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Fractional Ostrowski-type Inequalities via $(\alpha, \beta, \gamma, \delta)-$convex Function

Ali Hassan[1*], Asif R. Khan[2], Nazia Irshad[3] and Sumbul Khatoon[4]

Abstract. In this paper, we are introducing for the first time a generalized class named the class of $(\alpha, \beta, \gamma, \delta)-$convex functions of mixed kind. This generalized class contains many subclasses including the class of $(\alpha, \beta)-$convex functions of the first and second kind, $(s, r)-$convex functions of mixed kind, $s-$convex functions of the first and second kind, $P-$convex functions, quasi-convex functions and the class of ordinary convex functions. In addition, we would like to state the generalization of the classical Ostrowski inequality via fractional integrals, which is obtained for functions whose first derivative in absolute values is $(\alpha, \beta, \gamma, \delta)-$ convex function of mixed kind. Moreover, we establish some Ostrowski-type inequalities via fractional integrals and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $(\alpha, \beta, \gamma, \delta)-$ convex functions of mixed kind using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, the applications of special means will also be discussed.

1. Introduction

In almost every field of science, inequalities play an important role. Although it covers a wide range of topics but our focus is mainly on Ostrowski-type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as the Ostrowski inequality.

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Theorem 1.1 ([31])**.** *Let $\varphi : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|\varphi'(t)| \le M$, $\forall t \in (a, b)$. Then*

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)dt\right| \le (b-a)M\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right], \tag{1.1}$$

$\forall x \in (a, b)$.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [1, 5, 9–11, 18, 22, 26, 27, 29, 33, 34, 36].

The importance of convex functions for the generalization of integral inequalities due to the variety of their nature is already established. Integral inequalities are satisfied by many convex functions. Among these, most famous is the classical Ostrowski inequality [31]. To generalize the Ostrowski inequality, we need to generalize the concept of convex functions. In this way, we can easily see the generalizations and their particular cases. From the literature, we recall and introduce some definitions for various convex functions [4].

Definition 1.2 ([4])**.** The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex function, if

$$\tau\left(tx + (1-t)y\right) \le t\tau(x) + (1-t)\tau(y), \quad \forall x, y \in I, t \in [0, 1].$$

We recall here the definition of $P-$convex function from [19]:

Definition 1.3. Let $\tau : I \subset \mathbb{R} \to \mathbb{R}$ be a $P-$convex function, if τ is a non-negative and $\forall x, y \in I$ and $t \in [0, 1]$, we have

$$\tau\left(tx + (1-t)y\right) \le \tau(x) + \tau(y).$$

Here we also have the definition of quasi-convex function (for a detailed discussion see [21]).

Definition 1.4. The $\tau : I \subset \mathbb{R} \to \mathbb{R}$ is known as quasi-convex function, if

$$\tau(tx + (1-t)y) \le \max\{\tau(x), \tau(y)\}, \quad \forall x, y \in I, t \in [0, 1].$$

Now we present definitions of s-convex functions of the first kind as follows extracted from [30]:

Definition 1.5. Let $s \in (0, 1]$. The $\tau : I \subset [0, \infty) \to [0, \infty)$ is said to be a $s-$convex function of the first kind, if

$$\tau\left(tx + (1-t)y\right) \le t^s\tau(x) + (1-t^s)\tau(y), \quad \forall x, y \in I, t \in [0, 1].$$

Remark 1.6. If $s \to 0$, we get quasi-convexity (see Definition 1.4).

For the second kind of convexity we recall the definition from [30].

Definition 1.7. Let $s \in (0,1]$. The, $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be a $s-$convex function of the second kind, if

$$\tau\left(tx + (1-t)y\right) \leq t^s \tau(x) + (1-t)^s \tau(y), \quad \forall x, y \in I, t \in [0,1].$$

Remark 1.8. Further, if $s \to 0$, we easily get P-convexity (see Definition 1.3).

Definition 1.9 ([20])**.** Let $(\alpha, \beta) \in (0,1]^2$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be a $(\alpha,\beta)-$convex function of the first kind, if

$$\tau\left(tx + (1-t)y\right) \leq t^\alpha \tau(x) + (1-t^\beta)\tau(y), \quad \forall x, y \in I, t \in [0,1].$$

Definition 1.10 ([20])**.** Let $(\alpha, \beta) \in (0,1]^2$. The $\tau : I \subset [0,\infty) \to [0,\infty)$ is said to be a $(\alpha,\beta)-$convex function of the second kind, if

$$\tau\left(tx + (1-t)y\right) \leq t^\alpha \tau(x) + (1-t)^\beta \tau(y), \quad \forall x, y \in I, t \in [0,1].$$

Definition 1.11 ([32])**.** The Riemann-Liouville integral operator of order $\vartheta > 0$ with $a \geq 0$ is defined as

$$J_a^\vartheta \varphi(x) = \frac{1}{\Gamma(\vartheta)} \int_a^x (x-t)^{\vartheta-1} \varphi(t)dt,$$
$$J_a^0 \varphi(x) = \varphi(x).$$

Here $\Gamma(\vartheta) = \int_0^\infty e^{-u} u^{\vartheta-1} du$ is the Gamma function. In the case of $\vartheta = 1$, the fractional integral reduces to the classical integral.

