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A Fuzzy Solution of Fractional Differential Equations by Fuzzy Conformable Laplace Transforms

Atimad Harir[1*], Said Melliani[2] and L. Saadia Chadli[3]

Abstract. The fuzzy conformable Laplace transforms proposed in [8] are used to solve only fuzzy fractional differential equations of order $0 < \iota \leq 1$. In this article, under the generalized conformable fractional derivatives notion, we extend and use this method to solve fuzzy fractional differential equations of order $0 < \iota \leq 2$.

1. Introduction

Ordinary calculus is generalized to fractional calculus. This contains the function's arbitrary order derivative. Researchers in various fields including engineering, mathematics, and so on, , etc., have investigated and studied the topic [9–11, 15]. One of the most important to this field was the study of fuzzy fractional differential equations, generalized conformable differentiability, and fuzzy conformable Laplace transforms [17], which explored the problem extensively. It was later examined in [14], where the authors suggested some uses.

There are various definitions of fuzzy fractional differentiation and fuzzy integration. These include the fuzzy Riemann-Liouville formulation [2, 11], the fuzzy Caputo definition [11, 20], etc. In [10, 11] developed the conformable fractional derivative, which is a simple definition of the fractional derivative that corrects flaws in previous definitions. This new definition meets formulas for product derivative and quotient of two functions. [16, 19]. In [10] presented the fuzzy generalized conformable fractional derivative, which expanded and extended

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the Hukuhara differentiability idea for set valued mappings to the fuzzy mappings class [3, 18].

The authors of [8] identified a relationship between a fuzzy function's fuzzy Laplace transforms and its conformable fractional derivative of order $0 < \iota \leq 1$. They provided a numerical example to demonstrate the method's efficiency, although this example is of order $0 < \iota \leq 1$ for FFDEs.

This work aims improve and extend their method by establishing a relationship between a function's fuzzy conformable Laplace transforms and its conformable fractional derivative of order $1 < \iota \leq 2$, with the goal of solving conformable fuzzy fractional differential equations under generalized conformable differentiability.

2. Preliminaries

Let us denote by $\mathbf{F}(\mathbb{R}) = \{u : \mathbb{R} \to [0,1]\}$ the class of fuzzy subsets of the real axis satisfying the following properties:

(i) u is normal i.e, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex i.e for $x, y \in \mathbb{R}$ and $0 < \lambda \leq 1$;

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)],$$

(iii) u is upper semicontinuous;
(iv) $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact.

Then $\mathbf{F}(\mathbb{R})$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbf{F}(\mathbb{R})$. For $0 < \epsilon \leq 1$ denote $[u]^\epsilon = \{x \in \mathbb{R} | u(x) \geq \epsilon\}$, then from (i) to (iv) it follows that the ϵ-level sets $[u]^\epsilon \in P_K(\mathbb{R})$ for all $0 \leq \epsilon \leq 1$ is a bounded interval with a closed end that is symbolized by $[u]^\epsilon = [^1u^\epsilon, ^2u^\epsilon]$. By $P_K(\mathbb{R})$, we define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual, and we designate the family of all nonempty compact convex subsets of $\mathbb{R}$.

Theorem 2.1 ([20]). *If $u \in \boldsymbol{F}(\mathbb{R})$, then*

(i) $[u]^\epsilon \in P_K(\mathbb{R})$ *for all* $0 \leq \epsilon \leq 1$,
(ii) $[u]^{\epsilon_2} \subset [u]^{\epsilon_1}$ *for all* $0 \leq \epsilon_1 \leq \epsilon_2 \leq 1$,
(iii) $\{\epsilon_k\} \subset [0,1]$ *is a nondecreasing sequence which converges to* ϵ *and*

$$[u]^\epsilon = \bigcap_{k \geq 1} [u]^{\epsilon_k},$$

Conversely, if $A_\epsilon = \{[^1u^\epsilon, ^2u^\epsilon]; \epsilon \in (0,1]\}$ be a set of closed real intervals that confirms (i) and (ii), then $\{A_\epsilon\}$ defined a fuzzy number $u \in \boldsymbol{F}(\mathbb{R})$ such that
$[u]^\epsilon = A_\epsilon$ *for* $0 < \epsilon \leq 1$ *and* $[u]^0 = \overline{\bigcup_{0<\epsilon\leq 1} A_\epsilon} \subset A_0$.

Lemma 2.2 ([6])**.** *Let $u, v : \mathbb{R} \to [0,1]$ be the fuzzy sets.*
Then $u = v$ if and only if $[u]^\epsilon = [v]^\epsilon$ for all $\epsilon \in [0,1]$.

Definition 2.3 ([12])**.** In parametric form, a fuzzy number u is a pair $\left({}^1u^\epsilon, {}^2u^\epsilon\right)$ of functions ${}^1u^\epsilon, {}^2u^\epsilon$, $\epsilon \in [0,1]$, which satisfy the following requirements:

(i) ${}^1u^\epsilon$ is a right continuous at 0 and a bounded growing left continuous function in $(0,1]$.
(ii) ${}^2u^\epsilon$ is a right continuous at 0 and a bounded decreasing left continuous function in $(0,1]$.
(iii) ${}^1u^\epsilon \leq {}^2u^\epsilon$, $0 \leq \epsilon \leq 1$.

A crisp number k is simply represented by ${}^1u^\epsilon = {}^2u^\epsilon = k$.
For arbitrary $u = \left({}^1u^\epsilon, {}^2u^\epsilon\right)$, $v = \left({}^1v^\epsilon, {}^2v^\epsilon\right)$ and $\lambda > 0$ we define addition and scalar multiplication by λ see [6, 8]:

$$[u+v]^\epsilon = [{}^1u^\epsilon + {}^1v^\epsilon, {}^2u^\epsilon + {}^2v^\epsilon],$$

$$[\lambda u]^\epsilon = \lambda[u]^\epsilon = \begin{cases} [\lambda {}^1u^\epsilon, \lambda {}^2u^\epsilon], & \text{if} \quad \lambda \geq 0, \\ [\lambda {}^2u^\epsilon, \lambda {}^1u^\epsilon], & \text{if} \quad \lambda < 0. \end{cases}$$

Definition 2.4. Let $u, v \in \mathbf{F}(\mathbb{R})$. If $w \in \mathbf{F}(\mathbb{R})$ exists, such as $u = v + w$, w is known as the H-difference of u, v and is denoted $uominusv$.

