H(A,B)$ is $\dfrac{1}{5s}$-cocoercive with respect to A.

$$\begin{aligned}
&\langle H(u,Bx) - H(u,By), x - y\rangle \\
&\quad = \langle Bx - By, x - y\rangle \\
&\quad = \langle (-2sx_1 + 2sx_2, -2sx_1 - 2sx_2) \\
&\qquad -(-2sy_1 + 2sy_2, -2sy_1 - 2sy_2), (x_1 - y_1, x_2 - y_2)\rangle \\
&\quad = \langle -2s(x_1 - y_1) + 2s(x_2 - y_2), \\
&\qquad -2s(x_1 - y_1) - 2s(x_2 - y_2), (x_1 - y_1, x_2 - y_2)\rangle \\
&\quad = -2s(x_1 - y_1)^2 + 2s(x_1 - y_1)(x_2 - y_2) \\
&\qquad - 2s(x_1 - y_1)(x_2 - y_2) - 2s(x_2 - y_2)^2 \\
&\quad = -2s\left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right] \\
&\quad \geq -2s\|x - y\|^2,
\end{aligned}$$

which implies that

$$\langle H(u,Bx) - H(u,By), x - y\rangle \geq -2s\|x - y\|^2,$$

that is, $H(A,B)$ is $2s$-relaxed accretive with respect to B,

$$\begin{aligned}
&\|H(Ax,u) - H(Ay,u)\|^2 \\
&\quad = \|Ax - Ay\|^2 \\
&\quad = \langle Ax - Ay, Ax - Ay\rangle \\
&\quad = \langle (2sx_1 - 2sx_2, -2sx_1 + 3sx_2) - (2sy_1 - 2sy_2, -2sy_1 + 3sy_2), \\
&\qquad ((2sx_1 - 2sx_2, -2sx_1 + 3sx_2) - (2sy_1 - 2sy_2, -2sy_1 + 3sy_2))\rangle \\
&\quad = \langle 2s(x_1 - y_1) - 2s(x_2 - y_2), -2s(x_1 - y_1) + 3s(x_2 - y_2), \\
&\qquad (2s(x_1 - y_1) - 2s(x_2 - y_2), -2s(x_1 - y_1) + 3s(x_2 - y_2))\rangle \\
&\quad = 4s^2(x_1 - y_1)^2 - 4s^2(x_1 - y_1)(x_2 - y_2) - 4s^2(x_1 - y_1)(x_2 - y_2) \\
&\qquad + 4s^2(x_2 - y_2)^2 + 4s^2(x_1 - y_1)^2 - 6s^2(x_1 - y_1)(x_2 - y_2) \\
&\qquad - 6s^2(x_1 - y_1)(x_2 - y_2) + 9s^2(x_2 - y_2)^2 \\
&\quad = 8s^2(x_1 - y_1)^2 - 20s^2(x_1 - y_1)(x_2 - y_2) + 13s^2(x_2 - y_2)^2 \\
&\quad \leq 13s^2(x_1 - y_1)^2 + 13s^2(x_2 - y_2)^2,
\end{aligned}$$

which implies that

$$\|H(Ax,u) - H(Ay,u)\| \leq \sqrt{13}s\|x - y\|,$$

that is, $H(A,B)$ is $\sqrt{13}s$-Lipschitz continuous with respect to A.

$$\begin{aligned}
&\|H(u,Bx)-H(u,By)\|^2\\
&\quad=\|Bx-By\|^2\\
&\quad=\langle Bx-By, Bx-By\rangle\\
&\quad=\langle(-2sx_1+2sx_2,-2sx_1-2sx_2)-(-2sy_1+2sy_2,-2sy_1-2sy_2),\\
&\qquad((-2sx_1+2sx_2,-2sx_1-2sx_2)-(-2sy_1+2sy_2,-2sy_1-2sy_2))\rangle\\
&\quad=\langle-2s(x_1-y_1)+2s(x_2-y_2),-2s(x_1-y_1)-2s(x_2-y_2),\\
&\qquad(-2s(x_1-y_1)+2s(x_2-y_2),-2s(x_1-y_1)-2s(x_2-y_2))\rangle\\
&\quad=4s^2(x_1-y_1)^2-4s^2(x_1-y_1)(x_2-y_2)-4s^2(x_1-y_1)(x_2-y_2)\\
&\qquad+4s^2(x_2-y_2)^2+4s^2(x_1-y_1)^2+4s^2(x_1-y_1)(x_2-y_2)\\
&\qquad+4s^2(x_1-y_1)(x_2-y_2)+4s^2(x_2-y_2)^2\\
&\quad=8s^2(x_1-y_1)^2+8s^2(x_2-y_2)^2,
\end{aligned}$$

which implies that

$$\|H(u,Bx)-H(u,By)\|=\sqrt{8}s\|x-y\|,$$

that is, $H(A,B)$ is $\sqrt{8}s$-Lipschitz continuous with respect to B.

Definition 2.9. Let $f,g: X\to X$ be single-valued mappings and $M: X\times X\to 2^X$ be a set-valued mapping. Then

(i) $M(f,.)$ is said to be ω-strongly accretive regarding f with $\omega>0$, if

$$\langle u-v, J_q(x-y)\rangle\geq\omega\|x-y\|,^q\quad\forall x,y,w\in X,$$
$$u\in M(f(x),w), v\in M(f(y),w).$$

(ii) $M(.,g)$ is said to be τ-relaxed accretive regarding g with $\tau>0$, if

$$\langle u-v, J_q(x-y)\rangle\geq-\tau\|x-y\|^q,\quad\forall x,y,w\in X,$$
$$u\in M(w,g(x)), v\in M(w,g(y)).$$

(iii) $M(.,.)$ is said to be $\omega\tau$-symmetric accretive regarding f and g if $M(f,.)$ is ω-strongly accretive regarding f and $M(.,g)$ is τ-relaxed accretive regarding g with $\omega\geq\tau$ and $\omega=\tau$ if and only if $x=y$.

Definition 2.10 ($H(.,.)$-Mixed mappings)**.** Let $H: X\times X\to X$ and $A,B: X\to X$ be single-valued mapings and $M: X\times X\to 2^X$ be a set-valued mapping. Let $H(A,B)$ be μ-cocoercive with respect to A, γ-relaxed accretive with respect to B. Then M is said to be an $H(.,.)$-mixed mapping with respect to mappings A and B if

(i) M is $\omega\tau$-symmetric accretive regarding f and g;
(ii) $(H(A,B)+\rho M(f,g))(X)=X, \quad \forall \rho>0$.