Definition 1.12 ([32])**.** The Riemann-Liouville integrals $I_{a^+}^\vartheta \varphi$ and $I_{b^-}^\vartheta \varphi$ of $\varphi \in L_1[a,b]$ having order $\vartheta > 0$ with $0 \leq a < b$ are defined by

$$I_{a^+}^\vartheta \varphi(x) = \frac{1}{\Gamma(\vartheta)} \int_a^x (x-t)^{\vartheta-1} \varphi(t)dt, \quad a < x,$$

and

$$I_{b^-}^\vartheta \varphi(x) = \frac{1}{\Gamma(\vartheta)} \int_x^b (t-x)^{\vartheta-1} \varphi(t)dt, \quad x < b,$$

respectively. Note that $I_{a^+}^0 \varphi(x) = I_{b^-}^0 \varphi(x) = \varphi(x)$.

Theorem 1.13. *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $a, b \in I, a < b$, $\varphi' \in L_1[a,b]$ and for $\vartheta \geq 1$, montgomery identity for fractional integrals holds:*

$$\varphi(x) = \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta} J_a^\vartheta \varphi(b) - J_a^{\vartheta-1}(P_1(x,b)\varphi(b)) + J_a^\vartheta (P_1(x,b)\varphi'(b)),$$

where $P_1(x,t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t) = \begin{cases} \dfrac{t-a}{b-a}\dfrac{\Gamma(\vartheta)}{(b-x)^{\vartheta-1}}, & \text{if } t \in [a,x], \\ \\ \dfrac{t-b}{b-a}\dfrac{\Gamma(\vartheta)}{(b-x)^{\vartheta-1}}, & \text{if } t \in (x,b]. \end{cases}$$

Let $[a,b] \subseteq (0,+\infty)$, we may define special means as follows:

(a) The arithmetic mean
$$A = A(a,b) := \frac{a+b}{2};$$

(b) The geometric mean
$$G = G(a,b) := \sqrt{ab};$$

(c) The harmonic mean
$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}};$$

(d) The logarithmic mean
$$L = L(a,b) := \begin{cases} a, & \text{if } a = b, \\ \dfrac{b-a}{\ln b - \ln a}, & \text{if } a \neq b; \end{cases}$$

(e) The identric mean
$$I = I(a,b) := \begin{cases} a, & \text{if } a = b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b; \end{cases}$$

(f) The $p-$logarithmic mean
$$L_p = L_p(a,b) := \begin{cases} a, & \text{if } a = b, \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b; \end{cases}$$

where $p \in \mathbb{R} \setminus \{0,-1\}$.

The following lea has ee obtained in [7] and is necessary to prove our main theorems.

Lemma 1.14. *Let $\varphi : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b). If $\varphi' \in L_1[a,b]$, then $\forall x \in (a,b)$ the identity for fractional integrals holds:*

(1.2)
$$\left(\frac{(x-a)^{\vartheta}+(b-x)^{\vartheta}}{b-a}\right)\varphi(x) - \frac{\Gamma(\vartheta+1)}{b-a}\left[I_{x^-}^{\vartheta}\varphi(a) + I_{x^+}^{\vartheta}\varphi(b)\right]$$

$$= \frac{(x-a)^{\vartheta+1}}{b-a} \int_0^1 \frac{\varphi'(tx+(1-t)a)}{t^{-\vartheta}} dt - \frac{(b-x)^{\vartheta+1}}{b-a} \int_0^1 \frac{\varphi'(tx+(1-t)b)}{t^{-\vartheta}} dt.$$

Throughout this paper, we denote

$${}_a^b\sigma_\varphi^\vartheta(x) = \left(\frac{(x-a)^\vartheta + (b-x)^\vartheta}{b-a}\right)\varphi(x) - \frac{\Gamma(\vartheta+1)}{b-a}\left[I_{x^-}^\vartheta \varphi(a) + I_{x^+}^\vartheta \varphi(b)\right].$$

We also make use of Euler's beta function, which for $x, y > 0$ is defined as follows

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1} dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \end{aligned}$$

The main aim of our study is to generalize the Ostrowski inequality (1.1) for $(\alpha, \beta, \gamma, \delta)-$convex of mixed kind, which is discussed in Section 2. Moreover, we establish some Ostrowski-type inequalities for the class of functions whose derivatives in absolute values at certain powers are $(\alpha, \beta, \gamma, \delta)$-convex functions of mixed kind by using different techniques including Hölder's inequality [38] and power mean inequality. Also, we give the special cases of our results and applications of midpoint inequalities in special means. The last section gives us a conclusion with some remarks and future ideas to generalize the results.

