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On Relative Reproducing Kernel Banach Spaces: Definitions, Semi-Inner Product and Feature Maps

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Abstract. In this paper, a special class of relative reproducing kernel Banach spaces a semi-inner product is studied. We extend the concept of relative reproducing kernel Hilbert spaces to Banach spaces. We present these relative reproducing kernel Banach spaces in terms of the feature maps and establish the separability of the domains when they are separable. In addition, we prove some theorems concerning feature maps and reproducing kernel Banach spaces. And finally, the relative kernels are compared with the semi-inner ones.

1. Introduction

Reproducing kernel Hilbert spaces are Hilbert spaces of functions such that point evaluation functions are continuous [2, 12, 13]. In their generalization of Banach spaces, Zhang and Xu [16] define a reflexive Banach space of functions (function space) as a reproducing kernel Banach space if its dual space is isometric with a Banach space of functions and the point evaluation functions are continuous for both Banach space and its dual. In particular, they show that if $\varphi : X \to W$ is a map to a reflexive Banach space W and $\varphi^* : X \to W^*$ is a map to its dual so that linear span image of both maps are dense, then a reproducing kernel Banach space is determined with reproducing kernel $k(x,y) = \langle \varphi(x), \varphi^*(y) \rangle$. They called the maps as feature maps and the spaces W and W^* as the pair of feature spaces for reproducing kernel Banach spaces. In [1], Alpay and Jorgensen develop the reproducing kernel Hilbert space to

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relative reproducing kernel Hilbert spaces. A Hilbert space H of functions on the given set X is called a relative reproducing kernel Hilbert space if there exists a function $M_{x,y} \in H$ such that for every $f \in H$,

$$f(x) - f(y) = \langle f, M_{x,y} \rangle.$$

Alpay et al. in [1] introduced the notion of relative reproducing kernel Hilbert spaces to apply in electrical network models where the differences may represent voltage drops. We extend this definition to Banach spaces of functions with applying duality mapping. For a relative reproducing kernel Banach space of functions defined on a set X, the existence of the right and left relative reproducing kernels has been shown. Our goal is to apply feature maps to characterize the Banach spaces. To do this, we take two steps. Firstly, we show that for a Banach space V (not necessarily reflexive), if there exist two maps φ and ψ such that $\overline{span}\varphi(X) = V$ and $\overline{span}\psi(X) = V^*$, then there exists a relative reproducing kernel Banach space B with a dual space B^* endowed with a bilinear form associated the those feature maps. We called the maps primary and secondary feature maps for relative reproducing kernel Banach spaces. In the next step, we have a relative reproducing kernel Banach space and look for feature maps related to it. Then, we generalize the notion of relativity to semi-inner product spaces. We find feature maps in these spaces and compare the relative reproducing kernel and semi-inner product relative reproducing kernel in these spaces. According to Owhadi and Scovel [11], we establish the separability of relative reproducing kernel Banach spaces and find some results about Lipschitz function spaces and relative reproducing kernel Banach spaces. Finally, we show that if the feature space is reflexive, the relative reproducing kernel Banach space that is constructed by feature maps is a reproducing kernel Banach space, and as a result the evaluation function becomes continuous and so we continue with the reproducing kernel Banach spaces.

2. Reproducing Kernel Banach Spaces

In this section, we extend the idea of reproducing kernel Hilbert space to Banach spaces.

Definition 2.1. Let $X \subseteq R^d$ and H be a Hilbert space consisting of function $f : X \to \mathbb{C}$. H is called a reproducing kernel Hilbert space and a kernel function $K : X \times X \to \mathbb{C}$ is called a reproducing kernel for H if

(i) $K(., y) \in H$, for all $y \in X$,
(ii) $f(y) = \langle f, K(., y) \rangle_H$, for all $f \in H$ and $y \in X$,

where $\langle ., . \rangle_H$ is used to denote the inner product of H.

Let X be a set and $K : X \times X \to \mathbb{C}$ a function. Let us denote

$$B_0 = span\{K(t,.) : \; t \in X\}, \qquad B_0^\sharp = span\{K(.,t) : \; t \in X\}.$$

Furthermore, suppose that there is a norm $\|.\|_{B_0}$ in B_0 satisfying:

(I) The evaluation functionals are continuous of B_0.

(II) If $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $(B_0, \|.\|_{B_0})$ such that $f_n(y) \to 0$ for all $y \in X$, then $\|f_n\|_{B_0} \to 0$.

Then, there are $(B, \|.\|_B)$ and $(B^\sharp, \|.\|_{B^\sharp})$ Banach completions of B_0 and $B_0^\sharp$ respectively, such that $(B, B^\sharp)$ is a pair of reproducing kernel Banach space with the reproducing kernel K.