Proposition 2.11. *Let* $M : X\times X \to 2^X$ *be an* $H(.,.)$*-mixed mapping with respect to mappings* A *and* B*. If* A *is* α*-expansive and* $\mu>\gamma$ *with* $r=\mu\alpha^q-\gamma>(\omega-\tau)$*, then the following inequality holds:*

$$\langle u-v, J_q(x-y)\rangle \geq 0, \quad \forall (y,v)\in Graph(M(f,g)),$$
$$\textit{implies } (x,u)\in Graph(M(f,g)),$$

where

$$Graph(M(f,g))=\{(x,u)\in X\times X : u\in M(f(x),g(x))\}.$$

Proof. Assume on the contrary that there exists $(x_0,u_0)\notin Graph(M(f,g))$ such that

$$\langle u_0-v, J_q(x_0-y)\rangle \geq 0, \quad \forall (y,v)\in Graph(M(f,g)). \tag{2.1}$$

Since M is an $H(.,.)$-mixed mapping, we know that

$$\left(H(A,B)+\rho M(f,g)\right)(X)=X, \text{ holds for all } \rho>0.$$

So there exists $(x_1,u_1)\in Graph(M(f,g))$ such that

$$H(Ax_1,Bx_1)+\rho u_1 = H(Ax_0,Bx_0)+\rho u_0 \in X. \tag{2.2}$$

Now,

$$\rho u_0-\rho u_1 = H(Ax_1,Bx_1)-H(Ax_0,Bx_0)\in X,$$

which implies

$$\begin{aligned}&\langle \rho u_0-\rho u_1, J_q(x_0-x_1)\rangle\\ &\quad = -\left\langle H(Ax_0,Bx_0)-H(Ax_1,Bx_1), J_q(x_0-x_1)\right\rangle.\end{aligned}$$

Since M is $\omega\tau-$symmetric accretive regarding f and g, we obtain

$$\begin{aligned}(\omega-\tau)\|x_0-x_1\|^q &\leq \rho\left\langle u_0-u_1, J_q(x_0-x_1)\right\rangle\\ &= -\left\langle H(Ax_0,Bx_0)-H(Ax_1,Bx_1), J_q(x_0-x_1)\right\rangle\\ &= -\left\langle H(Ax_0,Bx_0)-H(Ax_1,Bx_0), J_q(x_0-x_1)\right\rangle\\ &\quad -\left\langle H(Ax_1,Bx_0)-H(Ax_1,Bx_1), J_q(x_0-x_1)\right\rangle.\end{aligned} \tag{2.3}$$

Since $H(A,B)$ is μ-cocoercive with respect to A, γ-relaxed accretive with respect to B, A is α-expansive, (2.3) implies

$$\begin{aligned}(\omega-\tau)\|x_0-x_1\|^q &\leq -\mu\|Ax_0-Ax_1\|^q+\gamma\|x_0-x_1\|^q\\ &\leq -\mu\alpha^q\|x_0-x_1\|^q+\gamma\|x_0-x_1\|^q\\ &\leq -(\mu\alpha^q-\gamma)\|x_0-x_1\|^q\\ &= -r\|x_0-x_1\|^q\leq 0, \quad r=(\mu\alpha^q-\gamma)\\ &\leq -(r-(\omega-\tau))\|x_0-x_1\|^q\end{aligned}$$

$$\leq 0.$$

This implies that $x_0 = x_1$. Since $r = (\mu\alpha^q - \gamma) > (\omega - \tau)$, we have $u_0 = u_1$, a contradiction. This completes the proof. □

Theorem 2.12. *Let $M : X \times X \to 2^X$ be an $H(.,.)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho(\omega - \tau)$, then $(H(A,B) + \rho M(f,g))^{-1}$ is single-valued.*

Proof. For any given $x \in X$, let $u, v \in H((A,B) + \rho M(f,g))^{-1}(x)$. It follows that

$$\begin{aligned} &- H(Au,Bu) + x \in \rho M(f,g)u, \\ &- H(Av,Bv) + x \in \rho M(f,g)v. \end{aligned}$$

Since M is $\omega\tau-$symmetric accretive rergarding f and g, we have

$$(\omega - \tau)\|u - v\|^q \leq \frac{1}{\rho} \left\langle -H(Au,Bu) + x - (-H(Av,Bv) + x), J_q(u - v)\right\rangle,$$

which implies

$$\begin{aligned} \rho(\omega - \tau)\|u - v\|^q &\leq \left\langle -H(Au,Bu) + x - (-H(Av,Bv) + x), J_q(u - v)\right\rangle \\ &= - \left\langle H(Au,Bu) - H(Av,Bv), J_q(u - v)\right\rangle \\ &= - \left\langle H(Au,Bu) - H(Av,Bu), J_q(u - v)\right\rangle \\ &\quad - \left\langle H(Av,Bu) - H(Av,Bv), J_q(u - v)\right\rangle, \end{aligned}$$

which is like (2.3). Hence, it follows that $\|u - v\| \leq 0$. This implies that $u = v$ and therefore $(H(A,B) + \rho M(f,g))^{-1}$ is single-valued. □

Definition 2.13. Let $M : X \times X \to 2^X$ be an $H(.,.)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho(\omega - \tau)$, then the proximal-point mapping is defined by

$$R^{H(.,.)}_{\rho,M(.,.)}(u) = (H(A,B) + \rho M(f,g))^{-1}(u), \quad \forall u \in X.$$

Now, we prove that the proximal-point mapping defined above is Lipschitz continuous.

Theorem 2.14. *Let $M : X \times X \to 2^X$ be an $H(.,.)$-mixed mapping with respect to mappings A and B. If A is α-expansive and $\mu > \gamma$ with $r = \mu\alpha^q - \gamma > \rho(\omega - \tau)$, then the proximal-point mapping $R^{H(.,.)}_{\rho,M(.,.)} : X \to X$ is $\frac{1}{r + \rho(\omega - \tau)}$-Lipschitz continuous, that is,*

$$\left\| R^{H(.,.)}_{\rho,M(.,.)}(u) - R^{H(.,.)}_{\rho,M(.,.)}(v) \right\| \leq \frac{1}{r + \rho(\omega - \tau)} \|u - v\|, \quad \forall u, v \in X,$$

or

$$\left\| R^{H(.,.)}_{\rho,M(.,.)}(u) - R^{H(.,.)}_{\rho,M(.,.)}(v) \right\| \leq L\|u - v\|, \quad \forall u, v \in X,$$

where

$$L = \frac{1}{r + \rho(\omega - \tau)}.$$

Proof. For the given points $u, v \in X$, it follows from Definition 2.13 that

$$\begin{aligned} R^{H(.,.)}_{\rho,M(.,.)}(u) &= (H(A,B) + \rho M(f,g))^{-1}(u), \\ R^{H(.,.)}_{\rho,M(.,.)}(v) &= (H(A,B) + \rho M(f,g))^{-1}(v). \end{aligned}$$

