Results on Discontinuity at Fixed Point for a New Class of F-Contractive Mappings

Pradip Debnath

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 4
Pages: 21-32

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.560141.1161

Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 4 (2023), 21-32
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.560141.1161

Results on Discontinuity at Fixed Point for a New Class of F-Contractive Mappings

Pradip Debnath

Abstract. The search for contractive definitions which do not compel the mapping to be continuous at fixed points remained an open problem for a long time. Several solutions to this open problem have been obtained in last two decades. The current paper, we aim to provide another new solution direction for the discontinuity study at fixed points using F-contractive mappings in a complete metric space. Several consequences of those new results are also provided. This manuscript consists of three main parts. In the first part, the notion of F-contractive mappings has been described. In the second part, discontinuity at the fixed point assuming continuity of the composition has been investigated, whereas in the third part, discontinuity at a fixed point without assuming continuity of the composition has been illustrated.

1. Introduction and Preliminaries

The famous Banach contraction principle guarantees the continuity of the mapping under consideration, whereas the study of fixed points for discontinuous mappings originated with the 1968-69 papers of Kannan [18, 19]. However, all the mapping that were known at that point of time were continuous at their respective fixed points, although they admitted points of discontinuity within their domains [4–6, 12]. In 1977, Rhoades [27] presented a comparative study of 250 contractive definitions and observed that many of those definitions did not imply the continuity of the mappings in their respective entire domains. Motivated by this, in 1988, Rhoades [28] posed the open problem of searching for contractive definitions which do not compel the mapping to be continuous at fixed

2020 *Mathematics Subject Classification.* 47H10; 54H25; 54E50.
Key words and phrases. Discontinuity, Fixed point, F-contraction, Complete metric space.
Received: 13 August 2022, Accepted: 07 March 2023.

points. In 1999, Pant [25] provided an affirmative answer to this search by producing functions that had discontinuities at their fixed points.

Recently, Debnath and Srivastava [14] studied common BPPs for multivalued contractive pairs of mappings. Debnath and Srivastava [15] also proved new extensions of Kannan's and Reich's theorems. Another Kannan-type contraction for multivalued asymptotic regular maps was presented by Debnath et al. [13]. An important application of fixed points of $F(\psi, \varphi)$-contractions to fractional differential equations was proposed by Srivastava et al. [32]. Srivastava et al. [31] also proposed the study of implicit functional differential inclusions of arbitrary fractional order. More relevant results, we refer to [1, 16, 29, 30].

Fixed point results for mappings with discontinuity are popular and interesting for their wide variety of applications in character recognition, neural networks and the solution of non-negative sparse approximation problems [9, 17, 20–23, 36]. Recently, several authors have attempted to provide solutions to the question posed by Rhodes from various points of view (see Bisht and Rakocević [3], Pant et al. [26], Tas and Ozgur [33], Ozgur and Tas [24]).

In the current paper, we aim to provide another new solution direction towards the study of discontinuity at fixed point using F-contractive mappings due to Wardowski in a complete metric space (MS).

In 2012, Wardowski [35] defined the concept of F-contraction as follows.

Definition 1.1. Let $F : (0, +\infty) \to (-\infty, +\infty)$ be a function which satisfies the following:

(F1) F is strictly increasing;

(F2) For each sequence $\{u_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$, $\lim\limits_{n\to+\infty} u_n = 0$ if and only if $\lim\limits_{n\to+\infty} F(u_n) = -\infty$;

(F3) There is $t \in (0, 1)$ such that $\lim\limits_{u\to 0^+} u^t F(u) = 0$.

Let $\mathcal{F}$ denote the class of all such functions F. If $(\mathcal{W}, \eta)$ is an MS, then a self-map $\Phi : \mathcal{W} \to \mathcal{W}$ is said to be an $F-$contraction if there exists $\delta > 0$, $F \in \mathcal{F}$, such that for all $\theta, \xi \in \mathcal{W}$,

$$\eta(\Phi\theta, \Phi\xi) > 0 \quad \Rightarrow \quad \delta + F(\eta(\Phi\theta, \Phi\xi)) \leq F(\eta(\theta, \xi)) \cdots (A).$$

Consider the function $F : \mathbb{R}^+ \to \mathbb{R}$ given by $F(\eta) = \ln \eta$. Then, it is easy to see that any mapping Φ satisfying the above inequality (A) is an F-contraction [35]. Further, $F(\eta) = \ln \eta + \eta, \eta > 0$, $F(\eta) = \ln(\eta^2 + \eta), \eta > 0$ can also provide examples of F-contractive mappings. It has been observed that F-contractions exhibit a faster rate of convergence towards the fixed point and as such, they are considered to provide better iteration schemes than many other contractions[35].