Definition 2.2. A reproducing kernel Banach space on X is a reflexive Banach space B of functions on X for which B^* is isometric to Banach space $B^\sharp$ of functions on X and the point evaluation is continuous on both B and $B^\sharp$.

As pointed out in [16], the identification $B^\sharp$ of B^* is not unique; we will refer to the dual space B^* of a reproducing kernel Banach space B as its chosen identification. It has been proved in [16] that there exists a reproducing kernel for a reproducing kernel Banach space as defined above. To this end, we denote by $\langle .,. \rangle : B \times B^* \to \mathbb{C}$ the evaluation map $\langle x, y^* \rangle = y^*(x)$ for $x \in B$ and $y^* \in B^*$. Let us notice that the mapping $\mathcal{L}$ from the Banach space $B^\sharp$ to the dual space B^* of B is defined as

$$(\mathcal{L}(g))(f) = \langle f, \mathcal{L}(g) \rangle := \langle f, g \rangle_K, \quad \text{for all } f \in B \text{ and } g \in B^\sharp,$$

is an embedding from $B^\sharp$ to B^*, i.e., it is an isometric and linear mapping. So, we can define $\phi : X \to B$ and $\phi^* : X \to B^*$ as

$$\phi(x) = K(x,.) \in B, \qquad \phi^*(y) = \mathcal{L}(K(.,y)) \in B^*,$$

and they satisfy

$$K(x,y) = \langle \phi(x), \phi^*(y) \rangle.$$

Notice that as B is a reflexive Banach space then for any bounded linear functional T on B^* there exists a unique $x \in B$ such that $T(y^*) = \langle x, y^* \rangle_B$ for each $y^* \in B^*$. The following result holds [16]:

Theorem 2.3. *Suppose that B is a reproducing kernel Banach space on X. Then there exists a unique function $K : X \times X \to \mathbb{C}$ such that the following statements hold:*

(a) *For every $t \in X$, $K(.,t) \in B^*$ and $f(t) = (f, K(.,t))_B$ for all $f \in B$.*

(b) *For every $t \in X$, $K(t,.) \in B$ and $f^*(t) = (K(t,.), f^*)_B$ for all $f^* \in B^*$.*

(c) *The linear span of $\{K(t,.) : \; t \in X\}$ is dense in B.*

(d) *The linear span of $\{K(.,t) : \; t \in X\}$ is dense in B^*.*

(e) *For all* $t, s \in X$, $K(t,s) = (K(t,.), K(.,s))_B$.

The function K in the above theorem is the reproducing kernel for the reproducing kernel Banach space B.

3. Relative Reproducing Kernel Banach Spaces

This section deals with the relative reproducing kernel on Banach spaces of functions by dual mapping. A relative reproducing kernel Banach space with dual mapping on $B \times B^*$ denoted by $\langle \cdot, \cdot \rangle$ is an extension of the relative reproducing kernel Hilbert space [1]. Influenced by the definition of reproducing kernel Banach space [6], it does not require the reflexivity condition. Throughout this paper, we assume that the input set X is non-empty.

Definition 3.1. Let B be a Banach space of functions on X whose dual space B^* is isometrically equivalent to the space of functions on X that we denote by $B^\sharp$.

(i) We say that $R_x(\cdot, y) \in B^*$ is a right relative reproducing kernel for B, if $f(x) - f(y) = \langle f, R_x(\cdot, y)\rangle$, for every $f \in B$ and $x, y \in X$;
(ii) We say that $L_y(x, \cdot) \in B^*$ is a left relative reproducing kernel for B, if $g(y) - g(x) = \langle L_y(x, \cdot), g\rangle$ for every $g \in B^*$ and $x, y \in X$.
(iii) If there are right and left relative reproducing kernels for B, then we say that B is a relative reproducing kernel Banach space.

Obviously, for every $x, y \in X$

(i) $R_x(\cdot, y) = -R_y(\cdot, x)$;
(ii) $L_y(x, \cdot) = -L_x(y, \cdot)$;
(iii) $\langle L_y(x, \cdot), R_x(\cdot, y)\rangle = R_x(y, y) + R_y(x, x) = L_y(x, x) + L_x(y, y)$.

If $x = y$, then $R_x(\cdot, y) = L_y(x, \cdot) = 0$. From now on, we suppose that $x \neq y$. Let F be a Banach space of functions on X. Define $S : X \times X \longrightarrow F^*$ such that for any $f \in F$

$$S_{x,y}(f) = f(x) - f(y), \quad (x, y \in X).$$

Theorem 3.2. *Let B be a reflexive Banach space of functions on X and $S_{x,y}$ be a continuous linear functional on both B and B^* for every $x, y \in X$. Then there exist unique $R_x(\cdot, y)$ and $L_y(x, \cdot)$ such that for every $f \in B, g \in B^*$ and for every $x, y \in X$:*

(i) $S_{x,y}(f) = f(x) - f(y) = \langle f, R_x(\cdot, y)\rangle$.
(ii) $S_{y,x}(g) = g(y) - g(x) = \langle L_y(x, \cdot), g\rangle$.