Let $w_1 = R^{H(.,.)}_{\rho,M(.,.)}(u)$ and $w_2 = R^{H(.,.)}_{\rho,M(.,.)}(v)$. This implies

$$\begin{aligned} &\frac{1}{\rho}\left(u - H(A(w_1), B(w_1))\right) \in M(f(w_1), g(w_1)), \\ &\frac{1}{\rho}\left(v - H(A(w_2), B(w_2))\right) \in M(f(w_2), g(w_2)). \end{aligned}$$

Since M is $\omega\tau-$symmetric accretive regarding f and g, we have

$$\begin{aligned} &\frac{1}{\rho}\left\langle \left(u - H(A(w_1), B(w_1))\right) - \left(v - H(A(w_2), B(w_2))\right), J_q(w_1 - w_2)\right\rangle \\ &\qquad \geq (\omega - \tau)\|w_1 - w_2\|^q, \\ &\frac{1}{\rho}\left\langle u - v - H(A(w_1), B(w_1)) + H(A(w_2), B(w_2)), J_q(w_1 - w_2)\right\rangle \\ &\qquad \geq (\omega - \tau)\|w_1 - w_2\|^q, \end{aligned}$$

which implies

$$\begin{aligned} &\langle u - v, J_q(w_1 - w_2)\rangle \\ &\quad \geq \langle H(A(w_1), B(w_1)) - H(A(w_2), B(w_2)), J_q(w_1 - w_2)\rangle \\ &\qquad + \rho(\omega - \tau)\|w_1 - w_2\|^q. \end{aligned}$$

Now, we have

$$\begin{aligned} &\|u - v\| \; \|w_1 - w_2\|^{q-1} \\ &\quad \geq \langle u - v, J_q(w_1 - w_2)\rangle \\ &\quad \geq \langle H(A(w_1), B(w_1)) - H(A(w_2), B(w_2)), J_q(w_1 - w_2)\rangle \\ &\qquad + \rho(\omega - \tau)\|w_1 - w_2\|^q \\ &\quad = \langle H(A(w_1), B(w_1)) - H(A(w_2), B(w_1)), J_q(w_1 - w_2)\rangle \\ &\qquad + \langle H(A(w_2), B(w_1)) - H(A(w_2), B(w_2)), J_q(w_1 - w_2)\rangle \\ &\qquad + \rho(\omega - \tau)\|w_1 - w_2\|^q. \end{aligned}$$

This implies

$$\begin{aligned} \|u - v\|\|w_1 - w_2\|^{q-1} \geq \mu\|A(w_1) - A(w_2)\|^q - \gamma\|w_1 - w_2\|^q \\ + \rho(\omega - \tau)\|w_1 - w_2\|^q \end{aligned}$$

$$\geq ((\mu\alpha^q - \gamma) + \rho(\omega - \tau)) \|w_1 - w_2\|^q$$
$$\geq (r + \rho(\omega - \tau)) \|w_1 - w_2\|^q$$

which implies

$$\|u - v\| \; \|w_1 - w_2\|^{q-1} \geq (r + \rho(\omega - \tau)) \|w_1 - w_2\|^q$$

which implies

$$\left\| R^{H(.,.)}_{\rho,M(.,.)}(u) - R^{H(.,.)}_{\rho,M(.,.)}(v) \right\| \leq \frac{1}{r + \rho(\omega - \tau)} \|u - v\|.$$

This completes the proof. □

Now, we formulate our main problem.

Let $\Omega \subset X$ be a nonempty open subset of X in which the parameter ϖ takes the values. Let for each $i = 1, 2$, $G_i : \Omega \times X \to X, F_i, f_i, g_i, A_i, B_i : X \to X$ and $H_i : X \times X \to X$ be single-valued mappings. Let $M_i : X \times X \to 2^X$ be $H_i(.,.)$-mixed mappings with respect to mappings A_i and B_i. We consider the following system of generalized mixed variational inclusion problem (in brief, SGMVIP): For given $\theta_i, \varpi_i \in X$, find $(x_1, x_2) \in X \times X$ such that

$$\begin{aligned} \theta_1 \in\, & H_1(A_1, B_1)(x_1) - H_2(A_2, B_2)(x_2) \\ & + \lambda_1\{F_1(x_2) + G_1(\varpi_1, x_2)\} + \lambda_1 M_1(f_1, g_1)(x_1) \\ \theta_2 \in\, & H_2(A_2, B_2)(x_2) - H_1(A_1, B_1)(x_1) \\ & + \lambda_2\{F_2(x_1) + G_2(\varpi_2, x_1)\} + \lambda_2 M_2(f_2, g_2)(x_2). \end{aligned} \tag{2.4}$$

Special Cases:

I. If in problem (2.4), $\theta_1 = \theta_2 = 0, H_1(A_1, B_1) = H_2(A_2, B_2) = I$ (identity mapping), $G_1 = G_2 \equiv 0$, then problem (2.4) reduces to the following problem: Find $(x_1, x_2) \in X \times X$ such that

$$\begin{aligned} & 0 \in x_1 - x_2 + \lambda_1\{F_1(x_2) + \lambda_1 M_1(f_1, g_1)(x_1)\} \\ & 0 \in x_2 - x_1 + \lambda_2\{F_2(x_1) + \lambda_2 M_2(f_2, g_2)(x_2)\}. \end{aligned} \tag{2.5}$$

This type of problem (2.5) has been considered and studied by Ceng *et al.*[5].

II. If in problem (2.4), $H_1(A_1, B_1) = H_2(A_2, B_2) = I$ (identity mapping), $G_1 = G_2 \equiv 0, M_1(f_1, g_1)(x_1) = M_2(f_2, g_2)(x_2) = \partial\delta_C(u), \quad \forall u \in X$, C is a nonempty closed and convex set in X and δ_C denotes the indicator function of closed convex set, C, i.e.,

$$\begin{aligned} \delta_C(g(u)) = 0; \quad & u \in C, \\ +\infty \quad & u \notin C. \end{aligned}$$

Then problem (2.4) reduces to the following problem:
Find $(x_1, x_2) \in C \times C$ such that

$$\begin{aligned} &\langle x_1 - x_2 + \lambda_1 F_1(x_2), u - x_1 \rangle \geq 0, \quad \forall u \in C, \\ &\langle x_2 - x_1 + \lambda_2 F_2(x_1), u - x_2 \rangle \geq 0, \quad \forall u \in C. \end{aligned} \tag{2.6}$$

This type of problem (2.6) has been considered and studied by Ceng and Shang [4].

3. Existence of Solution

First, we give the following lemma which guarentees the existence of solution of SGMVIP (2.4).