Define $d : \mathbf{F}(\mathbb{R}) \times \mathbf{F}(\mathbb{R}) \to \mathbb{R}_+ \cup \{0\}$ by the equation

$$d(u,v) = \sup_{\epsilon \in [0,1]} d_H([u]^\epsilon, [v]^\epsilon), \quad \text{for all } u, v \in \mathbf{F}(\mathbb{R}),$$

where d_H is the Hausdorff metric .

$$d_H([u]^\epsilon, [v]^\epsilon) = \max\left\{|{}^1u^\epsilon - {}^1v^\epsilon|, |{}^2u^\epsilon - {}^2v^\epsilon|\right\},$$

where $u = ({}^1u^\epsilon, {}^2u^\epsilon)$, $v = ({}^1v^\epsilon, {}^2v^\epsilon) \subset \mathbb{R}$ is utilized in Bede and Gal [3]. Then it's clear that d is a metric in $\mathbf{F}(\mathbb{R})$ and has the following properties [12]

(*i*) $d(u+w, v+w) = d(u,v), \quad \forall u, v, w \in \mathbf{F}(\mathbb{R})$;
(*ii*) $d(ku, kv) = |k| d(u,v),\ \forall k \in \mathbb{R},\ u, v \in \mathbf{F}(\mathbb{R})$;
(*iii*) $(d, \mathbf{F}(\mathbb{R}))$ is a complete metric space.

Definition 2.5 ([21])**.** Let $g : \mathbb{R} \to \mathbf{F}(\mathbb{R})$ be a fuzzy-valued function. If for arbitrary fixed $t_0 \in \mathbb{R}$ and $\tau > 0$ a $\delta > 0$ such that

$$|t - t_0| < \delta \quad \Rightarrow \quad d(g(t), g(t_0)) < \tau$$

then g is said to be continuous.

Definition 2.6 ([10])**.** Let $G : I \to \mathbf{F}(\mathbb{R})$ be a fuzzy function. ι^{th} order "fuzzy conformable fractional derivative" of G is defined by

$$G^{(\iota)}(t) = \lim_{\tau \to 0^+} \frac{G\left(t + \tau t^{1-\iota}\right) \ominus G(t)}{\tau}$$

$$= \lim_{\tau \to 0^+} \frac{G(t) \ominus G\left(t - \tau t^{1-\iota}\right)}{\tau},$$

for all $t > 0, \iota \in (0,1)$ If G is ι-differentiable in some I and $\lim_{t \to 0^+} G^{(\iota)}(t)$ exists, then

$$G^{(\iota)}(0) = \lim_{t \to 0^+} G^{(\iota)}(t),$$

and the limits (in the metric d).

Remark 2.7 ([10])**.** If G is ι-differentiable for all $\epsilon \in [0,1]$ and, then G_ϵ is ι-differentiable for all multi-valued mappings, and

$$G_\epsilon^{(\iota)} = \left[G_{(\iota)}(t)\right]^\epsilon .$$

The conformable fractional derivative of G_ϵ of order ι is represented by $G_\epsilon^{(\iota)}$. Because the existence of Hukuhara differences $[x]^\epsilon \ominus [y]^\epsilon, \quad \epsilon \in [0,1]$, does not necessitate the existence of H-differences, the reverse result does not hold. $x \ominus y$. is the result of $xminusy$.

Here $G_\epsilon^{(\iota)}$ is denoted the conformable fractional derivative of G_ϵ of order ι. The converse result doesn't hold, since the existence of Hukuhara differences $[x]^\epsilon \ominus [y]^\epsilon, \quad \epsilon \in [0,1]$, does not imply the existence of H-difference $x \ominus y$.

Definition 2.8 ([10])**.** Let $G : I \to \mathbf{F}(\mathbb{R})$ be a fuzzy function and $\iota \in (0,1]$. One says, G is $\iota^{(1)}$-differentiable at point $t > 0$ if there exists an element $G^{(\iota)}(t) \in \mathbf{F}(\mathbb{R})$ such that for all $\tau > 0$ sufficiently near to 0, there exist $G\left(t + \tau t^{1-\iota}\right) \ominus G(t), G(t) \ominus G\left(t - \tau t^{1-\iota}\right)$ and the limits (in the metric d)

$$\begin{aligned} \lim_{\tau \to 0^+} \frac{G\left(t + \tau t^{1-\iota}\right) \ominus G(t)}{\tau} &= \lim_{\tau \to 0^+} \frac{G(t) \ominus G\left(t - \tau t^{1-\iota}\right)}{\tau} \\ &= G^{(\iota)}(t), \end{aligned}$$

G is $\iota^{(2)}$-differentiable at $t > 0$ if for all $\tau < 0$ sufficiently near to 0, there exist $G\left(t + \tau t^{1-\iota}\right) \ominus G(t)$ and $G(t) \ominus G\left(t - \tau t^{1-\iota}\right)$

$$\begin{aligned} \lim_{\tau \to 0^-} \frac{G\left(t + \tau t^{1-\iota}\right) \ominus G(t)}{\tau} &= \lim_{\tau \to 0^-} \frac{F(t) \ominus G\left(t - \tau t^{1-\iota}\right)}{\tau} \\ &= G^{(\iota)}(t), \end{aligned}$$

If G is $\iota^{(n)}$-differentiable at $t > 0$, we denote its ι-derivatives, for $n = 1, 2$.

3. Fuzzy Conformable Laplace Transform

Definition 3.1 ([8])**.** The conformable fractional exponential function is defined for every $t \geq 0$ as

$$E_\iota(\Theta, t) = e^{\Theta \frac{t^\iota}{\iota}}, \tag{3.1}$$

where $\Theta \in \mathbb{R}$ and $0 < \iota \leq 1$.