Our main work is divided into two sections. In the first section, we establish a discontinuity at a fixed point result, assuming continuity of the composition of the mapping under consideration. In the second section, we develop a similar result where the continuity of the mapping composition can be dropped.

2. Discontinuity at Fixed Point Assuming Continuity of the Composition

In this section, we present a discontinuity at fixed point result assuming continuity of the composition of the mapping.

The following notation will be used.

$$\Delta_0(\theta,\xi) = \max\left\{\eta(\theta,\xi), \eta(\theta,\Phi\theta), \eta(\xi,\Phi\xi), \frac{\eta(\theta,\Phi\xi)+\eta(\xi,\Phi\theta)}{2}\right\}.$$

Theorem 2.1. *Let $(\mathcal{W},\eta)$ be a complete MS and $\Phi : \mathcal{W} \to \mathcal{W}$ be such that Φ^2 is continuous and there exists $\delta > 0$ and $F \in \mathcal{F}$ satisfying*

(i) *$\delta + F(\eta(\Phi\theta,\Phi\xi)) \leq F(\Gamma(\Delta_0(\theta,\xi)))$, where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ has the property $\Gamma(s) < s$ for each $s > 0$;*

(ii) *for a given $\epsilon > 0$, there exist $\kappa > 0$ such that $\epsilon < \Delta_0(\theta,\xi) < \epsilon+\kappa$ implies that $\eta(\theta,\xi) \leq \epsilon$.*

Then, Φ possesses a unique fixed point, say ω and $\lim\limits_{n\to\infty} \Phi^n\theta \to \omega$ for each $\theta \in \mathcal{W}$. Moreover, Φ happens to be discontinuous at ω if and only if $\lim\limits_{\theta\to\omega} \Delta_0(\theta,\omega) \neq 0$.

Proof. We fix $\theta_0 \in \mathcal{W}$ such that $\theta_0 \neq \Phi\theta_0$ and construct the sequence $\{\theta_n\}$ by $\theta_{n+1} = \Phi^n\theta_0 = \Phi\theta_n$. We denote $d_n = \eta(\theta_n,\theta_{n+1})$. Then $d_n = \eta(\theta_n,\theta_{n+1})) = \eta(\Phi\theta_{n-1},\Phi\theta_n)$.

Using (i) of the hypothesis, we have

$$\begin{aligned}\delta + F(\eta(\Phi\theta_{n-1},\Phi\theta_n)) &\leq F(\Gamma(\Delta_0(\theta_{n-1},\theta_n)))\\ &< F(\Delta_0(\theta_{n-1},\theta_n))\\ &= F(\max\{d_n,d_{n-1}\})\\ &= F(d_{n-1}).\end{aligned}$$

The above implies

$$\begin{aligned}\delta + F(d_n) < F(d_{n-1}) \quad &\Rightarrow \quad F(d_n) < F(d_{n-1}) - \delta\\ &\Rightarrow \quad d_n < d_{n-1}, \qquad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Thus $\{d_n\}$ is a strictly decreasing sequence of positive reals and hence converges to, say d. If possible, suppose that $d > 0$. Obviously, there exists $p \in \mathbb{N}$ such that for $n \geq p$, we have

$$d < d_n < d + \kappa. \tag{2.1}$$

Using (ii) of the hypothesis and the fact that $d_n < d_{n-1}$, we have that $d_n \leq d$ for all $n \geq p$, which contradicts (2.1). Thus, we have $d = 0$.

Next, we show that $\{\theta_n\}$ is Cauchy. Let $\epsilon > 0$. Without loss of generality, we may assume that $\kappa < \epsilon$. Since $d_n \to \infty$ as $n \to \infty$, there exists $q \in \mathbb{N}$ such that

$$d_n < \frac{\kappa}{2}, \quad \text{for all } n \geq q. \tag{2.2}$$

With the help of mathematical induction, we show that

$$\eta(\theta_q, \theta_{q+n}) < \epsilon + \frac{\kappa}{2}, \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

From (2.2), clearly (2.3) holds for $n = 1$. Suppose that (2.3) holds true for some n. Then, we obtain

$$\eta(\theta_q, \theta_{q+n+1}) \leq \eta(\theta_q, \theta_{q+1}) + \eta(\theta_{q+1}, \theta_{q+n+1}).$$