2. Generalization of Ostrowski Inequality via Fractional Integrals

Despite being a simple and ordinary concept, convexity has a great impact on our daily lives due to its widespread use in business and industry. In the solution of many real world problems the concept of convexity is very decisive. The problems, faced in constrained control and estimation are convex. Geometrically, a real-valued function is said to be convex if the line segment joining any two of its points lies on or above the graph of the function in Euclidean space. We introduce for the first time the class of $(s, r)-$convex and $(\alpha, \beta, \gamma, \delta)-$convex function of mixed kind.

Definition 2.1. Let $(s, r) \in (0, 1]^2$. The $\tau : I \subset [0, \infty) \to [0, \infty)$ is said to be an $(s, r)-$convex function of mixed kind, if

$$\tau(tx + (1-t)y) \leq t^{rs}\tau(x) + (1-t^r)^s \tau(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 2.2. Let $(\alpha, \beta, \gamma, \delta) \in (0, 1]^4$. The $\tau : I \subset [0, \infty) \to [0, \infty)$ is said to be an $(\alpha, \beta, \gamma, \delta)-$convex function of mixed kind, if

$$\tau(tx + (1-t)y) \leq t^{\alpha\gamma}\tau(x) + (1-t^\beta)^\delta \tau(y), \quad \forall x, y \in I, t \in [0, 1]. \tag{2.1}$$

Remark 2.3. In Definition 2.2, we have the following cases.

(i) If $\gamma=\delta=1$ in (2.1), we get $(\alpha,\beta)-$convex function of the first kind.
(ii) If $\beta=\gamma=1$ in (2.1), we get $(\alpha,\beta)-$convex function of the second kind.
(iii) If $\alpha=\delta=s$, $\beta=\gamma=r$, where $s,r\in[0,1]$ in (2.1), we get $(s,r)-$convex function of mixed kind.
(iv) If $\alpha=\beta=s$ and $\gamma=\delta=1$ where $s\in[0,1]$ in (2.1), we get $s-$convex function of the first kind.
(v) If $\alpha=\beta\to 0$, and $\gamma=\delta=1$, in (2.1), we get quasi$-$convex function.
(vi) If $\alpha=\delta=s$, $\beta=\gamma=1$ where $s\in[0,1]$ in (2.1), we get $s-$convex function of the second kind.
(vii) If $\alpha=\delta\to 0$, and $\beta=\gamma=1$, in (2.1), we get $P-$convex function.
(viii) If $\alpha=\beta=\gamma=\delta=1$ in (2.1), gives us ordinary convex function.

Theorem 2.4. *Let $\varphi:[a,b]\to\mathbb{R}$ be differentiable on (a,b), $\varphi':[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $\tau: I\subset\mathbb{R}\to\mathbb{R}$, be the $(\alpha,\beta,\gamma,\delta)-$convex function of mixed kind, then we have the inequalities:*

$$\tau\left[\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq (b-x)^{1-\vartheta}\left[\frac{(x-a)^{\alpha\gamma-1}}{(b-a)^{\alpha\gamma}}\int_a^x\tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}}{b-x}\int_x^b\tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right],\quad \forall x\in[a,b]. \tag{2.2}$$

Proof. Utilizing the Theorem 1.13, we get

$$\begin{aligned}&\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\\&=J_a^{\vartheta}(P_1(x,b)\varphi'(b))\\&=\frac{1}{\Gamma(\vartheta)}\int_a^b P_1(x,t)\frac{\varphi'(t)}{(b-t)^{1-\vartheta}}dt\end{aligned}\tag{2.3}$$

$$= \left(\frac{x-a}{b-a}\right)\left[\frac{(b-x)^{1-\vartheta}}{x-a}\int_a^x \frac{\{t-a\}\varphi'(t)}{(b-t)^{1-\vartheta}}dt\right]$$
$$+\left(1-\left(\frac{x-a}{b-a}\right)\right)\left[\frac{(b-x)^{1-\vartheta}}{b-x}\int_x^b \frac{\{t-b\}\varphi'(t)}{(b-t)^{1-\vartheta}}dt\right], \quad \forall x\in[a,b].$$

Next by using the $(\alpha,\beta,\gamma,\delta)-$convex function of mixed kind of $\tau : I \subset [0,\infty) \to \mathbb{R}$, we get

(2.4)

$$\tau\left[\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq \left(\frac{x-a}{b-a}\right)^{\alpha\gamma}\tau\left[\frac{(b-x)^{1-\vartheta}}{x-a}\int_a^x \frac{\{t-a\}\varphi'(t)}{(b-t)^{1-\vartheta}}dt\right]$$
$$+\left(1-\left(\frac{x-a}{b-a}\right)^{\beta}\right)^{\delta}\tau\left[\frac{(b-x)^{1-\vartheta}}{b-x}\int_x^b \frac{\{t-b\}\varphi'(t)}{(b-t)^{1-\vartheta}}dt\right], \quad \forall x\in[a,b].$$

