Further Operator and Norm Versions of Young Type Inequalities

Leila Nasiri and Mehdi Shams

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 4
Pages: 33-46

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.562013.1174

Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 4 (2023), 33-46
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.562013.1174

Further Operator and Norm Versions of Young Type Inequalities

Leila Nasiri[1]* and Mehdi Shams[2]

Abstract. In this note, first the better refinements of Young and its reverse inequalities for scalars are given. Then, several operator and norm versions according to these inequalities are established.

1. Introduction

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices and let $\|.\|$ denote any unitarily invariant norm on $\mathbb{M}_n$. So, $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and for all unitary matrices $U, V \in \mathbb{M}_n$. For $A = (a_{ij}) \in \mathbb{M}_n$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2 (A)},$$

where $s_1 (A) \geq s_2 (A) \geq \cdots \geq s_n (A)$ are the singular values of A, that is, the eigenvalues of the positive matrix. $|A| = (A^* A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. It is trivial that $\|.\|_2$ has the unitarily invariant property:

$$\|U_1 A U_2\|_2 = \|A\|_2 \,,$$

for all $A \in \mathbb{M}_n$ and all unitary matrices $U_1, U_2 \in \mathbb{M}_n$. The C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ define by $\mathbb{B}(\mathcal{H})$ (so that $||.||$ is the operator norm). We say an operator $A \in \mathbb{B}(\mathcal{H})$ is positive if $A \geq 0$. Also, $A \geq B \, (A \leq B)$ if $A - B \geq 0 \; (0 \leq B - A)$. The adjoint of the operator A and its absolute value is defined respectively

2010 *Mathematics Subject Classification.* 15A45, 15A60, 47A30.
Key words and phrases. Young inequality, Unitarily invariant norms, Positive operators and matrices.
Received: 12 September 2022, Accepted: 23 January 2023.
* Corresponding author.

by A^* by $|A|$, that $|A| = (A^*A)^{\frac{1}{2}}$. The famous Young inequality for scalars says that if $a, b \geq 0$ and $0 \leq \nu \leq 1$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b, \tag{1.1}$$

with equality holding if and only if $a = b$. For $\nu = \frac{1}{2}$, we get

$$\sqrt{ab} \leq \frac{a+b}{2},$$

which is called the fundamental arithmetic-geometric mean inequality. Kittaneh and Manasrah [8, 9] showed that for $a, b \geq 0$, $0 \leq \nu \leq 1$,

$$\begin{aligned} a^{1-\nu}b^{\nu} + r\left(\sqrt{a} - \sqrt{b}\right)^2 &\leq (1-\nu)\,a + \nu b \\ &\leq a^{1-\nu}b^{\nu} + s\left(\sqrt{a} - \sqrt{b}\right)^2, \end{aligned} \tag{1.2}$$

where $r = \min\{\nu, 1-\nu\}$ and $s = \max\{\nu, 1-\nu\}$ which the left-hand side of (1.2) is a refinement of (1.1) and the right-hand side of (1.2) is a improved reverse of (1.1). Hirzallah and Kittaneh [7] and He and Zou [6] showed that if $a, b \geq 0$, $0 \leq \nu \leq 1$, then, we have

$$\begin{aligned} \left(a^{1-\nu}b^{\nu}\right)^2 + r^2\,(a-b)^2 &\leq ((1-\nu)\,a + \nu b)^2 \\ &\leq \left(a^{1-\nu}b^{\nu}\right)^2 + s^2\,(a-b)^2, \end{aligned} \tag{1.3}$$

where $r = \min\{\nu, 1-\nu\}$ and $s = \max\{\nu, 1-\nu\}$. It should be mentioned that (1.3) refine (1.1) and it's reverse. After a short time, Zhao and Wu [14] presented further refinements and improvements of (1.1) and (1.2) as:

$$\begin{aligned} &r_0\left(\sqrt[4]{ab} - \sqrt{a}\right)^2 + \nu\left(\sqrt{a} - \sqrt{b}\right)^2 + a^{1-\nu}b^{\nu} \\ &\quad\leq (1-\nu)\,a + \nu b \\ &\quad\leq a^{1-\nu}b^{\nu} + (1-\nu)\left(\sqrt{a} - \sqrt{b}\right)^2 - r_0\left(\sqrt[4]{ab} - \sqrt{b}\right)^2, \quad 0 \leq \nu \leq \frac{1}{2}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} r_0&\left(\sqrt[4]{ab} - \sqrt{b}\right)^2 + (1-\nu)\left(\sqrt{a} - \sqrt{b}\right)^2 + a^{1-\nu}b^{\nu} \\ &\leq (1-\nu)\,a + \nu b \\ &\leq a^{1-\nu}b^{\nu} + \nu\left(\sqrt{a} - \sqrt{b}\right)^2 - r_0\left(\sqrt[4]{ab} - \sqrt{a}\right)^2, \quad \frac{1}{2} \leq \nu \leq 1, \end{aligned} \tag{1.5}$$

where $a, b \geq 0, r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$. They also obtained the following inequalities, that are the refinements of the two sides inequalities in (1.3).