It is enough to show that $\eta(\theta_{q+1}, \theta_{q+n+1}) \leq \epsilon$. For this purpose, we prove that $\Delta_0(\theta_q, \theta_{q+n}) \leq \epsilon + \kappa$, where

$$\Delta_0(\theta_q, \theta_{q+n}) = \max\Bigg\{\eta(\theta_q, \xi_{q+n}), \eta(\theta_q, \Phi\theta_q), \eta(\xi_{q+n}, \Phi\xi_{q+n}), \frac{\eta(\theta_q, \Phi\xi_{q+n}) + \eta(\xi_{q+n}, \Phi\theta_q)}{2}\Bigg\}.$$

Now we have that

$$\eta(\theta_q, \theta_{q+n}) < \epsilon + \frac{\kappa}{2}, \qquad \eta(\theta_q, \theta_{q+1}) < \frac{\kappa}{2}, \qquad \eta(\theta_{q+n}, \theta_{q+n+1}) < \frac{\kappa}{2}. \tag{2.4}$$

Further,

$$\begin{aligned} &\frac{\eta(\theta_q, \xi_{q+n+1}) + \eta(\xi_{q+1}, \theta_{q+n})}{2} \\ &\leq \frac{\eta(\theta_q, \theta_{q+n}) + \eta(\theta_{q+n+1}, \theta_{q+n}) + \eta(\theta_q, \theta_{q+1}) + \eta(\theta_q, \xi_{q+n})}{2} \\ &< \epsilon + \kappa. \end{aligned} \tag{2.5}$$

Thus, $\Delta_0(\theta_q, \theta_{q+n}) \leq \epsilon + \kappa$, so that by (ii) we have $\eta(\theta_{q+1}, \xi_{q+n+1}) \leq \epsilon$. This completes the induction.

Now, fix $p > 0$ and let $\epsilon = \frac{p}{2}$. Since $\kappa < \epsilon$, we have $\frac{\kappa}{2} < \frac{\epsilon}{2}$. Then

$$\begin{aligned} \eta(\theta_q, \theta_{q+n}) &< \epsilon + \frac{\kappa}{2} \quad \text{for all } n \in \mathbb{N} \\ &< \frac{p}{2} + \frac{p}{4} \\ &< \frac{p}{2} + \frac{p}{2} \\ &= p. \end{aligned}$$

Therefore, $\{\theta_n\}$ is Cauchy.

Since $\mathcal{W}$ is complete, there exists a point $\omega \in \mathcal{W}$ such that $\theta_n \to \omega$ as $n \to \infty$. Also, $\Phi\theta_n \to \omega$ and $\Phi^2\theta_n \to \omega$. Since Φ^2 is continuous, we have that $\Phi^2\theta_n \to \Phi^2\omega$.

If possible, suppose that $\omega \neq \Phi\omega$. Then

$$\begin{aligned}
\eta(\omega, \Phi\omega) &= \eta(\Phi^2\omega, \Phi\omega) \\
&\leq \Gamma(\Delta_0(\Phi\omega, \omega)) \\
&< \Delta_0(\Phi\omega, \omega) \\
&= \eta(\omega, \Phi\omega),
\end{aligned}$$

which is a contradiction. Hence, $\omega = \Phi\omega$.

To prove the uniqueness, let $\omega = \Phi\omega$ and $\nu = \Phi\nu$ with $\omega \neq \nu$. Thus, from (i),

$$\delta + F\left(\eta(\Phi\omega, \Phi\nu)\right) \leq F\left(\Gamma(\Delta_0(\omega, \nu))\right).$$

The above implies that

$$\eta\left(\Phi\omega, \Phi\nu\right) < \Gamma\left(\Delta_0(\omega, \nu)\right) < \Delta_0\left(\omega, \nu\right).$$

Thus, we have

$$\begin{aligned}
\eta\left(\omega, \nu\right) &< \max\left\{\eta(\omega, \nu), \eta(\omega, \omega), \eta(\nu, \nu), \frac{\eta(\omega, \nu) + \eta(\nu, \omega)}{2}\right\} \\
&= \eta\left(\omega, \nu\right),
\end{aligned}$$

which is a contradiction. Hence, $\omega = \nu$.

Last part of the proof is trivial and hence omitted. $\square$

Below we provide an example to validate Theorem 2.1.