Proof. Since B is reflexive, B^* is too. We know that for any continuous linear functional T on B^*, there exists a unique $L_{y,x} \in B$ such that

$$g(y) - g(x) = \langle L_{y,x}, g \rangle, \quad g \in B^*.$$

Define $L_y(x, \cdot) := L_{y,x}(\cdot)$ for every $x, y \in X$. Likewise, there exists $R_x(\cdot, y) \in B^*$ such that:

$$f(x) - f(y) = \langle f, R_x(\cdot, y) \rangle, \quad (f \in B, x, y \in X).$$

For the uniqueness of relative kernels, suppose that $R_x(\cdot, y)$ and $R'_x(\cdot, y)$ are in B^* such that for every $x, y \in X$ and $f \in B$ satisfy in (i). Then

$$\begin{aligned} f(x) - f(y) &= \langle f, R_x(\cdot, y) \rangle \\ &= \langle f, R'_x(\cdot, y) \rangle. \end{aligned}$$

The above equality implies that $\langle f, R_x(\cdot, y) - R'_x(\cdot, y) \rangle = 0$, for every $x, y \in X$ and $f \in B$. This means that

$$R_x(\cdot, y) = R'_x(\cdot, y).$$

The proof for $L_y(x, \cdot)$ is similar. □

Proposition 3.3. *If $R_x(\cdot, y)$ and $L_y(x, \cdot)$ are the right and left relative reproducing kernels of Banach space B, respectively, then for every x, y and $z \in X$:*

(i) $R_x(\cdot, y) = R_x(\cdot, z) + R_z(\cdot, y)$.
(ii) $L_y(x, \cdot) = L_y(z, \cdot) + L_z(x, \cdot)$.

Proof. Let $f \in B$, then for every $x, y, z \in X$

$$f(x) - f(y) = f(x) - f(z) + f(z) - f(y).$$

This implies (i). The proof is similar for (ii). □

For every $x, y \in X$ and fixed $z_0 \in X$, define

$$T_x := L_{z_0}(x, \cdot), \tag{3.1}$$

and

$$M_y := R_{z_0}(\cdot, y). \tag{3.2}$$

By the cases (i) and (ii) in Proposition 3.3, we have

$$L_y(x, \cdot) = T_x - T_y, \tag{3.3}$$

and

$$R_x(\cdot, y) = M_y - M_x. \tag{3.4}$$

A reproducing kernel Banach space with kernel $K(x, y)$ is a relative reproducing kernel on Banach spaces with right relative reproducing kernel $R_x(\cdot, y) = k(\cdot, y) - k(\cdot, x)$ and the left one as $L_y(x, \cdot) = k(x, \cdot) - k(y, \cdot)$ for all $x, y \in X$. Authors in [1], characterized the relative reproducing kernel on Hilbert spaces. In the next proposition, we find some similar results.

Proposition 3.4. *A Banach space B of functions on the set X is a relative reproducing kernel on Banach space with right and left relative reproducing kernels $R_x(\cdot, y)$ and $L_y(x, \cdot)$ respectively if and only if there exist functions such as $h_x : X \to B^*$, $d_y : X \to B$ and two linear functional $C : B \to \mathbb{C}$ and $D : B^* \to \mathbb{C}$ (possibly unbounded) such that for all $F \in B, G \in B^*$ and $x \in X$*

$$F(x) = \langle F, h_x \rangle + C(F),$$

and

$$G(y) = \langle d_y, G \rangle + D(G).$$

Proof. Similar to the proof of [1, Proposition 2.4], we define $R_x(\cdot, y) := h_x - h_y$ and $L_y(x, \cdot) := d_y - d_x$ for every $x, y \in X$, so B is a relative reproducing kernel on Banach space. On the other hand, take any fix $z_0 \in X$ and put $y = z_0$ in $f(x) - f(y) = \langle f, R_x(\cdot, y)\rangle$ and $x = z_0$ in $g(y) - g(x) = \langle L_y(x, \cdot), g\rangle$. These imply that

$$h_x = R_x(\cdot, z_0), \quad d_y = L_y(z_0, \cdot), \quad C(F) = F(z_0), \quad D(G) = G(z_0),$$

for every $x, y \in X$. □

By Proposition 3.4, we conclude that every relative reproducing kernel Hilbert space is a relative reproducing kernel on Banach space.