(1.6)
$$\begin{aligned}\left(a^{1-\nu}b^{\nu}\right)^2+\nu^2(a-b)^2+r_0\left(\sqrt{ab}-a\right)^2 &\le ((1-\nu)a+\nu b)^2\\ &\le \left(a^{1-\nu}b^{\nu}\right)^2+(1-\nu)^2(a-b)^2\\ &\quad -r_0\left(\sqrt{ab}-b\right)^2,\quad 0\le\nu\le\frac{1}{2},\end{aligned}$$

and

(1.7)
$$\begin{aligned}\left(a^{1-\nu}b^{\nu}\right)^2+(1-\nu)^2(a-b)^2+r_0\left(\sqrt{ab}-b\right)^2 &\le ((1-\nu)a+\nu b)^2\\ &\le \left(a^{1-\nu}b^{\nu}\right)^2+\nu^2(a-b)^2\\ &\quad -r_0\left(\sqrt{ab}-a\right)^2,\\ &\quad \frac{1}{2}\le\nu\le 1,\end{aligned}$$

where, $a,b\ge 0$, $r=\min\{\nu,1-\nu\}$ and $r_0=\min\{2r,1-2r\}$.

For two positive invertible operators $A,B\in\mathbb{B}(\mathcal{H})$ and $0\le\nu\le 1$, $\nu-$weighted arithmetic, geometric and harmonic means, are denoted, respectively by $A\nabla_\nu B$, $A\sharp_\nu B$ and $A!_\nu B$ and are defined respectively as [11]:

$$A\bigtriangledown_\nu B=(1-\nu)A+\nu B,$$
$$A\sharp_\nu B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}},$$

and

$$A!_\nu B=\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}.$$

In case $\nu=\frac{1}{2}$, we write $A\bigtriangledown B, A\sharp B$ and $A!B$, respectively. Furuta and Yanagida [3] and Furuta [4] proved that the following inequalities hold:

$$A!_\nu B\le A\sharp_\nu B\le A\bigtriangledown B, \tag{1.8}$$

where $0\le\nu\le 1$ and A and B are two positive invertible operators. The inequalities (1.8) are considered as the Heinz operator inequalities. In [8, 9] Kittaneh and Manasrah obtained the matrix versions of (1.2) in following form: If $B,C\in\mathbb{M}_n$ so that B is positive definite, C is invertible, $A=C^*C$, then

(1.9)
$$\begin{aligned}2s\left[A\bigtriangledown B-C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C\right] &\ge A\bigtriangledown_\nu B-C^*\left(C^{*-1}BC^{-1}\right)^{\nu}C\\ &\ge 2r\left[A\bigtriangledown B-C^*\left(C^{*-1}BC^{-1}\right)^{\frac{1}{2}}C\right],\end{aligned}$$

where $0 \leq \nu \leq 1, r = \min\{\nu, 1-\nu\}$ and $s = \max\{\nu, 1-\nu\}$. Zhao and Wu [14] extended numerical inequalities (1.4) and (1.5) to operator versions and improved the double inequalities in (1.9). Hirzallah and Kittaneh [7] and Kittaneh and Manasrah [8], respectively proved that if $A, B, X \in \mathbb{M}_n$ so that A and B are positive semidefinite, then

$$\left\|A^{1-\nu}XB^{\nu}\right\|_2^2 + r^2 \left\|AX - XB\right\|_2^2 \leq \left\|(1-\nu)AX + \nu XB\right\|_2^2, \tag{1.10}$$

and

$$\left\|(1-\nu)AX + \nu XB\right\|_2^2 \leq \left\|A^{1-\nu}XB^{\nu}\right\|_2^2 + s^2 \left\|AX - XB\right\|_2^2, \tag{1.11}$$

where, $0 \leq \nu \leq 1, r = \min\{\nu, 1-\nu\}$ and $s = \max\{\nu, 1-\nu\}$. Zhao and Wu presented in [14] norm versions of (1.6) and (1.7) for Hilbert-Schmidt norms, which are better than inequality (1.10) and (1.11). For further information about the Young inequality refer to [1, 2, 10, 12, 13]. The paper has four sections. Section 1 is devoted to the introduction. In section 2, we give some refinements for Young and its reverse inequalities. In section 3, we study the operator versions of these inequalities. In the end, matrix versions according the Hilbert-Schmidt norm are stated.