Example 2.2. Let $\mathcal{W} = [0, 4]$ be endowed with the usual metric η. Define $\Phi : \mathcal{W} \to \mathcal{W}$ by

$$\Phi\theta = \begin{cases} 2, & \text{if } \theta \leq 2 \\ 0, & \text{if } \theta > 2. \end{cases}$$

Then Φ satisfies condition (i) of Theorem 2.1 with

$$\Gamma s = \begin{cases} 1, & \text{if } s > 2 \\ \frac{s}{2}, & \text{if } s \leq 2. \end{cases}$$

Also, Φ satisfies condition (ii) of Theorem 2.1 with

$$\kappa(\epsilon) = \begin{cases} 1, & \text{if } \epsilon \geq 1 \\ 1 - \epsilon, & \text{if } \epsilon < 1. \end{cases}$$

We also observe that $\lim\limits_{\theta \to 2} \Delta_0(\theta, 2) \neq 0$ and Φ is discontinuous at the fixed point $\omega = 2$. However, $\Phi^2(\theta) = 2$ for all $\theta \in \mathcal{W}$ and hence, it is continuous (see Figure 1).

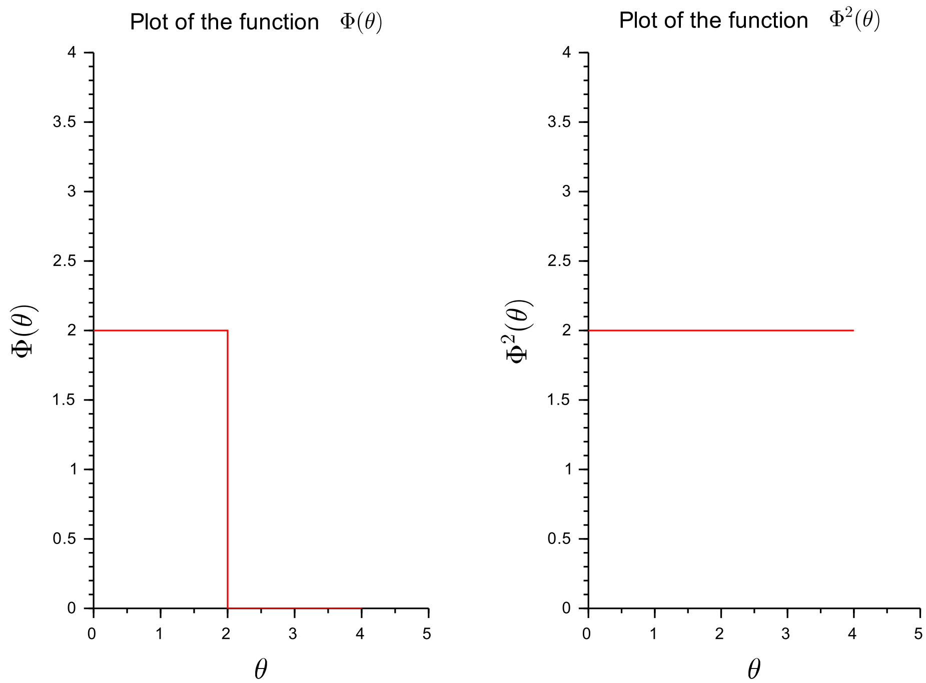

FIGURE 1. Plot of the functions Φ and Φ^2.

We have the following two consequences of Theorem 2.1.

Corollary 2.3. *Let $(\mathcal{W}, \eta)$ be a complete MS and $\Phi : \mathcal{W} \to \mathcal{W}$ be such that Φ^2 is continuous and there exists $\delta > 0$ and $F \in \mathcal{F}$ satisfying*

(i) $\delta + F(\eta(\Phi\theta, \Phi\xi)) \leq F(\Delta_0(\theta, \xi))$, *with* $\Delta_0(\theta, \xi) > 0$;
(ii) *for a given* $\epsilon > 0$, *there exists* $\kappa > 0$ *such that* $\epsilon < \Delta_0(\theta, \xi) < \epsilon + \kappa$ *implies that* $\eta(\Phi\theta, \Phi\xi) \leq \epsilon$.

Then, Φ possesses a unique fixed point, say ω and $\lim\limits_{n\to\infty} \Phi^n\theta \to \omega$ for each $\theta \in \mathcal{W}$. Moreover, Φ happens to be discontinuous at ω if and only if $\lim\limits_{\theta\to\omega} \Delta(\theta, \omega) \neq 0$.