Definition 3.2 ([8])**.** Let $0 < \iota \leq 1$ and $g(t)$ be continuous fuzzy-value function. Suppose that $E_\iota(-\Theta, t)g(t)$ is improper fuzzy Rimann-integrable on $[0, \infty)$, then $\int_0^\infty E_\iota(-\Theta, t)g(t)d_\iota t$ is called fractional fuzzy conformable Laplace transform of order ι starting from zero of g and is defined as:

$$\begin{aligned} \mathbf{L}_\iota\left[g(x)\right] &= \int_0^\infty E_\iota(-\Theta, t)g(t)d_\iota t, \quad \Theta > 0 \text{ and integer.} \\ &= \int_0^\infty E_\iota(-\Theta, t)g(t)t^{\iota-1}dt. \end{aligned} \tag{3.2}$$

Denote by $\mathcal{L}_\iota\left[g(t)\right]$ the classical fractional Laplace transform of order ι starting from zero of crisp function $g(t)$. From Proposition 2.1 in [21], we have

$$\int_0^\infty E_\iota(-\Theta, t)g(t)d_\iota t = \left(\int_0^\infty E_\iota(-\Theta, t)^1 g^\epsilon(t)d_\iota t, \int_0^\infty E_\iota(-\Theta, t)^2 g^\epsilon(t)d_\iota t\right),$$

then, we have:

$$\mathbf{L}_\iota\left[g(t)\right] = \left(\mathcal{L}_\iota\left[{}^1g^\epsilon(t)\right], \mathcal{L}_\iota\left[{}^2g^\epsilon(t)\right]\right),$$

where $\iota \in (0, 1]$ and

$$\mathcal{L}_\iota\left[{}^1g^\epsilon(t)\right] = \int_0^\infty E_\iota(-\Theta, t)^1 g^\epsilon(t)d_\iota t,$$

and

$$\mathcal{L}_\iota\left[{}^2g^\epsilon(t)\right] = \int_0^\infty E_\iota(-\Theta, t)^2 g^\epsilon(t)d_\iota t,$$

Theorem 3.3 ([8])**.** *Let $0 < \iota \leq 1$ and $g^{(\iota)}(t)$ be a conformable fractional integral fuzzy-value function and $g(t)$ is the primitive of $g^{(\iota)}(t)$ on $[0, \infty)$. Then*

(*i*) *if g is $\iota^{(1)}$-differentiable:*

$$\mathbf{L}_\iota\left[g^{(\iota)}(t)\right] = \Theta\mathbf{L}_\iota\left[g(t)\right] \ominus g(0), \tag{3.3}$$

(*ii*) *if g is $\iota^{(2)}$-differentiable:*

$$\mathbf{L}_\iota\left[g^{(\iota)}(t)\right] = (-g(0)) \ominus ((-\Theta)\mathbf{L}_\iota\left[g(t)\right]). \tag{3.4}$$

Theorem 3.4 ([8])**.** *Let $f(t)$, $g(t)$ be continuous fuzzy-valued functions, $\iota \in (0,1]$ and c_1, c_2 two real constants, then*

$$\mathbf{L}_\iota \left[c_1 f(t) + c_2 g(t)\right] = c_1 \mathbf{L}_\iota \left[f(t)\right] + c_2 \mathbf{L}_\iota \left[g(t)\right]. \tag{3.5}$$

4. Generalization of Conformable Fuzzy Laplace Transforms

In this section, we define conformable fractional derivatives of fractional order $0 < \iota \leq 2$ and we find fuzzy conformable Laplace transforms of the fractional order $0 < \iota \leq 2$ of fuzzy-valued function g.

Now, we introduce definitions and theoreme for $\iota \in (n, n+1]$ for some natural number n. For convenience, we concentrate on $\iota \in (1,2]$ case.

Definition 4.1. Let $G : I \to \mathbf{F}(\mathbb{R})$ be a fuzzy function and be n-differentiable at t, where $t > 0$. Then the fuzzy conformable fractional derivative of g of order ι is defined by

$$\begin{aligned} G^{(\iota)}(t) &= \lim_{\tau \to 0^+} \frac{G^{([\iota]-1)}\left(t + \tau t^{([\iota]-\iota)}\right) \ominus G^{([\iota]-1)}(t)}{\tau} \\ &= \lim_{\tau \to 0^+} \frac{G^{([\iota]-1)}(t) \ominus G^{([\iota]-1)}\left(t - \tau t^{([\iota]-\iota)}\right)}{\tau}, \end{aligned} \tag{4.1}$$

where $\iota \in (n, n+1)$ and $[\iota]$ is the smallest integer greater than or equal to ι. and the limits (in the metric d).

Theorem 4.2. *Let $G : I \to \boldsymbol{F}(\mathbb{R})$ and $\iota \in (1,2]$ and $n, m = 1, 2$. If G is (n,m)-differentiable and G is $\iota^{(n,m)}$-differentiable, then*

$$G^{(\iota^{(n,m)})}(t) = t^{2-\iota} D^{(2)}_{n,m} G(t). \tag{4.2}$$

Remark 4.3. [4] G is (n,m)-differentiable on I, if D^1_n exists on I and it be (m)-differentiable on I. The second derivatives of F are denoted by $D^{(2)}_{n,m} G(t)$ for $n, m = 1, 2$.