Corollary 3.5. *If B is a reflexive relative reproducing kernel on Banach space that point evaluation function is bounded in at least one point like $x_0 \in X$ on both B and B^*, then B is a reproducing kernel Banach space.*

Let B be a Banach space of functions. Define for every $f \in B$ and $g \in B^*$ the orthogonal spaces by

$$f^{\perp} := \{g \in B^*; \langle f, g\rangle = 0\}, \qquad {}^{\perp}g = \{f \in B; \langle f, g\rangle = 0\}.$$

Proposition 3.6. *Let B be a relative reproducing kernel on Banach space with the right and left relative kernels $R_x(\cdot, y)$ and $L_y(x, \cdot)$. Then the orthogonal spaces of $R_x(\cdot, y), L_y(x, \cdot)$ are the space of constant functions in B and B^*, accordingly. In particular, the linear span $R_x(\cdot, y)$ and $L_y(x, \cdot)$ are dense in B^* and B, respectively, if and only if B^* and B, contain no non-zero constant functions.*

Proof. For every $x, y \in X$

$${}^{\perp}R_x(\cdot, y) = \{f \in B; \langle f, R_x(\cdot, y)\rangle = 0\}.$$

This implies that $f(x) - f(y) = 0$, so f is constant in B and

$$L_y(x, \cdot)^{\perp} := \{g \in B^*; \langle L_y(x, \cdot), g\rangle = 0\}.$$

Hence, g is a constant function in B^*. For the last, assume that the constant functions in B and B^* are just zero functions and

$$\overline{span}\{L_y(x,\cdot); x, y \in X\} \neq B.$$

Then by Hahn-Banach theorem, there exists a nontrivial functional $f \in B^*$ such that $\langle L_y(x,\cdot), f\rangle = 0$. We get immediately that f is a constant function in B^*, so hypothesis $f = 0$, a contradiction. □

4. Relative Reproducing Kernel on Banach Space and Feature Maps

We start with the following lemma that is part of [16, Theorem 3]. Note that we have removed the reflexivity of the Banach space from the mentioned theorem because in the current case, we do not need that condition.

Lemma 4.1. *Let V be a Banach space with the dual space V^*. Suppose that there exist families of functions such $\{\psi_x\}_{x\in X} \in V^*$ and $\{\varphi_x\}_{x\in X} \in V$ on a given set X that (linear span) $\overline{span}\{\varphi_x\}_{x\in X} = V$ and $\overline{span}\{\psi_x\}_{x\in X} = V^*$. Define $I(f)(x) = \langle f, \psi_x\rangle$ and $J(g)(y) = \langle \varphi_y, g\rangle$ for all $f \in V$ and $g \in V^*$. Then the space*

$$B := \{I(f) : f \in V\},$$

with the norm $\|I(f)\|_B := \|f\|_V$ is a Banach space with the dual space

$$B^* := \{J(g) : g \in V^*\},$$

endowed with the norm

$$\|J(g)\|_{B^*} := \|g\|_{V^*}.$$

The bilinear form on $B \times B^*$ is

$$\langle I(f), J(g)\rangle := \langle f, g\rangle_V.$$

Proof. We show that I and J are one-one and the Banach space B^* is dual of B by bilinear form as defined. Assume that $I(f) = 0$. This implies that for all $x \in X$, $\langle f, \psi_x\rangle = 0$. By hypothesising on the ψ_x, $\langle f, g\rangle = 0$ for all $g \in V^*$ implying that $f = 0$. So B is a Banach space with the norm as mentioned. Similarly, so $J(g)$ is one-one and $B' := \{J(g) : g \in V^*\}$ is Banach space, too. We have

$$\begin{aligned} |\langle I(f), J(g)\rangle| &< \|f\|_V \|g\|_{V^*} \\ &= \|I(f)\|_B \|J(g)\|_{B^*}. \end{aligned}$$

Therefore, every function in B' is a continuous linear function on B, since the mappings $u \longmapsto \langle u, \psi(\cdot)\rangle_V$ is isometric from V to B. Whence, functions in B' exhaust all the continuous linear functionals on B, conclude that $B^* = B'$ with the above bilinear form and B^* is the dual of B. □

Theorem 4.2. *The Banach space B defined in Lemma 4.1 is a relative reproducing kernel on Banach space. Moreover, the right relative reproducing kernel for B is*

$$\begin{aligned} R_x(\cdot, y) &:= \langle \varphi(\cdot), \psi_x - \psi_y \rangle_V \\ &= J(\psi_x) - J(\psi_y), \end{aligned}$$

and the left relative reproducing kernel for B is

$$\begin{aligned} L_y(x, \cdot) &:= \langle \varphi_y - \varphi_x, \psi(\cdot) \rangle_V \\ &= I(\varphi_y) - I(\varphi_x), \end{aligned}$$

for every $x, y \in X$.