Lemma 3.1. *Let for each* $i = 1, 2$, $G_i, F_i, f_i, g_i, A_i, B_i, H_i$ *and* M_i *be same as in problem SGMVIP* (2.4). *Then* (x_1, x_2) *is a solution of SGMVIP* (2.4), *where* $(x_1, x_2) \in X \times X$ *if and only if it satisfies:*

$$\begin{aligned} x_1 &= R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \left\{ H_2(A_2, B_2)(x_2) - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) \right\}, \\ x_2 &= R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \left\{ H_1(A_1, B_1)(x_1) - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \right\}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} &= \left(H_1(A_1, B_1) + \lambda_1 M_1(f_1, g_1) \right)^{-1}, \\ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} &= \left(H_2(A_2, B_2) + \lambda_2 M_2(f_2, g_2) \right)^{-1}, \end{aligned}$$

and $\lambda_1, \lambda_2 > 0$ *are constants.*

Proof. Let

$$x_1 = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \left\{ H_2(A_2, B_2)(x_2) - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) \right\},$$

if and only if

$$\begin{aligned} x_1 = &\left(H_1(A_1, B_1) + \lambda_1 M_1(f_1, g_1) \right)^{-1} \\ &\left\{ H_2(A_2, B_2)(x_2) - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) \right\}, \end{aligned}$$

if and only if

$$\begin{aligned} &H_1(A_1, B_1)(x_1) + \lambda_1 M_1(f_1, g_1)(x_1) \\ &\quad = H_2(A_2, B_2)(x_2) - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right), \end{aligned}$$

if and only if

$$\begin{aligned} \theta_1 \in &H_1(A_1, B_1)(x_1) - H_2(A_2, B_2)(x_2) \\ &+ \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) + \lambda_1 M_1(f_1, g_1)(x_1). \end{aligned}$$

Similarly,

$$x_2 = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \left\{ H_1(A_1, B_1)(x_1) - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \right\},$$

if and only if

$$\begin{aligned}\theta_2 \in H_2(A_2,B_2)(x_2) &- H_1(A_1,B_1)(x_1)\\ &+ \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right) + \lambda_2 M_2(f_2,g_2)(x_2). \qquad \square\end{aligned}$$

Theorem 3.2. *Let X be a real q-uniformly smooth Banach space. Let for each $i = 1,2,\ j \in \{1,2\}\backslash i\ G_i : \Omega \times X \to X, F_i, f_i, g_i, A_i, B_i : X \to X$ and $H_i : X \times X \to X$ be single-valued mappings. Let $M_i : X \times X \to 2^X$ be $H_i(.,.)$-mixed mappings with respect to mappings A_i and B_i. Let $H_i(A_i,B_i)$ be s_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i, F_i be L_{F_i}-Lipschitz continuous and $\hbar_i$-strongly monotone with respect to $H_j(A_j,B_j)$. Further, suppose that G_i be $L_{G_{i_2}}$-Lipschitz continuous in the second argument and ξ_i-strongly monotone with respect to $H_j(A_j,B_j)$ in the second argument. In addition, assume that*

$$0 < \Gamma_1, \qquad \Psi_1 < 1, \tag{3.2}$$

where

$$\begin{aligned}\Gamma_1 &= L_1(\Delta_1 + \lambda_1 L_{G_{1_2}}),\\ \Delta_1 &= \left((s_2+t_2)^q + c_q\lambda_1^q L_{F_1}^q - q\lambda_1\hbar_1\right)^{\frac{1}{q}},\\ \Psi_1 &= L_2\left[\left((s_1+t_1)^q + c_q\lambda_2^q L_{G_{2_2}}^q - q\lambda_2\xi_2\right)^{\frac{1}{q}} + \lambda_2 L_{F_2}\right].\end{aligned}$$

Then SGMVIP (2.4) *has a solution.*

Proof. We first prove the existence of a solution. Define a mapping $K : X \to X$ by

$$\begin{aligned}&K(x_1)\\ &\quad= R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}\Big[H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big[H_1(A_1,B_1)(x_1)\\ &\qquad - \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\Big]\Big\}\\ &\qquad - \lambda_1\Big\{F_1\left\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1) - \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\right\}\\ &\qquad + G_1\left(\varpi_1, R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1) - \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\right)\Big\}\Big].\end{aligned}$$

From Lemma 3.1, for all $x_1, y_1 \in X$, it follows that

$$\begin{aligned}&\|K(x_1)-K(y_1)\|\\ &\quad= \Big\|R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}\Big[H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big[H_1(A_1,B_1)(x_1)\\ &\qquad - \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\Big]\Big\}\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&- \lambda_1 \Big\{ F_1 \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(x_1) \\
&- \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \Big] \Big\} \\
&+ G_1 \Big(\varpi_1, R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(x_1) \\
&- \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \Big] \Big) \Big\} \Big] \\
&- R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \Big[H_2(A_2, B_2) \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(y_1) \\
&- \lambda_2 \left(F_2(y_1) + G_2(\varpi_2, y_1) \right) \Big] \Big\} \\
&- \lambda_1 \Big\{ F_1 \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(y_1) \\
&- \lambda_2 \left(F_2(y_1) + G_2(\varpi_2, y_1) \right) \Big] \Big\} \\
&+ G_1 \Big(\varpi_1, R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(y_1) \\
&- \lambda_2 \left(F_2(y_1) + G_2(\varpi_2, y_1) \right) \Big] \Big) \Big\} \Big] \Big\| .
\end{aligned}
$$

Using Theorem 2.14, we have

$$
\begin{aligned}
&\|K(x_1) - K(y_1)\| \\
&\le L_1 \Big\| H_2(A_2, B_2) \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(x_1) \\
&\quad - \lambda_2 \Big(F_2(x_1) + G_2(\varpi_2, x_1) \Big) \Big] \Big\} \\
&\quad - H_2(A_2, B_2) \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(y_1) \\
&\quad - \lambda_2 \Big(F_2(y_1) + G_2(\varpi_2, y_1) \Big) \Big] \Big\} \\
&\quad - \lambda_1 \Big\{ F_1 \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(x_1) \\
&\quad - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \Big] \Big\} \\
&\quad - F_1 \Big\{ R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(y_1) \\
&\quad - \lambda_2 \left(F_2(y_1) + G_2(\varpi_2, y_1) \right) \Big] \Big\} \Big\} \Big\| \\
&\quad + L_1 \lambda_1 \Big\| G_1 \Big(\varpi_1, R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(x_1) \\
&\quad - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \Big] \Big) \\
&\quad - G_1 \Big(\varpi_1, R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \Big[H_1(A_1, B_1)(y_1) \\
&\quad - \lambda_2 \left(F_2(y_1) + G_2(\varpi_2, y_1) \right) \Big] \Big) \Big\| .
\end{aligned}
\tag{3.4}
$$