Proof. We present the details only for $n = m = 1$, since the other case is analogous. Let $h = \tau t^{2-\iota}$ in Definition (4.1), then $\tau = t^{\iota-2}\varpi$. Therefore, if $\tau > 0$ and $\epsilon \in [0,1]$, we have

$$\begin{aligned} &\left[D^1_1 G\left(t + \tau t^{2-\iota}\right) \ominus D^1_1 G(t)\right]^\epsilon \\ &\quad = \left[\left({}^1 g^\epsilon\right)'\left(t + \tau t^{2-\iota}\right) - \left({}^1 g^\epsilon\right)'(t), \left({}^2 g^\epsilon\right)'\left(t + \tau t^{2-\iota}\right) - \left({}^2 g^\epsilon\right)'(t)\right], \end{aligned}$$

Dividing by τ, we have

$$\frac{\left[D^1_1 G\left(t + \tau t^{2-\iota}\right) \ominus D^1_1 G(t)\right]^\epsilon}{\tau}$$

$$= \left[\frac{\left({}^1g^\epsilon\right)'\left(t+\tau t^{2-\iota}\right) - \left({}^1g^\epsilon\right)'(t)}{\tau}, \frac{\left({}^2g^\epsilon\right)'\left(t+\tau t^{2-\iota}\right) - \left({}^2g^\epsilon\right)'(t)}{\tau} \right],$$

and passing to the limit

$$\lim_{\tau\to 0^+} \frac{\left[D_1^1 G\left(t+\tau t^{2-\iota}\right) \ominus D_1^1 G(t)\right]^\epsilon}{\tau}$$
$$= \lim_{\tau\to 0^+} \left[\frac{\left({}^1g^\epsilon\right)'\left(t+\tau t^{2-\iota}\right) - \left({}^1g^\epsilon\right)'(t)}{\tau}, \frac{\left({}^2g^\epsilon\right)'\left(t+\tau t^{2-\iota}\right) - \left({}^2g^\epsilon\right)'(t)}{\tau} \right]$$
$$= \lim_{h\to 0^+} \left[\frac{\left({}^1g^\epsilon\right)'(t+\varpi) - \left({}^1g^\epsilon\right)'(t)}{t^{\iota-2}\varpi}, \frac{\left({}^2g^\epsilon\right)'(t+\varpi) - \left({}^2g^\epsilon\right)'(t)}{t^{\iota-2}\varpi} \right]$$
$$= t^{2-\iota} \lim_{\varpi\to 0^+} \left[\frac{\left({}^1g^\epsilon\right)'(t+\varpi) - \left({}^1g^\epsilon\right)'(t)}{\varpi}, \frac{\left({}^2g^\epsilon\right)'(t+\varpi) - \left({}^2g^\epsilon\right)'(t)}{\varpi} \right]$$
$$= t^{2-\iota} \left[\left({}^1g^\epsilon\right)''(t), \left({}^2g^\epsilon\right)''(t) \right].$$

Similarly, we obtain

$$\frac{\left[D_1^1 G(t) \ominus D_1^1 G\left(t-\tau t^{2-\iota}\right)\right]^\epsilon}{\tau}$$
$$= \left[\frac{\left({}^1g^\epsilon\right)'(t) - \left({}^1g^\epsilon\right)'\left(t-\tau t^{2-\iota}\right)}{\tau}, \frac{\left({}^2g^\epsilon\right)'(t) - \left({}^2g^\epsilon\right)'\left(t-\tau t^{2-\iota}\right)}{\tau} \right],$$

and passing to the limit and $\tau = t^{\iota-2}\varpi$ gives

$$G^{(\iota^{(1,1)})}(t) = t^{2-\iota} \left[\left({}^1g^\epsilon\right)''(t), \left({}^2g^\epsilon\right)''(t) \right]. \qquad \square$$

Theorem 4.4. *Let $g(t)$ be a continuous fuzzy-valued function such that $E_\iota(-\Theta, t)g(t)$ and $E_\iota(-\Theta, t)g^{(\iota)}(t)$ exist So $E_\iota(-\Theta, t)g^{(\iota)}(t)$ continuous for $\iota \in (0, 1]$, We distinguish the following cases:*

(a) *If $g(t)$ and $g^{(\iota)}(t)$ are (ι^1)-differentiable, then*

$$\mathbf{L}_\iota \left[g^{(2\iota)}(t)\right] = \left\{\Theta^2 \mathbf{L}_\iota[g(x)] \ominus \Theta g(0)\right\} \ominus g^{(\iota)}(0).$$

(b) *If $g(t)$ is (ι^1)-differentiable and $g^{(\iota)}(t)$ is (ι_2)-differentiable, then*

$$\mathbf{L}_\iota \left[g^{(2\iota)}(t)\right] = \left(-g^{(\iota)}(0)\right) \ominus \left\{-\Theta^2 \mathbf{L}_\iota[g(t)] \ominus (-\Theta g(0))\right\}.$$

(c) *If $g(t)$ is (ι_2)-differentiable and $g^{(\iota)}(t)$ is (ι_1)-differentiable, then*

$$\mathbf{L}_\iota \left[g^{(2\iota)}(t)\right] = \left\{(-\Theta g(0)) \ominus \left(-\Theta^2 \mathbf{L}_\iota[g(t)]\right)\right\} \ominus g^{(\iota)}(0).$$

(d) *If $g(t)$ is (ι^2)-differentiable and $g^{(\iota)}(t)$ is (ι^2)-differentiable, then*

$$\mathbf{L}_\iota\left[g^{(2\iota)}(t)\right] = \left(-g^{(\iota)}(0)\right) \ominus \left\{\Theta g(0) \ominus \Theta^2 \mathbf{L}_\iota[g(t)]\right\}.$$

Proof. (a) Let $0 < \iota \leq 1$, assume that $g(t)$ and $g^{(\iota)}(t)$ are (ι^1)-differentiable, then applying (3.3) to $f(t)$ and $g^{(\iota)}(t)$, respectively, we get

$$\mathbf{L}_\iota\left[g^{(\iota)}(t)\right] = \Theta \mathbf{L}_\iota[g(t)] \ominus g(0),$$

and

$$\mathbf{L}_\iota\left[g^{(2\iota)}(t)\right] = \Theta \mathbf{L}_\iota\left[g^{(\iota)}(t)\right] \ominus g^{(\iota)}(0).$$

Combining these identities yields

$$\begin{aligned}\mathbf{L}_\iota\left[g^{(\iota)}(t)\right] &= \Theta\left\{\Theta \mathbf{L}_\iota[g(t)] \ominus g(0)\right\} \ominus g^{(\iota)}(0),\\ &= \left\{\Theta^2 \mathbf{L}_\iota[g(t)] \ominus \Theta g(0)\right\} \ominus g^{(\iota)}(0).\end{aligned}$$