Corollary 2.4. *Let $(\mathcal{W}, \eta)$ be a complete MS and $\Phi : \mathcal{W} \to \mathcal{W}$ be such that Φ^2 is continuous and there exists $\delta > 0$ and $F \in \mathcal{F}$ satisfying*

(i) $\delta + F(\eta(\Phi\theta, \Phi\xi)) \leq F(\Gamma(\eta(\theta, \xi)))$, *where* $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *has the property that* $\Gamma(\eta(\theta, \xi)) < \eta(\theta, \xi)$ *for all* $\theta, \xi \in \mathcal{W}$;
(ii) *for a given* $\epsilon > 0$, *there exists* $\kappa > 0$ *such that for* $s > 0$, $\epsilon < s < \epsilon + \kappa$ *implies that* $\Gamma(s) \leq \epsilon$.

Then, Φ possesses a unique fixed point, say ω and $\lim_{n\to\infty} \Phi^n\theta \to \omega$ for each $\theta \in \mathcal{W}$.

3. Discontinuity at Fixed Point Without Assuming Continuity of the Composition

In this section, we drop the condition that Φ^2 is continuous. We shall use the following notation, which was recently used by Ozgur and Tas [24] to describe some interesting fixed-circle problems.

$$\Delta_1(\theta, \xi) = \max\left\{\eta(\theta,\xi), \eta(\theta,\Phi\theta), \eta(\xi,\Phi\xi), \left[\frac{\eta(\theta,\Phi\xi)+\eta(\xi,\Phi\theta)}{1+\eta(\theta,\Phi\xi)+\eta(\xi,\Phi\theta)}\right]\eta(\theta,\xi)\right\}.$$

Theorem 3.1. *Let $(\mathcal{W},\eta)$ be a complete MS and $\Phi : \mathcal{W} \to \mathcal{W}$ be such that there exists $\delta > 0$ and $F \in \mathcal{F}$ satisfying*

- (i) *$\delta + F(\eta(\Phi\theta, \Phi\xi)) \leq F(\Gamma(\Delta_1(\theta, \xi)))$, where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ has the property $\Gamma(s) < s$ for each $s > 0$;*
- (ii) *for a given $\epsilon > 0$, there exists $\kappa > 0$ such that $\epsilon < \Delta_1(\theta, \xi) < \epsilon + \kappa$ implies that $\eta(\Phi\theta, \Phi\xi) \leq \epsilon$.*

Then, Φ possesses a fixed point, say ω and $\lim_{n\to\infty} \Phi^n\theta \to \omega$ for each $\theta \in \mathcal{W}$. Moreover, Φ happens to be discontinuous at ω if and only if $\lim_{\theta\to\omega} \Delta_1(\theta, \omega) \neq 0$.

Proof. We fix $\theta_0 \in \mathcal{W}$ such that $\theta_0 \neq \Phi\theta_0$ and construct the sequence $\{\theta_n\}$ by $\theta_{n+1} = \Phi^n\theta_0 = \Phi\theta_n$. We denote $d_n = \eta(\theta_n, \theta_{n+1})$. Then $d_n = \eta(\theta_n, \theta_{n+1}) = \eta(\Phi\theta_{n-1}, \Phi\theta_n)$.

Now, we have

$$\begin{aligned}
(3.1)\quad \Delta(\theta_{n-1}, \theta_n) &= \max\Bigg\{\eta(\theta_{n-1},\theta_n), \eta(\theta_{n-1},\Phi\theta_{n-1}), \eta(\theta_n,\Phi\theta_n), \\
&\qquad \left[\frac{\eta(\theta_{n-1},\Phi\theta_n)+\eta(\theta_n,\Phi\theta_{n-1})}{1+\eta(\theta_{n-1},\Phi\theta_{n-1})+\eta(\theta_n,\Phi\theta_n)}\right]\eta(\theta_{n-1},\theta_n)\Bigg\} \\
&= \max\Bigg\{\eta(\theta_{n-1},\theta_n), \eta(\theta_{n-1},\theta_n), \eta(\theta_n,\theta_{n+1}), \\
&\qquad \left[\frac{\eta(\theta_{n-1},\theta_{n+1})+\eta(\theta_n,\theta_n)}{1+\eta(\theta_{n-1},\theta_n)+\eta(\theta_n,\theta_{n+1})}\right]\eta(\theta_{n-1},\theta_n)\Bigg\} \\
&= \max\Bigg\{\eta(\theta_{n-1},\theta_n), \eta(\theta_n,\theta_{n+1}),
\end{aligned}$$