Applying Jensen's integral inequality [12], we get the Inequality (2.2). □

Corollary 2.5. *In Theorem* 2.4, *one can see the following.*

(i) *If* $\gamma=\delta=1$, $\alpha\in[0,1]$ *and* $\beta\in(0,1]$, *in* (2.2), *then the fractional Ostrowski-type inequality for* $(\alpha,\beta)-$*convex functions of the first kind as follows:*

$$\tau\left[\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq (b-x)^{1-\vartheta}\left[\frac{(x-a)^{\alpha-1}}{(b-a)^{\alpha}}\int_a^x \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-a)^{\beta}-(x-a)^{\beta}}{(b-a)^{\beta}(b-x)}\int_x^b \tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

(ii) *If* $\beta=\gamma=1$, $\alpha\in[0,1]$ *and* $\delta\in[0,1]$, *in* (2.2), *then fractional Ostrowski-type inequality for* $(\alpha,\delta)-$*convex functions of the second kind as follows:*

$$\tau\left[\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq (b-x)^{1-\vartheta}\left[\frac{(x-a)^{\alpha-1}}{(b-a)^{\alpha}}\int_a^x \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{(b-x)^{\beta-1}}{(b-a)^{\beta}}\int_x^b \tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

(iii) *If* $\alpha = \delta = s$, $\beta = \gamma = r$, *where* $s \in [0,1]$ *and* $r \in (0,1]$ *in* (2.2), *then fractional Ostrowski-type inequality for* $(s,r)-$*convex functions of mixed kind as follows:*

$$\tau\left[\varphi(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b) + J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq \frac{(b-x)^{1-\vartheta}}{(b-a)^{rs}}\left[(x-a)^{rs-1}\int_a^x \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left. + \frac{((b-a)^r-(x-a)^r)^s}{b-x}\int_x^b \tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

(iv) *If* $\alpha = \beta = s$ *and* $\gamma = \delta = 1$, *where* $s \in (0,1]$ *in* (2.2), *then the Ostrowski inequality for the* $s-$*convex functions of the first kind as follows:*

$$\tau\left[\varphi(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b) + J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq \frac{(b-x)^{1-\vartheta}}{(b-a)^{s}}\left[(x-a)^{s-1}\int_a^x \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left. + \frac{(b-a)^s-(x-a)^s}{(b-x)}\int_x^b \tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

(v) *If* $\alpha = \beta \to 0$, *and* $\gamma = \delta = 1$, *in* (2.2), *we get quasi-convex function.*

$$\tau\left[\varphi(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b) + J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq \frac{(b-x)^{1-\vartheta}}{(x-a)}\left[\int_a^x \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right]. \tag{2.5}$$

(vi) *If* $\beta = \gamma = 1$, $\alpha = \delta = s$ *where* $s \in [0,1)$ (2.2), *then fractional Ostrowski-type inequality for* $s-$*convex functions of the second kind as follows:*

$$\tau\left[\varphi(x) - \frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b) + J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq \frac{(b-x)^{1-\vartheta}}{(b-a)^{s}}\left[(x-a)^{s-1}\int_a^x \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left. + (b-x)^{s-1}\int_x^b \tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

(vii) *If* $\alpha=\delta\to 0$ *and* $\beta=\gamma=1$ *in* (2.2), *then fractional Ostrowski-type inequality for the* $P-$*convex functions as follows:*

$$\tau\left[\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq(b-x)^{1-\vartheta}\left[\frac{1}{x-a}\int_a^x\tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left.+\frac{1}{b-x}\int_x^b\tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

(viii) *If* $\alpha=\beta=\gamma=\delta=1$, *in* (2.2), *then fractional Ostrowski-type inequality for the convex functions as follows:*

$$\tau\left[\varphi(x)-\frac{\Gamma(\vartheta)}{b-a}(b-x)^{1-\vartheta}J_a^{\vartheta}\varphi(b)+J_a^{\vartheta-1}(P_1(x,b)\varphi(b))\right]$$
$$\leq\frac{(b-x)^{1-\vartheta}}{b-a}\left[\int_a^x\tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt+\int_x^b\tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

Theorem 2.6. *Suppose all the assumptions of Lemma* 1.14 *hold. Additionally, assume that* $|\varphi'|$ *is* $(\alpha,\beta,\gamma,\delta)-$*convex function on* $[a,b]$ *and* $|\varphi'(x)|\leq M$, *then*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right|\leq M\left(\frac{1}{\vartheta+\alpha\gamma+1}+\frac{B\left(\frac{\vartheta+1}{\beta},\delta+1\right)}{\beta}\right)\ {}^{\vartheta}\kappa_a^b(x),\tag{2.6}$$

$\forall x\in(a,b)$, *where* ${}^{\vartheta}\kappa_a^b(x)=\frac{(x-a)^{\vartheta+1}+(b-x)^{\vartheta+1}}{b-a}$.