(b) Assume that $g(t)$ is (ι^1)-differentiable and $g^{(\iota)}(t)$ is (ι^2)-differentiable, then applying (3.3) and (3.4) to $g(t)$ and $g^{(\iota)}(t)$, respectively, we get

$$\mathrm{L}_\iota\left[g^{(\iota)}(t)\right] = \Theta \mathrm{L}_\iota[g(t)] \ominus f(0),$$

and

$$\mathrm{L}_\iota\left[g^{(2\iota)}(t)\right] = \left(-g^{(\iota)}(0)\right) \ominus (-\Theta)\mathrm{L}_\iota\left[g^{(\iota)}(t)\right].$$

The result of combining these identities is

$$\begin{aligned}\mathrm{L}_\iota\left[g^{(2\iota)}(t)\right] &= \left(-g^{(\iota)}(0)\right) \ominus (-\Theta)\{\Theta \mathrm{L}_\iota[g(t)] \ominus g(0)\}\\ &= \left(-g^{(\iota)}(0)\right) \ominus \left\{-\Theta^2 \mathrm{L}_\iota[g(t)] \ominus (-\Theta g(0))\right\}.\end{aligned}$$

(c) If $g(t)$ is (ι^2)-differentiable and $g^{(\iota)}(t)$ is (ι^1)-differentiable, then

$$\mathrm{L}_\iota\left[g^{(\iota)}(t)\right] = (-g(0)) \ominus (-\Theta)\mathrm{L}_\iota[g(t)],$$

and

$$\mathrm{L}_\iota\left[g^{(2\iota)}(t)\right] = \Theta \mathrm{L}_\iota\left[g^{(\iota)}(t)\right] \ominus g^{(\iota)}(0).$$

By combining of these identities, we get

$$\begin{aligned}\mathrm{L}_\iota\left[g^{(2\iota)}(t)\right] &= \Theta\{(-g(0)) \ominus (-\Theta)\mathbf{L}_\iota[g(t)]\} \ominus g^{(\iota)}(0)\\ &= \left\{(-\Theta g(0)) \ominus \left(-\Theta^2\right) \mathrm{L}_\iota[g(t)]\right\} \ominus g^{(\iota)}(0).\end{aligned}$$

(*d*) Assume that $g(t)$ and $g^{(\iota)}(t)$ are (ι^2)-differentiable, then

$$\mathrm{L}_\iota\left[g^{(\iota)}(t)\right] = (-g(0)) \ominus (-\Theta)\mathrm{L}_\iota[g(t)],$$

and

$$\mathrm{L}_\iota\left[g^{(2\iota)}(t)\right] = \left(-g^{(\iota)}(0)\right) \ominus (-\Theta)\mathrm{L}_\iota\left[g^{(\iota)}(t)\right],$$

When these identities are combined, the result is

$$\begin{aligned}\mathrm{L}_\iota\left[g^{(2\iota)}(t)\right] &= \left(-g^{(\iota)}(0)\right) \ominus (-\Theta)\{(-g(0)) \ominus (-\Theta)\mathrm{L}_\iota[g(t)]\}\\ &= \left(-g^{(\iota)}(0)\right) \ominus \left\{\Theta g(0) \ominus \Theta^2\mathrm{L}_\iota[g(t)]\right\}.\end{aligned}$$

□

5. Algorithem for Solving Fuzzy Fractional Differential Equations by Fuzzy Conformable Laplace Transform

Consider the fuzzy fractional differential equation:

$$\begin{cases} y^{(2\iota)}(t) = g\left(t, y(t), y^{(\iota)}(t)\right), & \\ y(0) = y_0, & y_0 = \left({}^1y_0, {}^2y_0\right) \in \mathbf{F}(\mathbb{R}),\\ y^{(\iota)}(0) = z_0, & z_0 = \left({}^1z_0, {}^2z_0\right) \in \mathbf{F}(\mathbb{R}),\end{cases}$$

where $y(t) = ({}^1y^\epsilon(t), {}^2y^\epsilon(t))$ is a fuzzy function of $t \geq 0$ and for all $\iota \in (0,1]$, $g\left(t, y(t), y^{(\iota)}(t)\right)$ is a fuzzy-valued function, which is linear with respect to $\left(y(t), y^{(\iota)}(t)\right)$. The fuzzy conformable Laplace transform is used to produce

$$\mathrm{L}_\iota\left[y^{(2\iota)}(t)\right] = \mathrm{L}_\iota\left[g\left(t, y(t), y^{(\iota)}(t)\right)\right]. \tag{5.1}$$

After that, we have the following options for solving (5.1): (a) Case I: If y and $y^{(\iota)}$ are (ι^1)-differentiable: $y^{(\iota)}(t) = \left(\left({}^1y^\epsilon\right)^{(\iota)}(t), \left({}^2y^\epsilon\right)^{(\iota)}(t)\right)$ and

$$\begin{aligned} y^{(2\iota)}(t) &= \left(\left(y_1^\epsilon\right)^{(2\iota)}(t), \left({}^2y^\epsilon\right)^{(2\iota)}(t)\right),\\ \mathrm{L}_\iota\left[y^{(2\iota)}(t)\right] &= \left\{\Theta^2\mathrm{L}_\iota[y(t)] \ominus \Theta y(0)\right\} \ominus y^{(\iota)}(0).\end{aligned}$$

Therefore

$$\mathrm{L}_\iota\left[g\left(t, y(t), y^{(\iota)}(t)\right)\right] = \left\{\Theta^2\mathrm{L}_\iota[y(t)] \ominus \Theta y_0\right\} \ominus z_0,$$

Hence

$$\begin{cases} \mathcal{L}_\iota\left[{}^1g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota[y_1^\epsilon(t)] - \Theta^1y_0^\epsilon -^1 z_0^\epsilon,\\ \mathcal{L}_\iota\left[{}^2g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota[y_2^\epsilon(t)] - \Theta^2y_0^\epsilon -^2 z_0^\epsilon,\end{cases} \tag{5.2}$$

where

${}^1g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)$

$$= \min\left\{g(t,u,v)/u \in \left({}^1y^\epsilon(t),{}^2y^\epsilon(t)\right); v \in \left(\left({}^1y^\epsilon\right)^{(\iota)}(t), \left({}^2y^\epsilon\right)^{(\iota)}(t)\right)\right\},$$

and

$$g_2^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)$$
$$= \max\left\{g(t,u,v)/u \in \left({}^1y^\epsilon(t),{}^2y^\epsilon(t)\right); v \in \left(\left({}^1y^\epsilon\right)^{(\iota)}(t), \left({}^2y^\epsilon\right)^{(\iota)}(t)\right)\right\}.$$