$$\left[\frac{\eta(\theta_{n-1},\theta_{n+1})}{1+\eta(\theta_{n-1},\theta_n)+\eta(\theta_n,\theta_{n+1})}\right]\eta(\theta_{n-1},\theta_n)\Bigg\}$$

$$= \max\Bigg\{\eta(\theta_{n-1},\theta_n),\eta(\theta_n,\theta_{n+1}),$$

$$\left[\frac{\eta(\theta_{n-1},\theta_n)+\eta(\theta_n,\theta_{n+1})}{1+\eta(\theta_{n-1},\theta_n)+\eta(\theta_n,\theta_{n+1})}\right]\eta(\theta_{n-1},\theta_n)\Bigg\}$$

$$= \max\left\{\eta(\theta_{n-1},\theta_n),\eta(\theta_n,\theta_{n+1})\right\}.$$

From (i) of the hypothesis, we have that

$$F(\eta(\Phi\theta,\Phi\xi)) \leq F(\Gamma(\Delta_1(\theta,\xi))), \quad \text{for } \delta > 0, \tag{3.2}$$

then

$$\eta(\Phi\theta,\Phi\xi) \leq \Gamma(\Delta_1(\theta,\xi)).$$

Putting $\theta = \theta_{n-1}$ and $\xi = \theta_n$ in (3.2), we have

$$\begin{aligned} \eta(\Phi\theta_{n-1},\Phi\theta_n) &\leq \Gamma(\Delta_1(\theta_{n-1},\theta_n)) \\ &\leq \Delta_1(\theta_{n-1},\theta_n). \end{aligned} \tag{3.3}$$

Now, if $\eta(\theta_{n-1},\theta_n) \leq \eta(\theta_n,\theta_{n+1})$, from (3.3) and (3.1), we obtain $\eta(\Phi\theta_{n-1},\Phi\theta_n) < \eta(\theta_n,\theta_{n+1})$, i.e., $\eta(\theta_n,\theta_{n+1}) < \eta(\theta_n,\theta_{n+1})$, which is a contradiction.

Hence, we must have

$$\eta(\theta_n,\theta_{n+1}) < \eta(\theta_{n-1},\theta_n).$$

Thus, from (3.1), we have $d_n < d_{n-1}$, i.e., $\{d_n\}$ is a strictly decreasing sequence of positive reals. Hence, $d_n \to d$ for some $d \in \mathbb{R}$ as $n \to \infty$. Using similar arguments as in the proof of Theorem 2.1, we can prove that $d = 0$.

Next, using exactly similar techniques as in [24] we can show that $\{\theta_n\}$ is Cauchy.

Since $(\mathcal{W},\eta)$ is complete, there exists $\omega \in \mathcal{W}$ such that $\theta_n \to \omega$ as $n \to \infty$. Also, we have $\Phi\theta_n \to \omega$ $\theta_n \to \omega$ as $n \to \infty$. If $\Phi\omega \neq \omega$, then

$$\delta + F(\eta(\Phi\omega,\Phi\theta_n)) \leq F(\Gamma(\Delta_1(\omega,\theta_n))),$$

then

$$F(\eta(\Phi\omega,\Phi\theta_n)) \leq F(\Gamma(\Delta_1(\omega,\theta_n))),$$

then

$$\begin{aligned} \eta(\Phi\omega,\Phi\theta_n) &\leq \Gamma(\Delta(\omega,\theta_n)) \\ &< \Delta_1(\omega,\theta_n) \\ &= \max\Bigg\{\eta(\omega,\theta_n),\eta(\omega,\Phi\omega), \end{aligned}$$

$$\left.\left[\frac{\eta(\omega,\Phi\theta_n)+\eta(\theta_n,\Phi\omega)}{1+\eta(\omega,\Phi\omega)+\eta(\theta_n,\Phi\theta_n)}\right]\eta(\omega,\theta_n)\right\}.$$

Letting $n \to \infty$, in the last inequality, we have

$$\eta(\Phi\omega,\omega) < \eta(\omega,\Phi\omega),$$

which is a contradiction. Hence, we must have $\Phi\omega = \omega$.

Last part of the proof is obvious. □

We have the following consequences of Theorem 3.1.