Proof. From Lemma 1.14 we have

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right|\leq\frac{(x-a)^{\vartheta+1}}{b-a}\int_0^1t^{\vartheta}\left|\varphi'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\vartheta+1}}{b-a}\int_0^1t^{\vartheta}\left|\varphi'(tx+(1-t)b)\right|dt.\tag{2.7}$$

Since $|\varphi'|$ is $(\alpha,\beta,\gamma,\delta)-$convex on $[a,b]$ and $|\varphi'(x)|\leq M$, we have

$$\int_0^1t^{\vartheta}\left|\varphi'(tx+(1-t)a)\right|dt\leq M\int_0^1t^{\vartheta}\left(t^{\alpha\gamma}+(1-t^{\beta})^{\delta}\right)dt,\tag{2.8}$$

and similarly

$$\int_0^1t^{\vartheta}\left|\varphi'(tx+(1-t)b)\right|dt\leq M\int_0^1t^{\vartheta}\left(t^{\alpha\gamma}+(1-t^{\beta})^{\delta}\right)dt.\tag{2.9}$$

By using inequalities (2.8) and (2.9) in (2.7), we get (2.6). □

Corollary 2.7. *In Theorem* 2.6, *one can see the following.*

(i) *If* $\gamma=\delta=1$, $\alpha\in[0,1]$ *and* $\beta\in(0,1]$ *in inequality* (2.6), *then the Ostrowski inequality for* $(\alpha,\beta)-$*convex functions of the first kind via fractional integrals are defined:*

$$\left|{}_a^b\sigma_\varphi^\vartheta(x)\right|\leq M\left(\frac{1}{\vartheta+\alpha+1}+\frac{B\left(\frac{\vartheta+1}{\beta},2\right)}{\beta}\right)\ {}^\vartheta\kappa_a^b(x).$$

(ii) *If* $\beta=\gamma=1$, $\alpha\in[0,1]$ *and* $\delta\in[0,1]$ *in inequality* (2.6), *then Ostrowski inequality for* $(\alpha,\delta)-$*convex functions of the second kind via fractional integrals are defined:*

$$\left|{}_a^b\sigma_\varphi^\vartheta(x)\right|\leq M\left(\frac{1}{\vartheta+\alpha+1}+B\left(\vartheta+1,\delta+1\right)\right)\ {}^\vartheta\kappa_a^b(x).$$

(iii) *If* $\alpha=\delta=s$, $\beta=\gamma=r$, *where* $s\in[0,1]$ *and* $r\in(0,1]$ *in inequality* (2.6), *then Ostrowski inequality for* $(s,r)-$*convex functions of mixed kind via fractional integrals are defined:*

$$\left|{}_a^b\sigma_\varphi^\vartheta(x)\right|\leq M\left(\frac{1}{\vartheta+rs+1}+\frac{B\left(\frac{\vartheta+1}{r},s+1\right)}{r}\right)\ {}^\vartheta\kappa_a^b(x).$$

(iv) *If* $\alpha=\beta=s$ *and* $\gamma=\delta=1$, *where* $s\in(0,1]$ *in inequality* (2.6), *then Ostrowski inequality for* $s-$*convex functions of the first kind via fractional integrals are defined:*

$$\left|{}_a^b\sigma_\varphi^\vartheta(x)\right|\leq M\left(\frac{1}{\vartheta+s+1}+\frac{B\left(\frac{\vartheta+1}{s},2\right)}{s}\right)\ {}^\vartheta\kappa_a^b(x).$$

(v) *If* $\alpha=\delta=s$, *and* $\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (2.6), *then Ostrowski inequality for* $s-$*convex functions of the second kind via fractional integrals are defined:*

$$\left|{}_a^b\sigma_\varphi^\vartheta(x)\right|\leq M\left(\frac{1}{\vartheta+s+1}+B\left(\vartheta+1,s+1\right)\right)\ {}^\vartheta\kappa_a^b(x).$$

(vi) *If* $\alpha=\delta=s$, *and* $\vartheta=\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (2.6), *then inequality* (2.1) *of Theorem* 2 *in [2] holds.*

(vii) *If* $\alpha=\delta=s$, *and* $\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (2.6), *then inequality* (2.6) *of Theorem* 7 *in [35] holds.*

(viii) *If* $\alpha=\delta\to0$, *and* $\beta=\gamma=1$ *in inequality* (2.6), *then Ostrowski inequality for* $P-$*convex functions via fractional integrals are defined:*

$$\left|{}_a^b\sigma_\varphi^\vartheta(x)\right|\leq M\left(\frac{1}{\vartheta+1}+B\left(\vartheta+1,1\right)\right)\ {}^\vartheta\kappa_a^b(x).$$

(ix) *If $\alpha = \beta = \gamma = \delta = 1$ in inequality* (2.6), *then Ostrowski inequality for convex functions via fractional integrals are defined:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq M\left(\frac{1}{\vartheta+2} + B\left(\vartheta+1,2\right)\right)\ {}^{\vartheta}\kappa_{a}^{b}(x).$$

(x) *If $\vartheta = \alpha = \beta = \gamma = \delta = 1$ in inequality* (2.6), *then Ostrowski inequality* (1.1) *for convex function holds.*

Theorem 2.8. *Suppose all the assumptions of Lemma* 1.14 *hold. Additionally, assume that $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex function on $[a,b], q \geq 1$ and $|\varphi'(x)| \leq M$, then*

(2.10)

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+\alpha\gamma+1} + \frac{B\left(\frac{\vartheta+1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_{a}^{b}(x),$$

$\forall x \in (a,b)$.