Proof. Clearly, by definition, B and B^* are function spaces. We show that $R_x(\cdot, y)$ is the right relative reproducing kernel and $L_y(x, \cdot)$ is the left one. For each $x, y \in X, u \in B$, put $u = I(f) = \langle f, \psi(\cdot) \rangle_V$ for some $f \in V$. Then

$$\begin{aligned} \langle u, R_x(\cdot, y) \rangle_B &= \big\langle \langle f, \psi(\cdot) \rangle_V, \langle \varphi(\cdot), \psi_x - \psi_y \rangle_V \big\rangle_B \\ &= \langle f, \psi_x - \psi_y \rangle \\ &= \langle f, \psi_x \rangle_V - \langle f, \psi_y \rangle_V \\ &= I(f)(x) - I(f)(y). \end{aligned}$$

Similarly, for every $t \in V^*$, define $g := \langle \varphi(\cdot), t \rangle$, then

$$\begin{aligned} \langle L_y(x, \cdot), g \rangle_B &= \big\langle \langle \varphi_y - \varphi_x, \psi(\cdot) \rangle_V, \langle \varphi(\cdot), t \rangle_V \big\rangle_B \\ &= \langle \varphi_y - \varphi_x, t \rangle_V \\ &= \langle \varphi_y, t \rangle_V - \langle \varphi_x, t \rangle_V \\ &= J(g)(y) - J(g)(x). \end{aligned}$$

These facts show that $L_y(x, \cdot)$ and $R_x(\cdot, y)$ are the left and right relative reproducing kernels, respectively. ☐

Theorem 4.3. *Assume that for every $x, y \in X$, $R_{x,y}$ and $L_{y,x}$ are functions on the set X. Then*

(i) *$R_{x,y}$ is the right relative reproducing kernel for some Banach spaceB,*
(ii) *$L_{y,x}$ is the left relative reproducing kernel for some Banach space B,*

for every $x, y \in X$ if, there exist some families of functions such as $\{\varphi_x\}_{x \in X} \subset V$ and $\{\psi_y\}_{y \in X} \subset V^*$, where V is a Banach space with dual space V^* that:

(1) $\overline{span}\{\varphi_x\}_{x \in X} = V$*;*
(2) $\overline{span}\{\psi_y\}_{y \in X} = V^*$*;*
(3) $\langle L_{y,x}, R_{x,y} \rangle = \langle \varphi_y - \varphi_x, \psi_x - \psi_y \rangle$.

The converse is true when the relative reproducing kernel on Banach space B and B^* contain no non-zero constant functions.

Proof. The first part has been shown in Theorem 4.2. Then

$$\begin{aligned}\langle L_y(x,\cdot), R_x(\cdot,y)\rangle_B &= \big\langle \langle \varphi_y - \varphi_x, \psi(\cdot)\rangle_V, \langle \varphi(\cdot), \psi_x - \psi_y\rangle_V \big\rangle_B \\ &= \langle \varphi_y - \varphi_x, \psi_x - \psi_y\rangle,\end{aligned}$$

for every $x, y \in X$. We put $R_{x,y}(\cdot) := R_x(\cdot, y)$ and $L_{y,x}(\cdot) := L_y(x,\cdot)$. For the converse, assume that $L_y(x,\cdot)$ and $R_x(\cdot,y)$ are the left and the right relative reproducing kernels. As equation (1) and (2), set for every $x, y \in X$:

$$\psi_{(y)} := M_y, \qquad \varphi_{(x)} := T_x.$$

By the Theorem 3.6, $\overline{span}\{L_y(x,\cdot) : x, y \in X\} = B$ and $\overline{span}\{R_x(\cdot,y) : x, y \in X\} = B^*$, because of that for every $x, y \in X$, $T_x - T_y \in span\{T_x\}_{x\in X}$. Thus $span\{T_x - T_y\} \subseteq span\{T_x\}_{x\in X} \subseteq B$. This means that $\overline{span}\{T_x\}_{x\in X} = B$. □

We call the mappings $\varphi : X \longrightarrow V$ and $\psi : x \longrightarrow V^*$ that maps $x \longmapsto \varphi_x$ and $y \longmapsto \psi_y$ in the Theorem 4.3 as the primary and secondary feature maps for relative reproducing kernel on Banach space B and the Banach spaces V the primary and V^* as secondary feature spaces.

Remark 4.4. If we assume that the Banach space V is reflexive, then by [16, Theorem 3], we see that the evaluation function is continuous on B and B^*. This implies that B is reproducing kernel on Banach space. However, we know that every reproducing kernel on Banach space is a relative. This is in accordance with Corollary 3.5, too.

In [11], there are some results about the separability of the image of the feature maps image and the corresponding reproducing kernel on Banach space. This lemma is used to prove our theorem about the separability of the relative reproducing kernel on Banach space. The proofs of the next lemma and the theorem are similar to [11], so we omit the proofs.