2. More Inequalities for Scalars

We begin this section with several theorems which refine the double inequalities (1.4)-(1.7). The main result of this section is the following, which are better than (1.4) and (1.5), respectively.

Theorem 2.1. *Let $a, b \geq 0$ and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If $0 \leq \nu \leq \frac{1}{4}$, then*

$$\begin{aligned}(1-\nu)a + \nu b \geq{}& a^{1-\nu}b^{\nu} + \nu\left(\sqrt{a} - \sqrt{b}\right)^2 \\ &+ 2\nu\left(\sqrt{a} - \sqrt[4]{ab}\right)^2 + r_0\left(\sqrt[4]{a\sqrt{ab}} - \sqrt{a}\right)^2,\end{aligned} \tag{2.1}$$

(ii) *If $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, then*

(2.2)

$$\begin{aligned}(1-\nu)a + \nu b \geq{}& a^{1-\nu}b^{\nu} + \nu\left(\sqrt{a} - \sqrt{b}\right)^2 + (1-2\nu)\left(\sqrt{a} - \sqrt[4]{ab}\right)^2 \\ &+ r_0\left(\sqrt[4]{a\sqrt{ab}} - \sqrt[4]{ab}\right)^2,\end{aligned}$$

where $r = \min\{2\nu, 1-2\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof. Firstly, we suppose $0 \le \nu \le \frac{1}{4}$. By applying (1.4), we have

$$\begin{aligned}(1-\nu)\,a+\nu b-\nu\left(\sqrt{a}-\sqrt{b}\right)^2 &= (1-2\nu)\,a+2\nu\sqrt{ab}\\ &\ge a^{1-\nu}b^{\nu}+2\nu\left(\sqrt{a}-\sqrt[4]{ab}\right)^2\\ &\quad+r_0\left(\sqrt[4]{a\sqrt{ab}}-\sqrt{a}\right)^2.\end{aligned}$$

That is,

$$\begin{aligned}(1-\nu)\,a+\nu b &\ge a^{1-\nu}b^{\nu}+\nu\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\quad+2\nu\left(\sqrt{a}-\sqrt[4]{ab}\right)^2+r_0\left(\sqrt[4]{a\sqrt{ab}}-\sqrt{a}\right)^2.\end{aligned}$$

Thus, (2.1) holds. Now, we suppose $\frac{1}{4} \le \nu \le \frac{1}{2}$. By (1.5), it follows that

$$\begin{aligned}(1-\nu)\,a+\nu b-\nu\left(\sqrt{a}-\sqrt{b}\right)^2 &= (1-2\nu)\,a+2\nu\sqrt{ab}\\ &\ge a^{1-\nu}b^{\nu}+(1-2\nu)\left(\sqrt{a}-\sqrt[4]{ab}\right)\\ &\quad+r_0\left(\sqrt[4]{a\sqrt{ab}}-\sqrt[4]{ab}\right)^2.\end{aligned}$$

i.e.,

$$\begin{aligned}(1-\nu)\,a+\nu b &\ge a^{1-\nu}b^{\nu}+\nu\left(\sqrt{a}-\sqrt{b}\right)^2\\ &\quad+(1-2\nu)\left(\sqrt{a}-\sqrt[4]{ab}\right)^2+r_0\left(\sqrt[4]{a\sqrt{ab}}-\sqrt{ab}\right)^2.\end{aligned}$$

Consequently, (2.2) holds. This completes the proof. ☐

Remark 2.2. With replacing a and b by their squares in (2.1) and (2.2), respectively, we have

$$\begin{aligned}(1-\nu)\,a^2+\nu b^2 &\ge \left(a^{1-\nu}b^{\nu}\right)^2+\nu\,(a-b)^2+2\nu\left(a-\sqrt{ab}\right)^2\\ &\quad+r_0\left(\sqrt[4]{a^3b}-a\right)^2,\quad 0\le\nu\le\frac{1}{4},\end{aligned}\tag{2.3}$$

$$\begin{aligned}(1-\nu)\,a^2+\nu b^2 &\ge \left(a^{1-\nu}b^{\nu}\right)^2+\nu\,(a-b)^2+(1-2\nu)\left(a-\sqrt{ab}\right)^2\\ &\quad+r_0\left(\sqrt[4]{a^3b}-\sqrt{ab}\right)^2,\quad \frac{1}{4}\le\nu\le\frac{1}{2},\end{aligned}\tag{2.4}$$

With considering (2.3) and (2.4), we get to the folowing result:

Corollary 2.3. *Let $a, b \geq 0$ and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If $0 \leq \nu \leq \frac{1}{4}$, then*

$$((1-\nu)a+\nu b)^2 \geq \left(a^{1-\nu}b^{\nu}\right)^2+\nu^2(a-b)^2+2\nu\left(a-\sqrt{ab}\right)^2 + r_0\left(\sqrt[4]{a^3b}-a\right)^2, \tag{2.5}$$