Proof. From the Lemma 1.14 and using power mean inequality, we have

(2.11)

$$\begin{aligned}&\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \\ &\leq \frac{(x-a)^{\vartheta+1}}{b-a}\left(\int_0^1 t^{\vartheta}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\vartheta}\left|\varphi'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^{\vartheta+1}}{b-a}\left(\int_0^1 t^{\vartheta}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\vartheta}\left|\varphi'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex on $[a,b]$, and $|\varphi'(x)| \leq M$, we get

$$\int_0^1 t^{\vartheta}\left|\varphi'\left(tx+(1-t)a\right)\right|^q dt \leq M^q\int_0^1 t^{\vartheta}\left(t^{\alpha\gamma}+(1-t^{\beta})^{\delta}\right)dt, \tag{2.12}$$

and

$$\int_0^1 t^{\vartheta}\left|\varphi'\left(tx+(1-t)b\right)\right|^q dt \leq M^q\int_0^1 t^{\vartheta}\left(t^{\alpha\gamma}+(1-t^{\beta})^{\delta}\right)dt. \tag{2.13}$$

Using the inequalities $(2.11)-(2.13)$, we get (2.10). □

Corollary 2.9. *In Theorem* 2.8, *one can see the following.*

(i) *If $\gamma = \delta = 1$, $\alpha \in [0,1]$ and $\beta \in (0,1]$ in inequality* (2.10), *then the Ostrowski inequality for $(\alpha,\beta)-$convex functions of the first* *kind via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+\alpha+1}+\frac{B\left(\frac{\vartheta+1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}} {}^{\vartheta}\kappa_{a}^{b}(x).$$

(ii) *If* $\beta=\gamma=1$, $\alpha\in[0,1]$ *and* $\delta\in[0,1]$ *in inequality* (2.10), *then Ostrowski inequality for* $(\alpha,\delta)-$*convex functions of the second kind via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+\alpha+1}+B\left(\vartheta+1,\delta+1\right)\right)^{\frac{1}{q}} {}^{\vartheta}\kappa_{a}^{b}(x).$$

(iii) *If* $\alpha=\delta=s$, $\beta=\gamma=r$, *where* $s\in[0,1]$ *and* $r\in(0,1]$ *in inequality* (2.10), *then Ostrowski inequality for* $(s,r)-$*convex functions of mixed kind via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+rs+1}+\frac{B\left(\frac{\vartheta+1}{r},s+1\right)}{r}\right)^{\frac{1}{q}} {}^{\vartheta}\kappa_{a}^{b}(x).$$

(iv) *If* $\alpha=\beta=s$ *and* $\gamma=\delta=1$, *where* $s\in(0,1]$ *in inequality* (2.10), *then Ostrowski inequality for* $s-$*convex functions of the first kind via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+s+1}+\frac{B\left(\frac{\vartheta+1}{s},2\right)}{s}\right)^{\frac{1}{q}} {}^{\vartheta}\kappa_{a}^{b}(x).$$

(v) *If* $\alpha=\delta=s$, *and* $\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (2.10), *then Ostrowski inequality for* $s-$*convex functions of the second kind via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+s+1}+B\left(\vartheta+1,s+1\right)\right)^{\frac{1}{q}} {}^{\vartheta}\kappa_{a}^{b}(x).$$

(vi) *If* $\alpha=\delta=s$, *and* $\vartheta=\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (2.10), *then inequality* (2.3) *of Theorem* 4 *in [2] holds.*

(vii) *If* $\alpha=\delta=s$, *and* $\beta=\gamma=1$, *where* $s\in(0,1]$ *in inequality* (2.10), *then inequality* (2.8) *of Theorem* 9 *in [35] holds.*

(viii) *If* $\alpha=\delta\to 0$, *and* $\beta=\gamma=1$ *in inequality* (2.10), *then Ostrowski inequality for* $P-$*convex functions via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+1}+B\left(\vartheta+1,1\right)\right)^{\frac{1}{q}} {}^{\vartheta}\kappa_{a}^{b}(x).$$

(ix) *If $\alpha = \beta = \gamma = \delta = 1$ in inequality (2.10), then Ostrowski inequality for convex functions via fractional integrals would be written:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\vartheta+2}+B\left(\vartheta+1,2\right)\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_{a}^{b}(x).$$

Theorem 2.10. *Suppose all the assumptions of Lemma 1.14 hold. Additionally, assume that $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex function on $[a,b], q>1$ and $|\varphi'(x)|\leq M$, then*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_{a}^{b}(x), \tag{2.14}$$

$\forall x\in(a,b)$, where $p^{-1}+q^{-1}=1$.