Lemma 4.5. *Let B be a relative reproducing kernel on Banach space of functions on a set X with primary and secondary feature Banach space V and V^* and feature maps φ and ψ. Then:*

(i) *if B^* contains no non-zero constant functions and $\varphi(X)$ is separable, then B is separable.*
(ii) *if B contains no non-zero constant functions and $\psi(X)$ is separable, then B^* is separable.*

Theorem 4.6. *Relative reproducing kernel B of function on X is separable if and only if there exists a (primary or secondary) feature map*

$\varphi : X \longrightarrow V$ or $\psi : X \longrightarrow V^*$ such that (X, d_φ)or (X, d_ψ) by

$$d_\varphi(x, y) := \|\varphi(x) - \varphi(y)\|_V, \qquad d_\psi(x, y) := \|\psi(x) - \psi(y)\|_{V^*},$$

is separable.

In Hein's paper, [8, Proposition 1], there is a result about the relation between the Banach space of real-valued Lipschitz functions on the set X and feature spaces of Banach space B. We show that there is an isomorphism between relative reproducing kernel on Banach space and the Banach space of real-valued Lipschitz functions:

Theorem 4.7. *Let B be a real relative reproducing kernel on Banach space of functions on the set X (contains no non-zero constant functions). Then there exists a primary feature map φ, the secondary one ψ for B and some Banach spaces F_B, F_{B^*} of real-valued Lipschitz functions on the (X, d_φ), (X, d_ψ), respectively, such that the maps $\Gamma_1 : B \longrightarrow F_B$ and $\Gamma_2 : B^* \longrightarrow F_{B^*}$ defined by $\Gamma_1(w)(\cdot) = \langle w, \psi(\cdot)\rangle$ and $\Gamma_2(w')(\cdot) = \langle \varphi(\cdot), w'\rangle$ are isometric isomorphisms with defined norm $\|\Gamma_1(w)\| = \|w\|$ and $\|\Gamma_2(w')\| = \|w'\|$.*

Proof. Put $\psi(y) := M_y$, $\varphi(x) := T_x$ for some fixed z_0 and every $x, y \in X$. Then

$$\begin{aligned}\Gamma_1(w_f)(y) - \Gamma_1(w_g)(y) &= \langle w_f, M_y\rangle - \langle w_g, M_y\rangle \\ &= \langle w_f - w_g, M_y\rangle.\end{aligned}$$

This implies that $(w_f - w_g)(y) = (w_f - w_g)(z_0)$ for every $y \in X$, so $w_f - w_g$ is constant. For the last, we have

$$\begin{aligned}|\Gamma_1(w)(x) - \Gamma_1(w)(y)| &= |\langle w, \psi(x) - \psi(y)\rangle| \\ &\leq \|w\|_B d_\psi(x, y).\end{aligned}$$

So, $\Gamma_1(w)$ is Lipschitz function. The proof for Γ_2 is similar. □

5. Relative Reproducing Kernel on Banach Space and Semi-Inner Products

In Theorem 4.2, we construct a relative reproducing kernel on Banach space without an inner product. Using semi-inner products, we find some results above reproducing kernel Banach spaces for relative ones. A semi-inner product on a vector space V is a function, defined by $[.,.]_V$, from $V \times V$ to $\mathbb{C}$ such that for all $x, y, z \in V$ and $\lambda \in \mathbb{C}$

(1) $[x + y, z]_V = [x, z]_V + [y, z]_V$,
(2) $[\lambda x, y]_V = \lambda[x, y]_V, [x, \lambda y]_V = \bar{\lambda}[x, y]_V$,
(3) $[x, x]_V > 0$ for $x \neq 0$,
(4) (Cauchy -schwartz) $|[x, y]_V|^2 \leq [x, x]_V[y, y]_V$.

For more information, refer to [7]. There is a well-known relationship between uniform Frećhet differentiability and uniform convexity. It states that a normed vector space is uniformly Frećhet differentiable if and only if its dual is uniformly convex. Therefore, if B is a uniformly convex and uniformly Frećhet differentiable Banach space, so is B^*, since B is reflexive. The important role of uniform convexity is explained in the next lemma.