(ii) *If $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, then*

$$((1-\nu)a+\nu b)^2 \geq \left(a^{1-\nu}b^{\nu}\right)^2+\nu^2(a-b)^2+(1-2\nu)\left(a-\sqrt{ab}\right)^2 + r_0\left(\sqrt[4]{a^3b}-\sqrt{ab}\right)^2, \tag{2.6}$$

where $r = \min\{2\nu, 1-2\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof. If $0 \leq \nu \leq \frac{1}{4}$. Then, we have from (2.3):

$$\begin{aligned} ((1-\nu)a+\nu b)^2-\nu^2(a-b)^2 &= (1-\nu)a^2+\nu b^2-\nu(a-b)^2 \\ &\geq \left(a^{1-\nu}b^{\nu}\right)^2+2\nu\left(a-\sqrt{ab}\right)^2 \\ &\quad + r_0\left(\sqrt[4]{a^3b}-a\right)^2. \end{aligned}$$

Hence,

$$((1-\nu)a+\nu b)^2 \geq \left(a^{1-\nu}b^{\nu}\right)^2+\nu^2(a-b)^2+2\nu\left(a-\sqrt{ab}\right)^2 + r_0\left(\sqrt[4]{a^3b}-a\right)^2.$$

Thus, (2.5) holds. By using (2.4) and similar calculations, we are able to obtain (2.6). □

Remark 2.4. Clearly, (2.5) and (2.6) are improvements of the first inequality in (1.6) and (1.7).

Theorem 2.5 is a refined reverse of (1.1) which refines the right-hand side of (1.4) and (1.5), respectively.

Theorem 2.5. *Let $a, b \geq 0$ and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If $0 \leq \nu \leq \frac{1}{4}$, then*

$$(1-\nu)a+\nu b \leq a^{1-\nu}b^{\nu}+(1-\nu)\left(\sqrt{a}-\sqrt{b}\right)^2 - 2\nu\left(\sqrt{b}-\sqrt[4]{ab}\right)^2 - r_0\left(\sqrt[4]{b\sqrt{ab}}-\sqrt{b}\right)^2, \tag{2.7}$$

(ii) *If* $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, *then*

(2.8)
$$(1-\nu)\,a+\nu b \leq a^{1-\nu}b^{\nu}+(1-\nu)\left(\sqrt{a}-\sqrt{b}\right)^2 \\ -(1-2\nu)\left(\sqrt{b}-\sqrt[4]{ab}\right)^2 - r_0\left(\sqrt[4]{b\sqrt{ab}}-\sqrt[4]{ab}\right)^2,$$

where $r=\min\{2\nu,1-2\nu\}$ *and* $r_0=\min\{2r,1-2r\}$.

Proof. Let $0 \leq \nu \leq \frac{1}{4}$. Then, by (1.4), we have

$$\begin{aligned} a^{1-\nu}b^{\nu} &+ (1-\nu)\left(\sqrt{a}-\sqrt{b}\right)^2-(1-\nu)\,a-\nu b \\ &= a^{1-\nu}b^{\nu}+(1-2\nu)\,b+2\nu\sqrt{ab}-2\sqrt{ab} \\ &\geq a^{1-\nu}b^{\nu}+b^{1-\nu}a^{\nu}+2\nu\left(\sqrt{b}-\sqrt[4]{ab}\right)^2+r_0\left(\sqrt[4]{b\sqrt{ab}}-\sqrt{b}\right)^2-2\sqrt{ab} \\ &\geq 2\sqrt{ab}+2\nu\left(\sqrt{b}-\sqrt[4]{ab}\right)^2+r_0\left(\sqrt[4]{b\sqrt{ab}}-\sqrt{b}\right)^2-2\sqrt{ab} \\ &= 2\nu\left(\sqrt{b}-\sqrt[4]{ab}\right)^2+r_0\left(\sqrt[4]{b\sqrt{ab}}-\sqrt{b}\right)^2. \end{aligned}$$

Therefore,

$$(1-\nu)\,a+\nu b \leq a^{1-\nu}b^{\nu}+(1-\nu)\left(\sqrt{a}-\sqrt{b}\right)^2-2\nu\left(\sqrt{b}-\sqrt[4]{ab}\right)^2 \\ - r_0\left(\sqrt[4]{b\sqrt{ab}}-\sqrt{b}\right)^2.$$

This estimate completes the proof of (2.7). The proof (2.8) is similar to inequality (2.7) by applying (1.5). □

Remark 2.6. By setting a and b by their squares in Theorems 2.2 respectively, we get

(2.9)
$$(1-\nu)\,a^2+\nu b^2 \leq \left(a^{1-\nu}b^{\nu}\right)^2+(1-\nu)\,(a-b)^2-2\nu\left(b-\sqrt{ab}\right)^2 \\ - r_0\left(\sqrt[4]{b^3a}-b\right)^2,$$

and

(2.10)
$$(1-\nu)\,a^2+\nu b^2 \leq \left(a^{1-\nu}b^{\nu}\right)^2+(1-\nu)\,(a-b)^2-(1-2\nu)\left(b-\sqrt{ab}\right)^2 \\ - r_0\left(\sqrt[4]{b^3a}-\sqrt{ab}\right)^2.$$

With applying (2.9) and (2.10), we obtain the following result that is a refinement of the second inequality in (1.6) and (1.7).