Proof. From Lemma 1.14 and using Hölder's inequality [38], we have

$$\begin{aligned}\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq{}& \frac{(x-a)^{\vartheta+1}}{b-a}\left(\int_0^1 t^{\vartheta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(b-x)^{\vartheta+1}}{b-a}\left(\int_0^1 t^{\vartheta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{2.15}$$

Since $|\varphi'|^q$ is $(\alpha,\beta,\gamma,\delta)-$convex and $|\varphi'(x)|\leq M$, we have

$$\int_0^1\left|\varphi'\left(tx+(1-t)a\right)\right|^q dt \leq M^q\int_0^1 t^{\alpha\gamma}+(1-t^{\beta})^{\delta}dt, \tag{2.16}$$

and

$$\int_0^1\left|\varphi'\left(tx+(1-t)b\right)\right|^q dt \leq M^q\int_0^1 t^{\alpha\gamma}+(1-t^{\beta})^{\delta}dt. \tag{2.17}$$

Using inequalities (2.15) – (2.17), we get (2.10). □

Corollary 2.11. *In Theorem 2.10, one can see the following.*

(i) *If $\gamma=\delta=1$, $\alpha\in[0,1]$ and $\beta\in(0,1]$ in inequality (2.14), then the Ostrowski inequality for $(\alpha,\beta)-$convex functions of the first kind via fractional integrals is as follows:*

$$\left|{}_{a}^{b}\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_{a}^{b}(x).$$

(ii) *If $\beta = \gamma = 1$, $\alpha \in [0,1]$ and $\delta \in [0,1]$ in inequality* (2.14), *then the Ostrowski inequality for $(\alpha,\delta)-$convex functions of the second kind via fractional integrals is as follows:*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+B\left(1,\delta+1\right)\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_a^b(x).$$

(iii) *If $\alpha = \delta = s$, $\beta = \gamma = r$, where $s \in [0,1]$ and $r \in (0,1]$ in inequality* (2.14), *then the Ostrowski inequality for $(s,r)-$convex functions of mixed kind via fractional integrals is as follows:*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{rs+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_a^b(x).$$

(iv) *If $\alpha = \beta = s$ and $\gamma = \delta = 1$, where $s \in (0,1]$ in inequality* (2.14), *then Ostrowski inequality for $s-$convex functions of the first kind via fractional integrals is as follows:*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}+\frac{B\left(\frac{1}{s},2\right)}{s}\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_a^b(x).$$

(v) *If $\alpha = \delta = s$, and $\beta = \gamma = 1$, where $s \in (0,1]$ in inequality* (2.14), *then Ostrowski inequality for $s-$convex functions of the second kind via fractional integrals is as follows:*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}+B\left(1,s+1\right)\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_a^b(x).$$

(vi) *If $\alpha = \delta = s$, and $\vartheta = \beta = \gamma = 1$, where $s \in (0,1]$ in inequality* (2.14), *then inequality* (2.2) *of Theorem* 3 *in [2] holds.*

(vii) *If $\alpha = \delta = s$, and $\beta = \gamma = 1$, where $s \in (0,1]$ in inequality* (2.14), *then inequality* (2.7) *of Theorem* 8 *in [35] holds.*

(viii) *If $\alpha = \delta \to 0$, and $\beta = \gamma = 1$ in inequality* (2.14), *then Ostrowski inequality for $P-$convex functions via fractional integrals is as follows:*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{(2)^{\frac{1}{q}}\,M}{(\vartheta p+1)^{\frac{1}{p}}}\ {}^{\vartheta}\kappa_a^b(x).$$

(ix) *If $\alpha = \beta = \gamma = \delta = 1$ in inequality* (2.14), *then Ostrowski inequality for convex functions via fractional integrals is as follows:*

$$\left|{}_a^b\sigma_{\varphi}^{\vartheta}(x)\right| \leq \frac{M}{(\vartheta p+1)^{\frac{1}{p}}}\left(\frac{1}{2}+B\left(1,2\right)\right)^{\frac{1}{q}}\ {}^{\vartheta}\kappa_a^b(x).$$

3. Applications of Midpoint Inequalities

If we replace φ by $-\varphi$ and $x = \frac{a+b}{2}$ in Theorem 2.4, what is gained?

Theorem 3.1. *Let $\varphi : [a,b] \to \mathbb{R}$ be differentiable on (a,b), $\varphi' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $\tau : I \subset \mathbb{R} \to \mathbb{R}$, be a $(\alpha,\beta,\gamma,\delta)-$convex function of mixed kind, then*

(3.1)

$$\tau\left[\frac{\Gamma(\vartheta)\left(\frac{b-a}{2}\right)^{1-\vartheta}}{b-a} J_a^{\vartheta}\varphi(b) - \varphi\left(\frac{a+b}{2}\right) - J_a^{\vartheta-1}\left(P_1\left(\frac{a+b}{2}, b\right)\varphi(b)\right)\right]$$
$$\leq \frac{2^{\vartheta-1}}{(b-a)^{\vartheta}}\left[\frac{1}{2^{\alpha\gamma-1}}\int_{\frac{a+b}{2}}^{a} \tau\left[\frac{(t-a)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right.$$
$$\left. + \frac{\left(2^{\beta}-1\right)^{\delta}}{2^{\beta\delta-1}}\int_{b}^{\frac{a+b}{2}} \tau\left[\frac{(t-b)\varphi'(t)}{(b-t)^{1-\vartheta}}\right]dt\right].$$