Theorem 5.1 (Riesz Representation Theorem)**.** *Suppose that B is a uniformly convex, uniformly Frećhet differentiable Banach space, then for each $f \in B^*$ there exists a unique $x \in B$ such that $f = x^*$, that is $f(y) = [y, x]_B$ for all $y \in B$. Moreover, $\|f\|_{B^*} = \|x\|_B$.*

The above Riesz Representation Theorem is desirable for relative reproducing kernel on Banach space, if $S_{x,y}$ defined as before is continued on both B and B^* for every x, y in a given set X. Note that the continuity of $S_{x,y}$ does not mean that the evaluation function is continuous. If this happens, every result is proved is the same as the reproducing kernel on Banach space [16]. Based on the above theorem and our earlier discussion, we shall investigate in the next subsection the relative reproducing kernel on Banach space. These kernels are both uniformly convex and uniformly Frećhet differentiable. We call these spaces uniform spaces. Let B be a uniform Banach space. According to the above theorem, $x \longmapsto x^*$ defines a bijection from B to B^* that preserves the norm. Note that this duality mapping is nonlinear. We call x^* the dual element of x. Since B^* is uniformly Frećhet differentiable, it has a unique semi-inner product which given by

$$[x^*, y^*]_{B^*} = [y, x]_B,$$

for all $x, y \in B$. We denote a uniform relative reproducing kernel on Banach space by a semi-inner product relative reproducing kernel on Banach space. We shall see that every semi-inner product reproducing kernel on Banach space is a semi-inner product relative reproducing kernel on Banach space.

Theorem 5.2. *Let B be a semi-inner product relative reproducing kernel on Banach space on X that $S_{x,y}$ is continuous function on both B and B^* with $R_x(\cdot, y)$ and $L_y(x, \cdot)$ as right and left relative reproducing kernels for B, respectively. Then there exists a unique function $K : X \times X \times X \to \mathbb{C}$ such that $\{K(x, y, \cdot) : \ x, y \in X\} \subset B$ and for all $f \in B$ and $x, y \in X$,*

$$f(x) - f(y) = [f, K(x, y, \cdot)]_B.$$

Moreover,

$$R_x(\cdot, y) = (K(x, y, \cdot))^*,$$

for all $x, y \in X$ *and for all* $f \in B$,

$$f^*(y) - f^*(x) = [L_y(x, \cdot), f]_B.$$

Proof. By continuity of $S_{x,y}$ on B and Riesz representation theorem, for each $x, y \in X$ there exists a function $K_{x,y} \in B$ such that

$$f(x) - f(y) = [f, K_{x,y}],$$

for all $f \in B$. We define $K : X \times X \times X \to \mathbb{C}$ by $K(x, y, z) = K_{x,y}(z)$ for all $x, y \in X$. We see that $K(x, y, \cdot) = K_{x,y}(\cdot) \in B$ and $f(x) - f(y) = [f, K(x, y, \cdot)]_B$ holds. The Riesz Representation Theorem states that such a function is unique, based on the uniqueness of the function. To prove the remaining claims, we have for every $f \in B$, $f^* \in B^*$ and $x, y \in X$,

$$f(x) - f(y) = \langle f, R_x(\cdot, y)\rangle_B,$$

and

$$f^*(y) - f^*(x) = \langle L_y(x, \cdot), f^*\rangle.$$

Then

$$\begin{aligned} f(x) - f(y) &= [f, K(x, y, \cdot)]_B \\ &= \langle f, R_x(\cdot, y)\rangle_B, \end{aligned}$$

for every $x, y \in X$. The above relation leads to

$$\begin{aligned} \langle f, (K(x, y, \cdot)^*)\rangle_B &= [f, K(x, y, \cdot)]_B \\ &= f(x) - f(y) \\ &= \langle f, R_x(\cdot, y)\rangle_B. \end{aligned}$$

This implies $R_x(\cdot, y) = (K(x, y, \cdot))^*$ and $\langle L_y(x, \cdot), f^*\rangle_B = [L_y(x, \cdot), f]_B$. Hence

$$f^*(y) - f^*(x) = [L_y(x, \cdot), f]_B,$$

for all $x, y \in X, f \in B$ and for $f^* \in B^*$. □

We call K is a semi-inner product relative reproducing kernel if $K(x, y, \cdot)^* = R_x(\cdot, y)$ and for abbreviate represent by $K_{x,y}$.
Now, we give a characterization of the semi-inner product relative reproducing kernel. For a map from X to a uniform space, we denote by φ^*, the mapping from X to V^* defined as

$$\varphi^*(X) = (\varphi(X))^*.$$

Theorem 5.3. *Let* V *be a uniform Banach space and* φ *be a mapping from* X *to* V *such that* $\overline{span}\varphi(X) = V$ *and* $\overline{span}\varphi^*(X) = V^*$*, then*