Corollary 2.7. *Let $a, b \geq 0$ and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If $0 \leq \nu \leq \frac{1}{4}$, then*

(2.11)
$$((1-\nu)a + \nu b)^2 \leq \left(a^{1-\nu}b^{\nu}\right)^2 + (1-\nu)^2 (a-b)^2 - 2\nu\left(b - \sqrt{ab}\right)^2 - r_0\left(\sqrt[4]{b^3 a} - b\right)^2,$$

(ii) *If $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, then*

(2.12)
$$((1-\nu)a + \nu b)^2 \leq \left(a^{1-\nu}b^{\nu}\right)^2 + (1-\nu)^2 (a-b)^2 - (1-2\nu)\left(b - \sqrt{ab}\right)^2 - r_0\left(\sqrt[4]{b^3 a} - \sqrt{ab}\right)^2,$$

where $r = \min\{2\nu, 1-2\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Proof. If $0 \leq \nu \leq \frac{1}{4}$. Then, we have (2.9)

$$\begin{aligned}((1-\nu)a + \nu b)^2 - (1-\nu)^2 (a-b)^2 &= (1-\nu)a^2 + \nu b^2 - (1-\nu)(a-b)^2 \\ &\leq \left(a^{1-\nu}b^{\nu}\right)^2 - 2\nu\left(b - \sqrt{ab}\right)^2 \\ &\quad - r_0\left(\sqrt[4]{b^3 a} - b\right)^2.\end{aligned}$$

by using from inequality (2.10) and the same technique as in (2.11), one can prove (2.12). Corollary 2.7 is proved. □

3. Some Inequalities for Operators

Our aim in this section is to present operator versions of the refined Young and its reverse inequalities according to the monotonicity property of operator functions. So, we need the following lemma to prove the main theorems (for more details one can see [5]).

Lemma 3.1. *Let $X \in \mathbb{B}(\mathcal{H})$ be hermitian and let f and g be continuous real functions such that $f(t) \geq g(t)$ on $Sp(X)$ the spectrum of (X). Then $f(X) \geq g(X)$.*

Now, we are ready to present operator versions of (2.1), (2.1), (2.7) and (2.8). We will prove the following theorem.

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be two positive invertible operators and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If* $0 \leq \nu \leq \frac{1}{4}$, *then*

$$A \bigtriangledown_\nu B \geq A\sharp_\nu B + 2\nu\left(A \bigtriangledown B - A\sharp B\right) + 2\nu\left(A + A\sharp B - 2\left(A\sharp_{\frac{1}{4}} B\right)\right) + r_0\left(A + A\sharp_{\frac{1}{4}} B - 2\left(A\sharp_{\frac{1}{8}} B\right)\right), \tag{3.1}$$

(ii) *If* $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, *then*

$$A \bigtriangledown_\nu B \geq B\sharp_{1-\nu} A + 2\upsilon\left(A \bigtriangledown B - B\sharp A\right) + (1-2\nu)\left(B\sharp A + B - 2\left(B\sharp_{\frac{3}{4}} A\right)\right) + r_0\left(B\sharp A + B\sharp_{\frac{3}{4}} A - 2\left(B\sharp_{\frac{5}{8}} A\right)\right), \tag{3.2}$$

where $r = \min\{2\nu, 1-2\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Proof. Letting $0 \leq \nu \leq \frac{1}{4}$. Then by putting $a = 1$ in (2.1), we obtain

$$(1-\nu) + \nu b \geq b^\nu + \nu\left(1 + b - 2\sqrt{b}\right) + 2\nu\left(\sqrt{b} + 1 - 2\sqrt[4]{b}\right) + r_0\left(1 + \sqrt[4]{b} - 2\sqrt[8]{b}\right), \tag{3.3}$$

for every $b > 0$. By lemma 3.1, for any positive operator X, (3.3) holds. Therefore, we have

$$(1-\nu)\, Id + \nu X \geq X^\nu + \nu\left(Id + X - 2\sqrt{X}\right) + 2\nu\left(\sqrt{X} + Id - 2\sqrt[4]{X}\right) + r_0\left(Id + \sqrt[4]{X} - 2\sqrt[8]{X}\right). \tag{3.4}$$