Corollary 3.2. *Let $(\mathcal{W},\eta)$ be a complete MS and $\Phi : \mathcal{W} \to \mathcal{W}$ be such that there exists $\delta > 0$ and $F \in \mathcal{F}$ satisfying*

(i) *$\delta + F(\eta(\Phi\theta,\Phi\xi)) \leq F(\Delta_1(\theta,\xi))$, for any $\theta,\xi \in \mathcal{W}$ with $\Delta_1(\theta,\xi) > 0$;*

(ii) *for a given $\epsilon > 0$, there exists $\kappa > 0$ such that $\epsilon < \Delta_1(\theta,\xi) < \epsilon + \kappa$ implies that $\eta(\Phi\theta,\Phi\xi) \leq \epsilon$.*

Then, Φ possesses a fixed point, say ω and $\lim_{n\to\infty} \Phi^n\theta \to \omega$ for each $\theta \in \mathcal{W}$. Moreover, Φ happens to be discontinuous at ω if and only if $\lim_{\theta\to\omega} \Delta_1(\theta,\omega) \neq 0$.

Corollary 3.3. *Let $(\mathcal{W},\eta)$ be a complete MS and $\Phi : \mathcal{W} \to \mathcal{W}$ be such that there exist $\delta > 0$ and $F \in \mathcal{F}$ satisfying*

(i) *$\delta + F(\eta(\Phi\theta,\Phi\xi)) \leq F(\Gamma(\eta(\theta,\xi)))$, where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ has the property that $\Gamma(\eta(\theta,\xi)) < \eta(\theta,\xi)$ for each $\eta(\theta,\xi) > 0$;*

(ii) *for a given $\epsilon > 0$, there exists $\kappa > 0$ such that for $s > 0$, $\epsilon < s < \epsilon + \kappa$ implies that $\Gamma(s) \leq \epsilon$.*

Then, Φ possesses a fixed point, say ω and $\lim_{n\to\infty} \Phi^n\theta \to \omega$ for each $\theta \in \mathcal{W}$.

4. Conclusion and Future Work

In this paper, we have established some new results on discontinuity at fixed points using F-contractive definitions. In [2], Bisht and Pant explained real-life situations where such discontinuity results could be applied. The well-known McCulloch-Pitts model is prevalent in Artificial Intelligence and Biology. This model devises particular algorithms for neural networks to reduce neuron deviation from its limiting equilibrium state. This type of equilibrium can be modeled using fixed points of specific mappings. All derived functions obtained by such a process display discontinuity caused by a jump in the biological operations, such as the threshold frequency. As such, applying of our results to neural networks applying feasible conditions is a suggested future work. The works listed in [7, 8, 34] are referred to for details of such models. Obtaining multivalued analogs of the current results using the framework as in [10, 11] is also some interesting suggested future work.

Acknowledgment. The author thanks the reviewers for their excellent constructive comments which resulted in the improvement of the manuscript.

References

1. P. Amiri and M.E. Samei, *Existence of Urysohn and Atangana-Baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators*, Chaos Solitons Fract., 165 (2022), Article ID: 112822.
2. R.K. Bisht and R.P. Pant, *A remark on discontinuity at fixed point*, J. Math. Anal. Appl., 445 (2017), pp. 1239-1242.
3. R.K. Bisht and V.Rakocević, *Generalized Meir-Keeler type contractions and discontinuity at fixed point*, Fixed Point Theory, 19 (2018), pp. 57-64.
4. D.W. Boyd and J.S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., 20 (1969), pp. 458-464.
5. L. Ćirić, *On contraction type mappings*, Math. Balkanica, 1 (1971), pp. 52-57.
6. L. Ćirić, *A generalization of Banach's contraction principle*, Proc. Math. Amer. Math. Soc., 45 (1974), pp. 267-273.
7. L.J. Cromme, *Fixed point theorems for discontinuous functions and applications*, Nonlinear Anal., 30 (1997), pp. 1527-1534.
8. L.J. Cromme and I. Diener, *Fixed point theorems for discontinuous functions and applications*, Math. Program., 51 (1991), pp. 257-267.
9. A. Das, S.A. Mohiuddine, A. Alotaibi and B.C. Deuri, *Generalization of Darbo-type theorem and application on existence of implicit fractional integral equations in tempered sequence spaces*, Alexandria Eng. J., 61 (2022), pp. 2010-2015.
10. P. Debnath, *Optimization through best proximity points for multivalued F-contractions*, Miskolc Math. Notes, 22 (2021), pp. 143-151.
11. P. Debnath, *Banach, Kannan, Chatterjea and Reich-type contractive inequalities for multivalued mappings and their common fixed points*, Math. Meth. Appl. Sci., 45 (2022), pp. 1587-1596.
12. P. Debnath, N. Konwar and S. Radenović, *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences*, Springer, Singapore, 2021.