(i) $B := \{[\varphi(\cdot), u]_V : u \in V\}$ *with* $[[\varphi(\cdot), u]_V, [\varphi(\cdot), w]_V]_B := [u, w]_V$,
(ii) $B^* := \{[v, \varphi(\cdot)]_V : v \in V\}$ *with* $[[v, \varphi(\cdot)]_V, [w, \varphi(\cdot)]_V]_{B^*} := [w, v]_V$,

are uniform Banach spaces and B^* is the dual of B with the bilinear form

$$\langle [\varphi(\cdot), u]_V, [v, \varphi(\cdot)]_V \rangle_B := [v, u]_V, \tag{5.1}$$

for all $u, w \in V$. Moreover, the semi-inner product relative reproducing kernel of B is given by

$$K(x, y, \cdot) = [\varphi(x) - \varphi(y), \varphi(\cdot)]_V. \tag{5.2}$$

Proof. We just show the (5.2), the others are such as the proof of the Theorem (4.2). Let $f \in B$, then there exists a unique $u \in V$ such that $f = [\varphi(\cdot), u]_V$. By the bilinear form (5.1), we have

$$\begin{aligned} f(x) - f(y) &= [\varphi(x), u]_V - [\varphi(y), u]_V \\ &= [\varphi(x) - \varphi(y), u]_V \\ &= \langle [\varphi(\cdot), u]_V, [\varphi(x) - \varphi(y), \varphi(\cdot)]_V \rangle_B \\ &= \langle f, [\varphi(x) - \varphi(y), \varphi(\cdot)]_V \rangle_B. \end{aligned}$$

for every $x, y \in X$. Compared with the semi-inner product relative reproducing kernel,

$$K(x, y, \cdot) = [\varphi(x) - \varphi(y), \varphi(\cdot)]_V{}^*.$$

On the other hand,

$$\begin{aligned} f(x) &= [\varphi(x), u]_V \\ &= [[\varphi(\cdot), \varphi(x)], [\varphi(\cdot), u]_V]_B \\ &= [[\varphi(\cdot), \varphi(x)], f]_B, \end{aligned}$$

for every $x \in X$ and $f \in B$. □

6. An Example of Relative Reproducing Kernel on Banach Space

Let $H^p(\mathbb{C}_+)$ be a set of all analytic functions on an open half plane denoted by $\mathbb{C}_+$. $H^p(\mathbb{C}_+)$ is a Banach space isomorphic to a closed subspace of $L^p(\mathbb{R})$ denoted by $H^p(\mathbb{R})$. $H^1(\mathbb{C}_+)$ is a non-reflexive Banach space that contains no non-zero constant functions and its dual space is the space of bound mean oscillation functions. For more details, see [3, 10]. Let V be the space $H^1(\mathbb{C}_+)$. Put $X := \mathbb{C}_+$. Suppose that C and D are everywhere defined unbounded linear functions with complex values. There exist families of functions like $\{\varphi_x\}_{x\in X} \in V$ and $\{\psi_y\}_{y\in X} \in V^*$ that

$$\overline{span}\{\varphi_x - \varphi_y\} = V, \qquad \overline{span}\{\psi_y - \psi_x\} = V^*.$$

It should be noted that such families exist, and that polynomials with the form of $\sum_{k=0}^{n} a_k e^{ik\theta}$ are dense in $H^1(\mathbb{C}_+)$. Define

$$\rho(f)(x) := \langle f, \psi_x \rangle + C(f), \quad (f \in V).$$
$$\omega(g)(y) := \langle \varphi_y, g \rangle + D(g), \quad (g \in V^*).$$

Define $B := \{\rho(f) : f \in V\}$ and $B^* = \{\omega(g) : g \in V^*\}$. Clearly the linear space B^* is dual space of B. By bilinear form

$$\langle \rho(f), \omega(g) \rangle = \langle f, g \rangle.$$

Theorem 6.1. *The Banach space B is a relative reproducing kernel Banach space, but it is not a reproducing kernel Banach space.*

Proof. For every $f \in V$ and $x, y \in X$, we have

$$\begin{aligned}\rho(f)(x) - \rho(f)(y) &= \langle f, \psi_x \rangle - \langle f, \psi_y \rangle \\ &= \langle f, \psi_x - \psi_y \rangle \\ &= \langle \rho(f), \omega(\psi_x - \psi_y) \rangle \\ &= \langle \rho(f), R_{x,y} \rangle.\end{aligned}$$

It is similar for B^* where $L_{y,x} = \rho(\varphi_y - \varphi_x)$.

Now, we show that this space is not a reproducing kernel Banach space. Suppose that B is a reproducing kernel Banach space, then there exists a function $K : X \times X \to \mathbb{C}$ such that $\rho(f)(x) = \langle \rho(f), K(\cdot, x) \rangle = \langle \rho(f), \omega(g_x) \rangle$ for some $g_x \in V^*$, Since $K(\cdot, x)$ belongs to B^*.

$$\begin{aligned}\rho(f)(x) &= \langle \rho(f), \omega(g_x) \rangle \\ &= \langle f, g_x \rangle.\end{aligned}$$

So that

$$C(f) = \langle f, g_x \rangle - \langle f, \psi_x \rangle,$$

so $C(f)$ is bounded. This is a contradiction, because we assumed C is unbounded. □

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