By putting $X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ in (3.4), we get

$$\begin{aligned}(1-\nu)\, Id + \nu\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) \geq{}& \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\nu \\ &+ \nu\left(Id + \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) - 2\sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}}\right) \\ &+ 2\nu\left(\sqrt{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} + Id - 2\sqrt[4]{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}}\right) \\ &+ r_0\left(Id + \sqrt[4]{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}} - 2\sqrt[8]{A^{-\frac{1}{2}} B A^{-\frac{1}{2}}}\right).\end{aligned} \tag{3.5}$$

By multiplying both sides of (3.5) by $A^{\frac{1}{2}}$, we have

$$A \bigtriangledown_\nu B \geq A\sharp_\nu B + 2\nu\left(A \bigtriangledown B - A\sharp B\right) + 2\nu\left(A + A\sharp B - 2\left(A\sharp_{\frac{1}{4}} B\right)\right)$$

$$+ r_0 \left(A + A\sharp_{\frac{1}{4}} B - 2 \left(A\sharp_{\frac{1}{8}} B \right) \right).$$

Now, let $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, then by replacing $b = 1$ in (2.2), we have

(3.6)
$$\begin{aligned}(1-\nu)\, a + \nu \geq a^{1-\nu} + \nu \left(1 + a - 2\sqrt{a}\right) + (1-2\nu) \left(\sqrt{a} + a - 2\sqrt[4]{a^3}\right) \\ + r_0 \left(\sqrt{a} + \sqrt[4]{a^3} - 2\sqrt[8]{a^5}\right),\end{aligned}$$

for any $a > 0$. By lemma 3.1, for any positive operator X, (3.6) holds. Thus,

$$\begin{aligned}(1-\nu)\, X + \nu Id \geq X^{1-\nu} &+ \nu \left(Id + X - 2\sqrt{X}\right) \\ &+ (1-2\nu) \left(\sqrt{X} + X - 2\sqrt[4]{X^3}\right) \\ &+ r_0 \left(\sqrt{X} + \sqrt{X^3} - 2\sqrt[8]{X^5}\right).\end{aligned} \tag{3.7}$$

By insertting $X = B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ in (3.7) and then multiplying both sides by $B^{\frac{1}{2}}$, we get,

$$\begin{aligned}A \bigtriangledown_\nu B \geq B\sharp_{1-\nu} A &+ 2\upsilon \left(A \bigtriangledown B - B\sharp A\right) \\ &+ (1-2\nu) \left(B\sharp A + B - 2 \left(B\sharp_{\frac{3}{4}} A\right)\right) \\ &+ r_0 \left(B\sharp A + B\sharp_{\frac{3}{4}} A - 2 \left(B\sharp_{\frac{5}{8}} A\right)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}A \bigtriangledown_\nu B \geq B\sharp_{1-\nu} A &+ 2\nu \left(A \bigtriangledown B - B\sharp A\right) \\ &+ (1-2\nu) \left(B + B\sharp A - 2 \left(B\sharp_{\frac{3}{4}} A\right)\right) \\ &+ r_0 \left(B\sharp A + B\sharp_{\frac{3}{4}} A - 2 \left(B\sharp_{\frac{5}{8}} A\right)\right).\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let $A, B \in \mathbb{B}(\mathcal{H})$ so that A and B are two positive invertible operators and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If $0 \leq \nu \leq \frac{1}{4}$, then*

$$\begin{aligned}A \bigtriangledown_\nu B \leq A\sharp_\nu B &+ 2(1-\nu) \left(A \bigtriangledown B - A\sharp B\right) \\ &- 2\nu \left(A + A\sharp B - 2 \left(A\sharp_{\frac{3}{4}} B\right)\right) \\ &- r_0 \left(A + A\sharp_{\frac{3}{4}} B - 2 \left(A\sharp_{\frac{7}{8}} B\right)\right),\end{aligned} \tag{3.8}$$

(ii) *If* $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, *then*

$$A \bigtriangledown_{\nu} B \leq B \sharp_{1-\nu} A + 2(1-\nu)(A \bigtriangledown B - B \sharp A) - (1-2\nu)\left(B \sharp A + B - 2\left(B \sharp_{\frac{1}{4}} A\right)\right) - r_0\left(B \sharp A + B \sharp_{\frac{1}{4}} A - 2\left(B \sharp_{\frac{3}{8}} A\right)\right), \tag{3.9}$$

where $r = \min\{2\nu, 1-2\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Proof. In view of Theorem 2.2, and by proceeding with similar calculations as used in Theorem 3.2, we obtain inequalities (3.8) and (3.9). $\square$

4. Better Inequalities for Matrices

In this section, we establish norm inequalities for the Hilbert-Schmidt norm based on Corollaries 2.3 and 2.7.