13. P. Debnath, Z.D. Mitrović and H.M. Srivastava, *Fixed points of some asymptotically regular multivalued mappings satisfying a Kannan-type condition*, Axioms, 10 (2021), Article ID: 24.
14. P. Debnath and H.M. Srivastava, *Global optimization and common best proximity points for some multivalued contractive pairs of mappings*, Axioms, 9 (2020), Article ID: 102.
15. P. Debnath and H.M. Srivastava, *New extensions of Kannan's and Reich's fixed point theorems for multivalued maps using Wardowski's technique with application to integral equations*, Symmetry, 12 (2020), Article ID: 1090.
16. S. Etemad, S. Rezapour and M.E. Samei, *α-ψ-contractions and solutions of a q-fractional differential inclusion with three-point boundary value conditions via computational results*, Adv. Differ. Equ., 42 (2020), Article ID: 218.
17. M. Forti and P. Nistri, *Global convergence of neural networks with discontinuous neuron activations*, IEEE Trans. Circuits Syst. I, Fundam. Theory Appl., 50 (2003), pp. 1421-1435.
18. R. Kannan, *Some results on fixed points*, Bull. Calc. Math. Soc., 60 (1968), pp. 71-77.
19. R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly, 76 (1969), pp. 405-408.
20. S.A. Mohiuddine, A. Das and A. Alotaibi, *Existence of solutions for nonlinear integral equations in tempered sequence spaces via generalized Darbo-type theorem*, J. Funct. Spaces, 2022 (2022), Article ID: 4527439.
21. X. Nie and W.X. Zheng, *On multistability of competitive neural networks with discontinuous activation functions*, 4th Australian Control Conference (AUCC), 2014, pp. 245-250.
22. X. Nie and W.X. Zheng, *Multistability of neural networks with discontinuous non-monotonic piecewise linear activation functions and time-varying delays*, Neural Networks, 65 (2015), pp. 65-79.
23. X. Nie and W.X. Zheng, *Multistability of neural networks with discontinuous non-monotonic piecewise linear activation functions and time-varying delays*, IEEE Transactions On Cybernatics, 46 (2015), pp. 679-693.
24. N. Ozgur and N. Tas, *New discontinuity results at fixed point on metric spaces*, J. Fixed Point Theory Appl., 23 (2021).
25. R. Pant, *Discontinuity and fixed points*, J. Math. Anal. Appl., 240 (1999), pp. 284-289.
26. R.P. Pant, N. Ozgur and N. Tas, *On discontinuity problem at fixed point*, Bull. Malays. Math. Sci. Soc., 43 (2020), pp. 499-517.

27. B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226 (1977), pp. 257-290.
28. B.E. Rhoades, *Contractive definitions and continuity*, Contemp. Math., 42 (1988), pp. 233-245.
29. M.E. Samei, *Some fixed point results on intuitionistic fuzzy metric spaces with a graph*, Sahand Commun. Math. Anal., 13 (2019), pp. 141-152.
30. M.E. Samei, *Convergence of an iterative scheme for multifunctions on fuzzy metric spaces*, Sahand Commun. Math. Anal., 15 (2019), pp. 91-106.
31. H.M. Srivastava, A.M.A. El-Sayed, H.H.G. Hashem and Sh.M. Al-Issa, *Analytical investigation of nonlinear hybrid implicit functional differential inclusions of arbitrary fractional orders*, RACSAM, 2022 (2015), Article ID: 26, pp. 1-19.
32. H.M. Srivastava, A. Shehata and S.I. Moustafa, *Some fixed point theorems for $f(\psi, \varphi)$-contractions and their application to fractional differential equations*, Russian J. Math. Phys., 27 (2020), pp. 385-398.
33. N. Tas and N. Ozgur, *A new contribution to discontinuity at fixed point*, Fixed Point Theory, 20 (2019), pp. 715-728.
34. M.J. Todd, *The Computation of Fixed Points and Applications*, Springer-Verlag, New York (1976).
35. D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric space*, Fixed Point Theory Appl., 2012 (2012), Article ID: 94.
36. H. Wu and C. Shan, *Stability analysis for periodic solution of BAM neural networks with discontinuous neuron activations and impulses*, Appl. Math. Modelling, 33 (2017), pp. 2564-2574.

Department of Applied Science and Humanities, Assam University, Silchar, Cachar, Assam - 788011, India.

Email address: debnath.pradip@yahoo.com; pradip.debnath@aus.ac.in