Since $H_2(A_2,B_2)$ is s_2-Lipschitz continuous with respect to A_2 and t_2-Lipschitz continuous with respect to B_2, F_1 is L_{F_1}-Lipschitz continuous and $\hbar_1$-strongly monotone with respect to $H_2(A_2,B_2)$ and from Lemma 2.4, it follows that

$$\begin{aligned}
&\Big\| H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\
&\quad - H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\\
&\quad - \lambda_1\Big\{F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\
&\quad - F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\Big\}\Big\|^q\\
&\quad \leq \Big\|H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big[H_1(A_1,B_1)(x_1)\\
&\qquad - \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\Big]\Big\}\\
&\qquad - H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big[H_1(A_1,B_1)(y_1)\\
&\qquad - \lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\Big]\Big\}\Big\|^q\\
&\qquad + c_q\lambda_1^q\Big\|F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\
&\qquad - F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\Big\|^q\\
&\qquad - q\lambda_1\Big\langle F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\
&\qquad - F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\},\\
&\qquad J_q\Big(H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big[H_1(A_1,B_1)(x_1)\\
&\qquad - \lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\Big]\Big\}\\
&\qquad - H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big[H_1(A_1,B_1)(y_1)\\
&\qquad - \lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\Big]\Big\}\Big)\Big\rangle\\
&\quad \leq (s_2+t_2)^q\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\
&\qquad - R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|^q\\
&\qquad + c_q\lambda_1^qL_{F_1}^q\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\
&\qquad - R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|^q\\
&\qquad - q\lambda_1\hbar_1\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]
\end{aligned}$$

$$\begin{aligned}
&- R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|^q\\
&\leq \Big((s_2+t_2)^q+c_q\lambda_1^q L_{F_1}^q-q\lambda_1\hbar_1\Big)\\
&\quad\times\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\
&\quad- R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|^q.
\end{aligned}$$

This implies

(3.5)
$$\begin{aligned}
&\Big\|H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\
&\quad- H_2(A_2,B_2)\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\\
&\quad- \lambda_1\Big\{F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\
&\quad- F_1\Big\{R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\Big\}\Big\|\\
&\quad\leq \Big((s_2+t_2)^q+c_q\lambda_1^q L_{F_1}^q-q\lambda_1\hbar_1\Big)^{\frac{1}{q}}\\
&\qquad\times\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\
&\qquad- R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|.
\end{aligned}$$

Again using Theorem 2.14, we have

(3.6)
$$\begin{aligned}
&\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\
&\qquad- R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|\\
&\quad\leq L_2\Big\|\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\
&\qquad- \left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|\\
&\quad\leq L_2\Big\|H_1(A_1,B_1)(x_1)-H_1(A_1,B_1)(y_1)\\
&\qquad- \lambda_2\left(G_2(\varpi_2,x_1)-G_2(\varpi_2,y_1)\right)\Big\|+L_2\lambda_2\left\|F_2(x_1)-F_2(y_1)\right\|.
\end{aligned}$$

Since $H_1(A_1,B_1)$ is s_1-Lipschitz continuous with respect to A_1 and t_1-Lipschitz continuous with respect to B_1, G_2 is $L_{G_{2_2}}$-Lipschitz continuous in the second argument, ξ_2-strongly monotone with respect to $H_1(A_1,B_1)$ in the second argument and using Lemma 2.4, it follows that

$$\begin{aligned}&\|H_1(A_1,B_1)(x_1)-H_1(A_1,B_1)(y_1)-\lambda_2\left(G_2(\varpi_2,x_1)-G_2(\varpi_2,y_1)\right)\|^q\\ &\quad\leq \|H_1(A_1,B_1)(x_1)-H_1(A_1,B_1)(y_1)\|^q\\ &\qquad + c_q\lambda_2^q\|G_2(\varpi_2,x_1)-G_2(\varpi_2,y_1)\|^q\\ &\qquad - q\lambda_2\Big\langle G_2(\varpi_2,x_1)-G_2(\varpi_2,y_1),\\ &\qquad J_q\left(H_1(A_1,B_1)(x_1)-H_1(A_1,B_1)(y_1)\right)\Big\rangle\\ &\quad\leq \Big((s_1+t_1)^q+c_q\lambda_2^qL_{G_{2_2}}^q-q\lambda_2\xi_2\Big)\|x_1-y_1\|^q.\end{aligned}$$

This implies

(3.7)

$$\begin{aligned}&\|H_1(A_1,B_1)(x_1)-H_1(A_1,B_1)(y_1)-\lambda_2\left(G_2(\varpi_2,x_1)-G_2(\varpi_2,y_1)\right)\|\\ &\quad\leq \Big((s_1+t_1)^q+c_q\lambda_2^qL_{G_{2_2}}^q-q\lambda_2\xi_2\Big)^{\frac{1}{q}}\ \|x_1-y_1\|.\end{aligned}$$

Also by L_{F_2}-Lipschitz continuity of F_2, we have

$$\|F_2(x_1)-F_2(y_1)\|\leq L_{F_2}\ \|x_1-y_1\|. \tag{3.8}$$

From (3.6),(3.7) and (3.8), it follows that

$$\begin{aligned}&\Big\|R_{\lambda_2,M_2(.,.)}^{H_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\\ &\qquad - R_{\lambda_2,M_2(.,.)}^{H_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\|\\ &\qquad\leq L_2\left[\Big((s_1+t_1)^q+c_q\lambda_2^qL_{G_{2_2}}^q-q\lambda_2\xi_2\Big)^{\frac{1}{q}}+\lambda_2L_{F_2}\right]\|x_1-y_1\|.\end{aligned} \tag{3.9}$$

From (3.5) and (3.9), it follows that

(3.10)

$$\begin{aligned}&\Big\|H_2(A_2,B_2)\Big\{R_{\lambda_2,M_2(.,.)}^{H_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\ &\quad - H_2(A_2,B_2)\Big\{R_{\lambda_2,M_2(.,.)}^{H_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\\ &\quad - \lambda_1\Big\{F_1\Big\{R_{\lambda_2,M_2(.,.)}^{H_2(.,.)}\left[H_1(A_1,B_1)(x_1)-\lambda_2\left(F_2(x_1)+G_2(\varpi_2,x_1)\right)\right]\Big\}\\ &\quad - F_1\Big\{R_{\lambda_2,M_2(.,.)}^{H_2(.,.)}\left[H_1(A_1,B_1)(y_1)-\lambda_2\left(F_2(y_1)+G_2(\varpi_2,y_1)\right)\right]\Big\}\Big\}\Big\|\\ &\leq \Big((s_2+t_2)^q+c_q\lambda_1^qL_{F_1}^q-q\lambda_1\hbar_1\Big)^{\frac{1}{q}}\\ &\quad\times L_2\left[\Big((s_1+t_1)^q+c_q\lambda_2^qL_{G_{2_2}}^q-q\lambda_2\xi_2\Big)^{\frac{1}{q}}+\lambda_2L_{F_2}\right]\|x_1-y_1\|\end{aligned}$$

$\leq \Delta_1\Psi_1 \;\|x_1 - y_1\|.$

Again using $L_{G_{1_2}}$-Lipschitz continuity of G_1 in the second argument and (3.9), we have