Assume that this leads to

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = \Psi_1^\epsilon(\Theta), \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \Omega_1^\epsilon(\Theta), \end{cases}$$

where the couple $(\Psi_1^\epsilon(\Theta), \Omega_1^\epsilon(\Theta))$ is a solution of the system (5.2). The inverse conformable Laplace transform is used to obtain

$$\begin{cases} {}^1y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Psi_1^\epsilon(\Theta)\right], \\ {}^2y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Omega_1^\epsilon(\Theta)\right]. \end{cases},$$

(b) Case II: If y is (ι^1)-differentiable and $y^{(\iota)}$ is (ι^2)-differentiable:

$$y^{(\iota)}(t) = \left(\left({}^1y^\epsilon\right)^{(\iota)}, \left({}^2y^\epsilon\right)^{(\iota)}\right)$$

and $y^{(2\iota)}(t) = \left(\left({}^1y^\epsilon\right)^{(2\iota)}(t), \left({}^2y^\epsilon\right)^{(2\iota)}(t)\right)$ and

$$\mathbf{L}_\iota\left[y^{(2\iota)}(t)\right] = \left(-y^{(\iota)}(0)\right) \ominus \left\{-\Theta^2\mathbf{L}_\iota[y(t)] \ominus (-\Theta y(0))\right\}.$$

Therefore

$$\mathbf{L}_\iota\left[g\left(t, y(t), y^{(\iota)}(t)\right)\right] = (-z_0) \ominus \left\{-\Theta^2\mathbf{L}_\iota[y(t)] \ominus (-\Theta y_0)\right\}.$$

Hence

$$\begin{cases} \mathcal{L}_\iota\left[{}^2g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota[{}^1y^\epsilon(t)] - \Theta^1y_0^\epsilon -^1 z_0^\epsilon, \\ \mathcal{L}_\iota\left[{}^1g^\epsilon\left(t, y(t), y'(t)\right)\right] = \Theta^2\mathcal{L}_\iota[{}^2y^\epsilon(t)] - \Theta^2y_0^\epsilon -^2 z_0^\epsilon, \end{cases} \tag{5.3}$$

that this implies

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = \Psi_2^\epsilon(\Theta), \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \Omega_2^\epsilon(\Theta), \end{cases}$$

where $(\Psi_2^\epsilon(\Theta), \Omega_2^\epsilon(\Theta))$ is a solution of the system (5.3). We may reach this result by applying the inverse conformable Laplace transform:

$$\begin{cases} {}^1y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Psi_2^\epsilon(\Theta)\right], \\ {}^2y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Omega_2^\epsilon(\Theta)\right]. \end{cases}$$

(c) Case III: If y is (ι^2)-differentiable and $y^{(\iota)}$ is (ι^1)-differentiable:

$$y^{(\iota)}(t) = \left(\left({}^1y^\epsilon\right)^{(\iota)}(t), \left({}^2y^\epsilon\right)^{(\iota)}(t)\right),$$
$$y^{(2\iota)}(t) = \left(\left({}^1y^\epsilon\right)^{(2\iota)}(t), \left({}^2y^\epsilon\right)^{(2\iota)}(t)\right),$$

$$\mathbf{L}_\iota\left[y^{(2\iota)}(t)\right] = \left\{(-\Theta y(0)) \ominus \left(-\Theta^2\mathbf{L}_\iota[y(t)]\right)\right\} \ominus y^{(\iota)}(0).$$

Therefore

$$\mathbf{L}_\iota\left[g\left(t, y(t), y^{(\iota)}(t)\right)\right] = \left\{(-\Theta y(0)) \ominus \left(-\Theta^2\mathbf{L}_\iota[y(t)]\right)\right\} \ominus y^{(\iota)}(0).$$

Hence

$$\text{(5.4)} \qquad \begin{cases} \mathcal{L}_\iota\left[{}^2g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota[y_1^\epsilon(t)] - \Theta^1 y_0^\epsilon -^2 z_0^\epsilon, \\ \mathcal{L}_\iota\left[{}^1g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota[{}^2y^\epsilon] - \Theta^2 y_0^\epsilon -^1 z_0^\epsilon, \end{cases}$$

that this leads to

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = \Psi_3^\epsilon(\Theta), \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \Omega_3^\epsilon(\Theta), \end{cases}$$

where $(\Psi_3^\epsilon(\Theta), \Omega_3^\epsilon(\Theta))$ is a solution of the system (5.4). By using the inverse Laplace transform, we get

$$\begin{cases} {}^1y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Psi_3^\epsilon(\Theta)\right], \\ {}^2y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Omega_3^\epsilon(\Theta)\right]. \end{cases}$$

(d) Case IV: If y and $y^{(\iota)}$ are (ι^2)-differentiable:

$$y^{(\iota)}(t) = \left(\left(y_1^\epsilon\right)^{\iota)}(t), \left({}^2y^\epsilon\right)^{\iota)}(t)\right),$$

$$y^{(2\iota)}(t) = \left(\left({}^1y^\epsilon\right)^{2\iota)}(t), \left({}^1y^\epsilon\right)^{2\iota)}(t)\right),$$

$$\mathbf{L}_\iota\left[y^{(2\iota)}(t)\right] = \left(-y^{(\iota)}(0)\right) \ominus \left\{\Theta y(0) \ominus \Theta^2\mathbf{L}_\iota[y(t)]\right\}.$$

Therefore

$$\mathbf{L}_\iota\left[g\left(t, y(t), y^{(\iota)}(t)\right)\right] = \left(-y^{(\iota)}(0)\right) \ominus \left\{\Theta y(0) \ominus \Theta^2\mathbf{L}_\iota[y(t)]\right\}.$$