Remark 3.2. In Theorem 3.1, if we put $\vartheta = 1$ in (3.1), we get

(3.2)

$$\tau\left(\frac{1}{b-a}\int_a^b \varphi(t)dt - \varphi\left(\frac{a+b}{2}\right)\right)$$
$$\leq \frac{1}{b-a}\left[\frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau[(a-t)\varphi'(t)]dt + \frac{\left(2^{\beta}-1\right)^{\delta}}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^{b} \tau[(b-t)\varphi'(t)]dt\right].$$

Remark 3.3. Assume that $\tau : I \subset [0,\infty) \to \mathbb{R}$ be an $(\alpha,\beta,\gamma,\delta)-$convex function of mixed kind:

(i) If $\vartheta = 1$, and $\varphi(t) = \frac{1}{t}$ in inequality (3.1) where $t \in [a,b] \subset (0,\infty)$, then

$$(b-a)\tau\left[\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right]$$
$$\leq \frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{t-a}{t^2}\right]dt + \frac{\left(2^{\beta}-1\right)^{\delta}}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^{b} \tau\left[\frac{t-b}{t^2}\right]dt.$$

(ii) If $\vartheta = 1$, and $\varphi(t) = -\ln t$ in inequality (3.1), where $t \in [a,b] \subset (0,\infty)$, then

$$(b-a)\tau\left[\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right]$$
$$\leq \frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}} \tau\left[\frac{t-a}{t}\right]dt + \frac{\left(2^{\beta}-1\right)^{\delta}}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^{b} \tau\left[\frac{t-b}{t}\right]dt.$$

(iii) If $\vartheta = 1, \varphi(t) = t^p$, and $p \in \mathbb{R}\backslash\{0,-1\}$ in inequality (3.1), where $t \in [a,b] \subset (0,\infty)$, then

$$(b-a)\tau\left[L_p^p(a,b) - A^p(a,b)\right]$$
$$\leq \frac{1}{2^{\alpha\gamma-1}}\int_a^{\frac{a+b}{2}}\tau\left[\frac{p\,(a-t)}{t^{1-p}}\right]dt + \frac{\left(2^\beta-1\right)^\delta}{2^{\beta\delta-1}}\int_{\frac{a+b}{2}}^{b}\tau\left[\frac{p\,(b-t)}{t^{1-p}}\right]dt.$$

Remark 3.4. In Theorem 2.8, one can see the following.

(i) Let $x = \frac{a+b}{2}, \vartheta = 1, 0 < a < b$, $q \geq 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(t) = t^n$ in (2.10). Then

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{M\,(b-a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2} + \frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

(ii) Let $x = \frac{a+b}{2}, \vartheta = 1, 0 < a < b$, $q \geq 1$ and $\varphi : (0,1] \to \mathbb{R}$, $\varphi(t) = -\ln t$ in (2.10). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M\,(b-a)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2} + \frac{B\left(\frac{2}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

Remark 3.5. In Theorem 2.10, one can see the following.

(i) Let $x = \frac{a+b}{2}, \vartheta = 1, 0 < a < b$, $p^{-1} + q^{-1} = 1$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(t) = t^n$ in (2.14). Then

$$|A^n(a,b) - L_n^n(a,b)| \leq \frac{M\,(b-a)}{2\,(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1} + \frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

(ii) Let $x = \frac{a+b}{2}, \vartheta = 1, 0 < a < b$, $p^{-1}+q^{-1} = 1$ and $\varphi : (0,1] \to \mathbb{R}$, $\varphi(t) = -\ln t$ in (2.14). Then

$$\left|\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right| \leq \frac{M\,(b-a)}{2\,(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1} + \frac{B\left(\frac{1}{\beta},\delta+1\right)}{\beta}\right)^{\frac{1}{q}}.$$

4. Conclusion and Remarks

4.1. **Conclusion.** The Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in the literature. In this paper, we presented the generalized notion of $(\alpha,\beta,\gamma,\delta)-$convex functions of mixed kind. This class of functions contains many important classes, including the class of $(\alpha,\beta)-$convex functions of the first and second kind [20], $(s,r)-$convex functions of the mixed kind [3], $s-$convex functions of the first and second kind

[8], $P-$convex functions, quasi-convex functions and the class of convex functions. In Section 2, we presented our main result, the generalization of Ostrowski inequality [35] via fractional integral. We also presented other results using various techniques, including Hölder's inequality [38] and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

4.2. **Remarks and Future Ideas.**

- (i) One may also do similar work by using various classes of convex functions.
- (ii) One may do similar work to generalize all results stated in this research work by applying weights.
- (iii) One may also present all results obtained in this research work by higher order derivatives.
- (iv) One may also state all results stated in this research work by multi-variable functions and generalized fractional integral operators.
- (v) One may also do similar work by using various generalized forms for the Korkine's and Montgomery identities, improved power means inequality, Hölder's Iscan inequality, Jensen's integral inequality with weights are generalized fuzzy metric spaces on the set of all fuzzy numbers.

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