(3.11)

$$\begin{aligned}&\Big\|G_1\left(\varpi_1, R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1) - \lambda_2\left(F_2(x_1) + G_2(\varpi_2,x_1)\right)\right]\right)\\&\quad - G_1\left(\varpi_1, R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1) - \lambda_2\left(F_2(y_1) + G_2(\varpi_2,y_1)\right)\right]\right)\Big\|\\&\quad\leq L_{G_{1_2}}\Big\|R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(x_1) - \lambda_2\left(F_2(x_1) + G_2(\varpi_2,x_1)\right)\right]\\&\qquad - R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left[H_1(A_1,B_1)(y_1) - \lambda_2\left(F_2(y_1) + G_2(\varpi_2,y_1)\right)\right]\Big\|\\&\quad\leq L_{G_{1_2}}L_2\left[\left((s_1+t_1)^q + c_q\lambda_2^q L^q_{G_{2_2}} - q\lambda_2\xi_2\right)^{\frac{1}{q}} + \lambda_2 L_{F_2}\right]\|x_1 - y_1\|.\end{aligned}$$

Combining (3.4)-(3.11), it follows that

$$\begin{aligned}&\|K(x_1) - K(y_1)\|\\&\quad\leq L_1(\Delta_1 + \lambda_1 L_{G_{1_2}})\Psi_1 \;\|x_1 - y_1\|\\&\quad\leq \Gamma_1\Psi_1 \;\|x_1 - y_1\|.\end{aligned}$$

where $\Gamma_1 = L_1(\Delta_1 + \lambda_1 L_{G_{1_2}})$. Since $0 < \Gamma_1, \Psi_1 < 1$, from (3.2), it follows that K is a contractive mapping. Therefore, there exists $x_1 \in X$ such that $K(x_1) = x_1$. Let

$$x_2 = R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left\{H_1(A_1,B_1)(x_1) - \lambda_2\left(F_2(x_1) + G_2(\varpi_2,x_1)\right)\right\}.$$

Therefore, from the definition of K, we have

$$\begin{aligned}x_1 &= R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}\left\{H_2(A_2,B_2)(x_2) - \lambda_1\left(F_1(x_2) + G_1(\varpi_1,x_2)\right)\right\},\\x_2 &= R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\left\{H_1(A_1,B_1)(x_1) - \lambda_2\left(F_2(x_1) + G_2(\varpi_2,x_1)\right)\right\}.\end{aligned}$$

Thus it follows from Lemma 3.1 that (x_1, x_2) is a solution of (2.4). ☐

4. ALGORITHM AND CONVERGENCE ANALYSIS

Now, we discuss the following two-step iterative algorithm which contains a number of iterative algorithms as special cases for finding the approximate solution of SGMVIP (2.4).

Iterative Algorithm 4.1. For arbitrarily chosen initial point $x_1^0 \in X$, compute the sequences $\{x_1^n\}, \{x_2^n\}$ such that

$$\begin{aligned}x_1^{n+1} = (1 - \beta^n - \delta^n)x_1^n + \beta^n R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}\Big\{&H_2(A_2,B_2)(x_2^n)\\&- \lambda_1\left(F_1(x_2^n) + G_1(\varpi_1,x_2^n)\right)\Big\} + \delta^n z_1^n,\end{aligned}$$

$$
\begin{aligned}
x_2^n = (1-\sigma^n-\nu^n)x_1^n + \sigma^n R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big\{&H_1(A_1,B_1)(x_1^n)\\
&- \lambda_2\left(F_2(x_1^n)+G_2(\varpi_2,x_1^n)\right)\Big\} + \nu^n z_2^n,
\end{aligned}
$$

where $\{\beta^n\},\{\delta^n\},\{\sigma^n\},\{\nu^n\}\subset[0,1],\{z_1^n\},\{z_2^n\}$ are bounded sequences in X, $0\le\beta^n+\delta^n\le1, 0\le\sigma^n+\nu^n\le1,\quad$ for all $n\ge0$.

If $F_1=F_2=0$, then the Iterative Algorithm 4.1 reduces to the following algorithm.

Iterative Algorithm 4.2. For arbitrarily chosen initial point $x_1^0\in X$, compute the sequences $\{x_1^n\},\{x_2^n\}$ such that

$$
\begin{aligned}
x_1^{n+1} &= (1-\beta^n-\delta^n)x_1^n + \beta^n R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}\Big\{H_2(A_2,B_2)(x_2^n)\\
&\quad - \lambda_1 G_1(\varpi_1,x_2^n)\Big\} + \delta^n z_1^n,\\
x_2^n &= (1-\sigma^n-\nu^n)x_1^n + \sigma^n R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big\{H_1(A_1,B_1)(x_1^n)\\
&\quad - \lambda_2 G_2(\varpi_2,x_1^n)\Big\} + \nu^n z_2^n,
\end{aligned}
$$

where $\{\beta^n\},\{\delta^n\},\{\sigma^n\},\{\nu^n\}\subset[0,1],\{z_1^n\},\{z_2^n\}$ are bounded sequences in X, $0\le\beta^n+\delta^n\le1, 0\le\sigma^n+\nu^n\le1,\quad$ for all $n\ge0$.

If $\delta^n=0,\nu^n=0$, then the Iterative Algorithm 4.1 reduces to the following algorithm.

Iterative Algorithm 4.3. For arbitrarily chosen initial point $x_1^0\in X$, compute the sequences $\{x_1^n\},\{x_2^n\}$ such that

$$
\begin{aligned}
x_1^{n+1} &= (1-\beta^n)x_1^n + \beta^n R^{H_1(.,.)}_{\lambda_1,M_1(.,.)}\Big\{H_2(A_2,B_2)(x_2^n)\\
&\quad - \lambda_1\left(F_1(x_2^n)+G_1(\varpi_1,x_2^n)\right)\Big\}\\
x_2^n &= (1-\sigma^n)x_1^n + \sigma^n R^{H_2(.,.)}_{\lambda_2,M_2(.,.)}\Big\{H_1(A_1,B_1)(x_1^n)\\
&\quad - \lambda_2\left(F_2(x_1^n)+G_2(\varpi_2,x_1^n)\right)\Big\},
\end{aligned}
$$

where $\{\beta^n\},\{\sigma^n\}\subset[0,1],\quad\forall n\ge0$.

Now, we give the convergence analysis of the sequences generated by the Iterative Algorithm 4.1.