Hence

$$\text{(5.5)} \qquad \begin{cases} \mathcal{L}_\iota\left[{}^1g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota\left[\left({}^1y^\epsilon(t)\right)\right] - \Theta^1 y_0^\epsilon -^2 z_0^\epsilon, \\ \mathcal{L}_\iota\left[{}^2g^\epsilon\left(t, y(t), y^{(\iota)}(t)\right)\right] = \Theta^2\mathcal{L}_\iota\left[\left({}^2y^\epsilon(t)\right)\right] - \Theta^2 y_0^\epsilon -^1 z_0^\epsilon, \end{cases}$$

that this implies

$$\begin{cases} \mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] = \Psi_4^\epsilon(\Theta), \\ \mathcal{L}_\iota\left[{}^2y^\epsilon(t)\right] = \Omega_4^\epsilon(\Theta), \end{cases}$$

where $(\Psi_4^\epsilon(\Theta), \Omega_4^\epsilon(\Theta))$ is a solution of the system (5.5) the inverse Laplace transform is used to obtain

$$\begin{cases} y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Psi_4^\epsilon(\Theta)\right], \\ y^\epsilon(t) = \mathcal{L}_\iota^{-1}\left[\Omega_4^\epsilon(\Theta)\right]. \end{cases}$$

The following cases must be discussed: (1) Case (I.1): If $\eta \geq 0$ and $\beta \geq 0$, then the system (5.2) is equivalent to

$$\begin{cases} \Theta^2\mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] - \Theta^1 y_0^\epsilon -^1 z_0^\epsilon = (\eta + \beta\Theta)\mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] - b^1 y^\epsilon(t) + \lambda, \\ \Theta^2\mathcal{L}_\iota\left[{}^2y^\epsilon(t)\right] - \Theta^2 y_0^\epsilon -^2 z_0^\epsilon = (\eta + \beta\Theta)\mathcal{L}_\iota[{}^2y^\epsilon(t)] - b^2 y_0^\epsilon + \lambda. \end{cases}$$

By consequence

$$\begin{cases} \Psi_1^\epsilon(\Theta) = \mathcal{L}_\iota \left[{}^1y^\epsilon(t)\right] = \frac{(\Theta-\beta)^1y_0^\epsilon + {}^1z_0^\epsilon + \lambda}{\Theta^2-\beta\Theta-\eta}, \\ \Omega_1^\epsilon(\Theta) = \mathcal{L}_\iota \left[{}^2y^\epsilon(t)\right] = \frac{(\Theta-\beta)^2y_0^\epsilon + {}^2z_0^\epsilon + \lambda}{\Theta^2-\beta\Theta-\eta}. \end{cases}$$

(2) Case (I.2): If $\eta \geq 0$ and $\beta < 0$, then (5.2) is equivalent to the system:

$$\begin{cases} \left(\Theta^2-\eta\right)\mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] - \beta\Theta\mathcal{L}_\iota\left[{}^2y^\epsilon(t)\right] = \Theta^1y_0^\epsilon +^1 z_0^\epsilon - \beta^2y_0^\epsilon + \lambda, \\ \left(\Theta^2-\eta\right)\mathcal{L}_\iota\left[{}^2y^\epsilon(t)\right] + \beta\Theta\mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] = \Theta^2y_0^\epsilon +^2 z_0^\epsilon - \beta^1y_0^\epsilon + \lambda. \end{cases}$$

Denote

$$\begin{cases} \Lambda^\epsilon(\Theta) = \Theta^1y_0^\epsilon +^1 z_0^\epsilon - \beta^2y_0^\epsilon + \lambda, \\ \Delta^\epsilon(\Theta) = \Theta^2y_0^\epsilon +^2 z_0^\epsilon - \beta^1y_0^\epsilon + \lambda. \end{cases} \tag{5.6}$$

Hence

$$\begin{cases} \Psi_1^\epsilon(\Theta) = \mathcal{L}[y_\epsilon^\epsilon(t)] = \frac{\left(\Theta^2-\eta\right)\Lambda^\epsilon(\Theta)+\beta\Theta\Delta^\epsilon(\Theta)}{\left(\Theta^2-\eta\right)^2+(\beta\Theta)^2}, \\ \Omega_1^\epsilon(\Theta) = \mathcal{L}_\iota[y_\epsilon^\epsilon(t)] = \frac{\left(\Theta^2-\eta\right)\Delta^\epsilon(\Theta)-\beta\Theta\Lambda^\epsilon(\Theta)}{\left(\Theta^2-\eta\right)^2+(\beta\Theta)^2} \end{cases}$$

(3) Case (I.3): If $\eta < 0$ and $\beta \geq 0$, then (5.4) is equivalent to the system:

$$\begin{cases} \left(\Theta^2-\beta\Theta\right)\mathcal{L}_\iota[{}^1y^\epsilon(t)] - \eta\mathcal{L}_\iota[{}^2y^\epsilon(t)] = \Theta^1y_0^\epsilon +^1 z_0^\epsilon - \beta^1y_0^\epsilon + \lambda, \\ \left(\Theta^2-\beta\Theta\right)\mathcal{L}_\iota[{}^2y^\epsilon(t)] - \eta\mathcal{L}_\iota[{}^1y^\epsilon(t)] = \Theta^2y_0^\epsilon(\epsilon) +^2 z_0^\epsilon - \beta^1y_0^\epsilon + \lambda. \end{cases}$$

Therefore

$$\begin{cases} \Psi_1^\epsilon(\Theta) = \mathcal{L}_\iota[{}^1y^\epsilon(t)] = \frac{\left(\Theta^2-\beta\Theta\right)\Lambda^\epsilon(\Theta)+\eta\Delta^\epsilon(\Theta)}{\left(\Theta^2-\beta\Theta\right)^2+\eta^2}, \\ \Omega_1^\epsilon(\Theta) = \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \frac{\left(\Theta^2-\eta\right)\Delta^\epsilon(\Theta)+\eta\Lambda^\epsilon(\Theta)}{\left(\Theta^2-\beta\Theta\right)^2+\eta^2}. \end{cases}$$