Theorem 4.1. *Let* $A, B, X \in \mathbb{M}_n$ *so that* A *and* B *are positive semidefinite and* $0 \leq \nu \leq \frac{1}{2}$.

(i) *If* $0 \leq \nu \leq \frac{1}{4}$, *then*

$$\|(1-\nu)AX + \nu XB\|_2^2 \geq \left\|A^{1-\nu}XB^{\nu}\right\|_2^2 + v^2\|AX - XB\|_2^2 + 2v\left(\|AX\|_2^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right) + r_0\left(\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2 + \|AX\|_2^2 - 2\left\|A^{\frac{7}{8}}XB^{\frac{1}{8}}\right\|_2^2\right), \tag{4.1}$$

(ii) *If* $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, *then*

$$\|(1-\nu)AX + \nu XB\|_2^2 \geq \left\|A^{1-\nu}XB^{\nu}\right\|_2^2 + v^2\|AX - XB\|_2^2 + (1-2v)\left(\|AX\|_2^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right) + r_0\left(\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2 + \left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\left\|A^{\frac{5}{8}}XB^{\frac{3}{8}}\right\|_2^2\right), \tag{4.2}$$

where $r = \min\{2\nu, 1-2\nu\}$ *and* $r_0 = \min\{2r, 1-2r\}$.

Proof. To prove Theorem 4.1, we consider the spectral theorem. Since A, B are positive semidefinite. So by spectral theorem, there exist unitary matrices U and V so that $A = UDU^*$ and $B = VEV^*$, where $D = diag(\lambda_1, \ldots, \lambda_n)$ and $E = diag(\mu_1, \ldots, \mu_n)$, with $\lambda_i, \mu_j \geq 0$, $1 \leq i, j \leq n$. Let $Y = U^*XV = (y_{ij})$, then we have

$$AX + XB = U[(\lambda_i + \mu_j) \circ y_{ij}]V^*,$$
$$AX - XB = U[(\lambda_i - \mu_j) \circ y_{ij}]V^*,$$
$$A^{\frac{1}{2}}XB^{\frac{1}{2}} = U\left[(\lambda_i\mu_j)^{\frac{1}{2}} \circ y_{ij}\right]V^*,$$

$$A^v XB^{1-v} + A^{1-v} XB^v = U\left[\left(\lambda_i^v \mu_j^{1-v} + \lambda_i^{1-v} \mu_j^v\right) \circ y_{ij}\right] V^*.$$

If $0 \leq \nu \leq \frac{1}{4}$, then according to (2.5) and the unitary invariance of $\|.\|_2$, we obtain

$$\begin{aligned}
\|(1-\nu)\,AX + \nu XB\|_2^2 &= \sum_{i,j=1}^{n} \left((1-\nu)\,\lambda_i + \nu\mu_j\right)^2 |y_{ij}|^2 \\
&\geq \sum_{i,j=1}^{n} \left(\lambda_i^{1-\nu}\mu_j^{\nu}\right)^2 |y_{ij}|^2 + \nu^2 \sum_{i,j=1}^{n} (\lambda_i - \mu_j)^2 \\
&\quad + 2\nu \sum_{i,j=1}^{n} \left(\lambda_i - \sqrt{\lambda_i \mu_j}\right)^2 \\
&\quad + r_0 \sum_{i,j=1}^{n} \left(\sqrt[4]{\lambda_i^3 \mu_j} - \lambda_i\right)^2 \\
&= \left\|A^{1-\nu} XB^{\nu}\right\|_2^2 + v^2 \|AX - XB\|_2^2 \\
&\quad + 2v\left(\|AX\|_2^2 + \left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|_2^2 - 2\left\|A^{\frac{3}{4}} XB^{\frac{1}{4}}\right\|_2^2\right) \\
&\quad + r_0\left(\left\|A^{\frac{3}{4}} XB^{\frac{1}{4}}\right\|_2^2 + \|AX\|_2^2 - 2\left\|A^{\frac{7}{8}} XB^{\frac{1}{8}}\right\|_2^2\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|(1-\nu)\,AX + \nu XB\|_2^2 \geq{} & \left\|A^{1-\nu} XB^{\nu}\right\|_2^2 + v^2 \|AX - XB\|_2^2 \\
& + 2v\left(\|AX\|_2^2 + \left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|_2^2 - 2\left\|A^{\frac{3}{4}} XB^{\frac{1}{4}}\right\|_2^2\right) \\
& + r_0\left(\left\|A^{\frac{3}{4}} XB^{\frac{1}{4}}\right\|_2^2 + \|AX\|_2^2 - 2\left\|A^{\frac{7}{8}} XB^{\frac{1}{8}}\right\|_2^2\right).
\end{aligned}$$