Theorem 4.4. *Let X be a real q-uniformly smooth Banach space. Let for each $i=1,2,\ j\in\{1,2\}\backslash i$ $G_i:\Omega\times X\to X, F_i,f_i,g_i,A_i,B_i:X\to X$ and $H_i:X\times X\to X$ be single-valued mappings. Let $M_i:X\times X\to2^X$ be $H_i(.,.)$-mixed mappings with respect to mappings A_i and B_i. Let $H_i(A_i,B_i)$ be s_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz* *continuous with respect to B_i, F_i be L_{F_i}-Lipschitz continuous and $\hbar_i$-strongly monotone with respect to $H_j(A_j, B_j)$. Further, suppose that G_i be $L_{G_{i_2}}$-Lipschitz continuous in the second argument and ξ_i-strongly monotone with respect to $H_j(A_j, B_j)$ in the second argument. Suppose the sequences $\{x_1^n\}, \{x_2^n\}$ generated by above Iterative Algorithm 4.1 and satisfies*

$$\sum_{n=0}^{\infty} \beta^n = \infty, \qquad \sum_{n=0}^{\infty} \delta^n < \infty, \quad \sigma^n \to 1,$$

and $0 < \Psi_1, \Psi_2 < 1$

where

$$\Psi_1 = L_2 \left[\left((s_1 + t_1)^q + c_q \lambda_2^q L_{G_{2_2}}^q - q\lambda_2 \xi_2 \right)^{\frac{1}{q}} + \lambda_2 L_{F_2} \right],$$
$$\Psi_2 = L_1 \left[\left((s_2 + t_2)^q + c_q \lambda_1^q L_{G_{1_2}}^q - q\lambda_1 \xi_1 \right)^{\frac{1}{q}} + \lambda_1 L_{F_1} \right].$$

Then the sequences $\{x_1^n\}, \{x_2^n\}$ generated by above Iterative Algorithm 4.1 converge strongly to x_1, x_2 where x_1, x_2 are solutions of SGMVIP (2.4)*.*

Proof. Since $(x_1, x_2) \in X \times X$ is a solution of SGMVIP (2.4), from Lemma 3.1, we have

$$x_1 = R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \left\{ H_2(A_2, B_2)(x_2) - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) \right\},$$
$$x_2 = R^{H_2(.,.)}_{\lambda_2, M_2(.,.)} \left\{ H_1(A_1, B_1)(x_1) - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \right\}.$$

Let $P = \sup_{n \geq 0} \left\{ \sup_{n \geq 0} \|z_1^n - x_1\|, \sup_{n \geq 0} \|z_2^n - x_2\|, \|x_1 - x_2\| \right\}.$

Using Iterative Algorithm 4.1, Lemma 3.1 and Theorem 2.14, we have

(4.1)

$$\begin{aligned}
&\|x_1^{n+1} - x_1\| \\
&\quad = \Big\| \Big\{ (1 - \beta^n - \delta^n) x_1^n + \beta^n R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \Big\{ H_2(A_2, B_2)(x_2^n) \\
&\qquad - \lambda_1 \left(F_1(x_2^n) + G_1(\varpi_1, x_2^n) \right) \Big\} + \delta^n z_1^n \Big\} \\
&\qquad - \Big\{ (1 - \beta^n - \delta^n) x_1 + \beta^n R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \Big\{ H_2(A_2, B_2)(x_2) \\
&\qquad - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) \Big\} + \delta^n x_1 \Big\} \Big\| \\
&\quad \leq \Big\| (1 - \beta^n - \delta^n)(x_1^n - x_1) + \beta^n \Big\{ R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \Big\{ H_2(A_2, B_2)(x_2^n) \\
&\qquad - \lambda_1 \left(F_1(x_2^n) + G_1(\varpi_1, x_2^n) \right) \Big\} - R^{H_1(.,.)}_{\lambda_1, M_1(.,.)} \Big\{ H_2(A_2, B_2)(x_2) \\
&\qquad - \lambda_1 \left(F_1(x_2) + G_1(\varpi_1, x_2) \right) \Big\} \Big\} + \delta^n (z_1^n - x_1) \Big\|
\end{aligned}$$

$$\begin{aligned}&\leq (1-\beta^n-\delta^n)\|x_1^n-x_1\|+\beta^n L_1\Big\|H_2(A_2,B_2)(x_2^n)-H_2(A_2,B_2)(x_2)\\&\quad-\lambda_1\left(G_1(\varpi_1,x_2^n)-G_1(\varpi_1,x_2)\right)\Big\|+\beta^n L_1\lambda_1\left\|F_1(x_2^n)-F_1(x_2)\right\|\\&\quad+\delta^n\|z_1^n-x_1\|.\end{aligned}$$

Since G_1 is $L_{G_{1_2}}$-Lipschitz continuous in the second argument and ξ_1-strongly monotone in the second argument with respect to $H_2(A_2, B_2)$, $H_2(A_2, B_2)$ is s_2-Lipschitz continuous with respect to A_2 and t_2-Lipschitz continuous with respect to B_2, using Lemma 2.4, it follows that

(4.2)

$$\begin{aligned}&\left\|H_2(A_2,B_2)(x_2^n)-H_2(A_2,B_2)(x_2)-\lambda_1\left(G_1(\varpi_1,x_2^n)-G_1(\varpi_1,x_2)\right)\right\|^q\\&\leq\left\|H_2(A_2,B_2)(x_2^n)-H_2(A_2,B_2)(x_2)\right\|^q\\&\quad+c_q\lambda_1^q\left\|G_1(\varpi_1,x_2^n)-G_1(\varpi_1,x_2)\right\|^q-q\lambda_1\Big\langle G_1(\varpi_1,x_2^n)-G_1(\varpi_1,x_2),\\&\quad J_q\left(H_2(A_2,B_2)(x_2^n)-H_2(A_2,B_2)(x_2)\right)\Big\rangle\\&\leq\Big((s_2+t_2)^q+c_q\lambda_1^qL_{G_{1_2}}^q-q\lambda_1\xi_1\Big)\left\|x_2^n-x_2\right\|^q,\end{aligned}$$

then

$$\begin{aligned}&\left\|H_2(A_2,B_2)(x_2^n)-H_2(A_2,B_2)(x_2)-\lambda_1\left(G_1(\varpi_1,x_2^n)-G_1(\varpi_1,x_2)\right)\right\|\\&\quad\leq\Big((s_2+t_2)^q+c_q\lambda_1^qL_{G_{1_2}}^q-q\lambda_1\xi_1\Big)^{\frac{1}{q}}\ \|x_2^n-x_2\|.\end{aligned}$$