This estimate completes the proof of (4.1). With the help of (2.6) and the same method as used in (4.1), one can deduce (4.2). □

Theorem 4.2. *Let $A, B, X \in \mathbb{M}_n$ so that A and B are positive semi-definite and $0 \leq \nu \leq \frac{1}{2}$.*

(i) *If $0 < \nu \leq \frac{1}{4}$, then*

$$\begin{aligned}
\|(1-\nu)\,AX + \nu XB\|_2^2 \leq{} & \left\|A^{1-v} XB^{v}\right\|_2^2 + (1-\nu)^2 \|AX - XB\|_2^2 \\
& - 2\nu\left(\|XB\|_2^2 + \left\|A^{\frac{1}{2}} XB^{\frac{1}{2}}\right\|_2^2 - 2\left\|A^{\frac{1}{4}} XB^{\frac{3}{4}}\right\|\right) \\
& - r_0\left(\|XB\|_2^2 + \left\|A^{\frac{1}{4}} XB^{\frac{3}{4}}\right\|_2^2 - 2\left\|B^{\frac{7}{8}} XA^{\frac{1}{8}}\right\|_2^2\right),
\end{aligned}$$

(ii) *If* $\frac{1}{4} \leq \nu \leq \frac{1}{2}$, *then*

$$\begin{aligned}\|(1-\nu)AX+\nu XB\|_2^2 \leq & \left\|A^{1-v}XB^{v}\right\|_2^2+(1-v)^2\|AX-XB\|_2^2\\ & -(1-2\nu)\left(\|XB\|_2^2+\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2-2\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right)\\ & -r_0\left(\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\left\|B^{\frac{3}{4}}XA^{\frac{1}{4}}\right\|_2^2-2\left\|A^{\frac{3}{8}}XB^{\frac{5}{8}}\right\|_2^2\right),\end{aligned}$$

where $r=\min\{2\nu,1-2\nu\}$ *and* $r_0=\min\{2r,1-2r\}$.

Proof. Considering Corollary (2.7), the unitarily invariant property of $\|.\|_2$ and in the same way as in Theorem 4.1, we are able to prove Theorem 4.2. □

5. Declarations

Conflicts of interest/Competing interests. The authors declare that they have no competing interests.

Code availability. Not applicable.

Authors' contributions. All authors contributed to the study conception and design. The first draft of the manuscript was written by Mehdi Shams and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript. All authors read and approved the final manuscript.

Funding. Not applicable.

Availability of data and material. Not applicable.

References

1. S. Furuichi, *On refined Young inequalities and reverse inequalities*, J. Math. Inequal., 5 (1) (2011), pp. 21-31.
2. S. Furuichi, *Refined Young inequalities with Specht's ratio*, J. Egyptian Math. Soc., 20 (2012), pp. 46-49.
3. T. Furuta and M. Yanagida, *Generalized means and convexity of inversion for positive operators*, Amer. Math. Monthly, 105 (1998), pp. 258-259.
4. T. Furuta, *Invitation to Linear Operators: From Matrix to Bounded Linear Operators on a Hilbert Space*, Taylor and Francis, 2002.
5. T. Furuta and J.M. Hot, *Mond-Pecaric Method in operator inequalities*, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space, Zagreb: Element. 2005.
6. C.J. He and L.M. Zou, *Some inequalities involving unitarily invariant norms*, Math. Inequal. Appl., 12 (4) (2012), pp. 767-776.

7. O. Hirzallah and F. Kittaneh, *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl., 308 (2000), pp. 77-84.
8. F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., 361 (2010), pp. 262-269.
9. F. Kittaneh and Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear A. 59 (9) (2011), pp. 1031-1037.
10. F. Kittaneh, M. Krnic, N. Lovricevic and J. Pecaric, *Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators*, Publ. Math. Debrecen 80 (3-4) (2012), pp. 465-478.
11. R. Mikic; J. Pečarić, *Inequalities of Ando's Type for n-convex Functions* , Sahand Commun. Math. Anal., 17 (2020), pp. 139-159.
12. L. Nasiri and M. Shakoori, *Reverses of the Young type inequalities with the classical Kantorovich constant*, Results Math. 74 (16) (2019), pp.1-10.
13. L. Nasiri and W. Liao, *The new reverses of Young type inequalities for numbers*, operators and matrices, Oper. Matrices, 12 (4) (2018), pp. 1063-1071.
14. J. Zhao and J. Wu, *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. Appl., 421 (2) (2015), pp. 1779-1789.

[1] Department of Mathematics and computer science, Faculty of science, Lorestan University, Khorramabad, Iran.

Email address: `nasiri.le@fs.lu.ac.ir`

[2] Department of Statistics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran.

Email address: `mehdishams@kashanu.ac.ir`