Also, since F_1 is L_{F_1}-Lipschitz continuous, we have

$$\|F_1(x_2^n)-F_1(x_2)\|\leq L_{F_1}\ \|x_2^n-x_2\|. \tag{4.3}$$

Combining (4.1)-(4.3), we have

(4.4)

$$\begin{aligned}&\|x_1^{n+1}-x_1\|\\&\quad\leq(1-\beta^n-\delta^n)\|x_1^n-x_1\|\\&\qquad+\beta^nL_1\left\{\Big((s_2+t_2)^q+c_q\lambda_1^qL_{G_{1_2}}^q-q\lambda_1\xi_1\Big)^{\frac{1}{q}}+\lambda_1L_{F_1}\right\}\|x_2^n-x_2\|\\&\qquad+\delta^nP,\end{aligned}$$

then

$$\|x_1^{n+1}-x_1\|\leq(1-\beta^n)\|x_1^n-x_1\|+\beta^n\Psi_2\|x_2^n-x_2\|+\delta^nP,$$

where

$$\Psi_2=L_1\left\{\Big((s_2+t_2)^q+c_q\lambda_1^qL_{G_{1_2}}^q-q\lambda_1\xi_1\Big)^{\frac{1}{q}}+\lambda_1L_{F_1}\right\}.$$

Next, consider

(4.5)

$$\begin{aligned}
&\|x_2^n - x_2\| \\
&\quad = \Big\| \Big\{ (1-\sigma^n-\nu^n)x_1^n + \sigma^n R^{H_2(.,.)}_{\lambda_2,M_2(.,.)} \Big\{ H_1(A_1,B_1)(x_1^n) \\
&\qquad - \lambda_2 \left(F_2(x_1^n) + G_2(\varpi_2, x_1^n) \right) \Big\} + \nu^n z_2^n \Big\} \\
&\qquad - \Big\{ (1-\sigma^n-\nu^n)x_2 + \sigma^n R^{H_2(.,.)}_{\lambda_2,M_2(.,.)} \Big\{ H_1(A_1,B_1)(x_1) \\
&\qquad - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \Big\} + \nu^n x_2 \Big\} \Big\| \\
&\quad \leq \Big\| (1-\sigma^n-\nu^n)(x_1^n - x_2) \\
&\qquad + \sigma^n \Big\{ R^{H_2(.,.)}_{\lambda_2,M_2(.,.)} \left\{ H_1(A_1,B_1)(x_1^n) - \lambda_2 \left(F_2(x_1^n) + G_2(\varpi_2, x_1^n) \right) \right\} \\
&\qquad - R^{H_2(.,.)}_{\lambda_2,M_2(.,.)} \left\{ H_1(A_1,B_1)(x_1) - \lambda_2 \left(F_2(x_1) + G_2(\varpi_2, x_1) \right) \right\} \Big\} \\
&\qquad + \nu^n (z_2^n - x_2) \Big\| \\
&\quad \leq (1-\sigma^n-\nu^n)\|x_1^n - x_2\| + \sigma^n L_2 \Big\| H_1(A_1,B_1)(x_1^n) - H_1(A_1,B_1)(x_1) \\
&\qquad - \lambda_2 \left(G_2(\varpi_2, x_1^n) - G_2(\varpi_2, x_1) \right) \Big\| \\
&\qquad + \sigma^n L_2 \lambda_2 \left\| F_2(x_1^n) - F_2(x_1) \right\| + \nu^n \|z_2^n - x_2\|.
\end{aligned}$$

Since G_2 is $L_{G_{2_2}}$-Lipschitz continuous in the second argument and ξ_2-strongly monotone in the second argument with respect to $H_1(A_1, B_1)$, $H_1(A_1, B_1)$ is s_1-Lipschitz continuous with respect to A_1 and t_1-Lipschitz continuous with respect to B_1, using Lemma 2.4, and following the same procedure as in (4.2), we have

(4.6)

$$\begin{aligned}
&\left\| H_1(A_1,B_1)(x_1^n) - H_1(A_1,B_1)(x_1) - \lambda_2 \left(G_2(\varpi_2, x_1^n) - G_2(\varpi_2, x_1) \right) \right\| \\
&\qquad \leq \Big((s_1+t_1)^q + c_q \lambda_2^q L^q_{G_{2_2}} - q\lambda_2 \xi_2 \Big)^{\frac{1}{q}} \ \|x_1^n - x_1\|.
\end{aligned}$$

Also, since F_2 is L_{F_2}-Lipschitz continuous, we have

$$\|F_2(x_1^n) - F_2(x_1)\| \leq L_{F_2} \ \|x_1^n - x_1\|. \tag{4.7}$$

Combining (4.5)-(4.7), we have

$$\begin{aligned}
&\|x_2^n - x_2\| \\
&\quad \leq (1-\sigma^n-\nu^n)\|x_1^n - x_2\|
\end{aligned}$$

$$+\sigma^n L_2\left\{\left((s_1+t_1)^q+c_q\lambda_2^q L_{G_{2_2}}^q-q\lambda_2\xi_2\right)^{\frac{1}{q}}+\lambda_2 L_{F_2}\right\}\|x_1^n-x_1\|+\nu^n P,$$

which implies

$$\|x_2^n-x_2\|\le(1-\sigma^n-\nu^n)\|x_1^n-x_2\|+\sigma^n\Psi_1\|x_1^n-x_1\|+\nu^n P, \tag{4.8}$$

where

$$\Psi_1=L_2\left\{\left((s_1+t_1)^q+c_q\lambda_2^q L_{G_{2_2}}^q-q\lambda_2\xi_2\right)^{\frac{1}{q}}+\lambda_2 L_{F_2}\right\}.$$

This implies

$$\begin{aligned}\|x_2^n-x_2\|&\le(1-\sigma^n-\nu^n)\|x_1^n-x_1\|+\sigma^n\Psi_1\|x_1^n-x_1\|\\&\quad+(1-\sigma^n-\nu^n)\|x_1-x_2\|+\nu^n P\\&\le(1-\sigma^n-\nu^n)\|x_1^n-x_1\|+\sigma^n\Psi_1\|x_1^n-x_1\|\\&\quad+(1-\sigma^n-\nu^n)P+\nu^n P\\&\le(1-\nu^n)\|x_1^n-x_1\|+(1-\sigma^n)P\\&\le\|x_1^n-x_1\|+(1-\sigma^n)P.\end{aligned}\tag{4.9}$$

Using (4.9) in (4.4), we have

$$\begin{aligned}\|x_1^{n+1}-x_1\|&\le(1-\beta^n)\|x_1^n-x_1\|+\beta^n\Psi_2(\|x_1^n-x_1\|+(1-\sigma^n)P)+\delta^n P\\&\le(1-\beta^n(1-\Psi_2))\|x_1^n-x_1\|+\beta^n(1-\sigma^n)P+\delta^n P.\end{aligned}\tag{4.10}$$

Let

$$\begin{aligned}a^n&=\|x_1^n-x_1\|, & d^n&=\beta^n(1-\Psi_2),\\ b^n&=\beta^n(1-\sigma^n)P, & c^n&=\delta^n P.\end{aligned}$$

Therefore, by Lemma 2.5, we have $a^n=\|x_1^n-x_1\|\to 0\ as\ n\to\infty$. This implies from (4.9) that $\|x_2^n-x_2\|\to 0\ as\ n\to\infty$.

This completes the proof. □

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