(4) Case (I .4): If $\eta < 0$ and $\beta < 0$, then (5.5) is equivalent to the system:

$$\begin{cases} \Theta^2\mathcal{L}_\iota[{}^1y^\epsilon(t)] - (\eta+\beta\Theta)\mathcal{L}_\iota[{}^2y^\epsilon(t)] = \Theta^1y_0^\epsilon +^1 z_0^\epsilon - \beta^2y_0^\epsilon + \lambda, \\ \Theta^2\mathcal{L}_\iota[{}^2y^\epsilon(t)] - (a+b\Theta)\mathcal{L}_\iota[{}^1y^\epsilon(t)] = \Theta^2y_0^\epsilon +^2 z_0^\epsilon - \beta^1y_0^\epsilon + \lambda. \end{cases}$$

Therefore

$$\begin{cases} \Psi_1^\epsilon(\Theta) = \mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] = \frac{\Theta^2\Lambda^\epsilon(\Theta)+(\eta+\beta\Theta)\Delta^\epsilon(\Theta)}{\Theta^4+(\eta+\beta\Theta)^2}, \\ \Omega_1^\epsilon(\Theta) = \mathcal{L}_\iota\left[{}^2y^\epsilon(t)\right] = \frac{\Theta^2\Delta^\epsilon(\Theta)+(\eta+\beta\Theta)\Lambda^\epsilon(\Theta)}{\Theta^4+(\eta+\beta\Theta)^2}. \end{cases}$$

Here $\Lambda^\epsilon(\Theta)$ and $\Delta^\epsilon(\Theta)$ are given by (5.6).

Similarly, the respective expressions of $\Psi_2^\epsilon(\Theta)$, $\Omega_2^\epsilon(\Theta)$, $\Psi_3^\epsilon(\Theta)$, $\Omega_3^\epsilon(\Theta)$, $\Psi_4^\epsilon(\Theta)$, $\Omega_4^\epsilon(\Theta)$ can be computed.

Example 5.1 ([5])**.** Consider the simple harmonic vibration equation

$$\begin{cases} y^{(2\iota)}(x) + \omega^2y(x) = \sigma_0, \\ y(0,\epsilon) = (\epsilon-1, 1-\epsilon), \\ y'(0,\epsilon) = (\epsilon-1, 1-\epsilon). \end{cases}$$

where $\sigma_0 = (\epsilon, 2-\epsilon)$, $\quad 1 < 2\iota \leq 2 \quad$ and $\quad \omega = 1$.

Case I: If $y(t)$ and $y^{(\iota)}(t)$ are (ι^1)-differentiable, then

$$\begin{cases} \left({}^1y^\epsilon\right)^{(2\iota)}(t) + \left({}^1y^\epsilon\right)(t) = \epsilon, \\ \left({}^2y^\epsilon\right)^{(2\iota)}(t) + \left({}^2y^\epsilon\right)(t) = 2-\epsilon. \end{cases}$$

Therefore

$$\begin{cases} \mathcal{L}_\iota\left[\left({}^1y^\epsilon\right)^{(2\iota)}(t)\right] + \mathcal{L}_\iota[\left({}^1y^\epsilon\right)(t)] = \frac{\epsilon}{\Theta}, \\ \mathcal{L}_\iota\left[\left({}^2y^\epsilon\right)^{(2\iota)}(t)\right] + \mathcal{L}_\iota\left[\left({}^2y^\epsilon\right)(t)\right] = \frac{2-\epsilon}{\Theta}. \end{cases}$$

Using Theorem 4.4 , we get

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = (\epsilon-1)\frac{\Theta+1}{\Theta^2+1} + \epsilon\left(\frac{1}{\Theta} - \frac{\Theta}{\Theta^2+1}\right), \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = (1-\epsilon)\frac{\Theta+1}{\Theta^2+1} + (2-\epsilon)\left(\frac{1}{\Theta} - \frac{\Theta}{\Theta^2+1}\right). \end{cases}$$

By using the inverse Laplace transform, we deduce

$$\begin{cases} {}^1y^\epsilon(t) = \epsilon\left(1+\sin(\frac{t^\iota}{\iota})\right) - \sin(\frac{t^\iota}{\iota}) - \cos(\frac{t^\iota}{\iota}), \\ {}^2y^\epsilon(t) = (2-\epsilon)\left(1+\sin(\frac{t^\iota}{\iota})\right) - \sin(\frac{t^\iota}{\iota}) - \cos(\frac{t^\iota}{\iota}). \end{cases}$$

In this case, no solution exists, since $y^{(\iota)}(t)$ is not an (ι^1)-differentiable fuzzy-valued function [1].

Case II: If $y(t)$ is (ι^1)-differentiable and $y^{(\iota)}(t)$ is (ι^2)-differentiable, then

$$\begin{cases} \mathcal{L}_\iota\left[\left({}^2y^\epsilon\right)^{(2\iota)}(t)\right] + \mathcal{L}_\iota\left[{}^1y^\epsilon(t)\right] = \frac{\epsilon}{\Theta}, \\ \mathcal{L}_\iota\left[\left({}^1y^\epsilon\right)^{(2\iota)}(t)\right] + \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \frac{2-\epsilon}{\Theta}. \end{cases}$$

Using Theorem 4.4 , we get

$$\begin{cases} \Theta^2\mathcal{L}_\iota[{}^2y^\epsilon(t)] + \mathcal{L}_\iota[{}^1y^\epsilon(t)] = (1-\epsilon)(\Theta+1) + \frac{\epsilon}{\Theta}, \\ \Theta^2\mathcal{L}_\iota[{}^1y^\epsilon(t)] + \mathcal{L}_\iota[{}^2y^\epsilon(t)] = (\epsilon-1)(\Theta+1) + \frac{2-\epsilon}{\Theta}. \end{cases}$$

Thus

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = \epsilon\left(\frac{1}{2(\Theta-1)} - \frac{1}{2(\Theta+1)} + \frac{1}{\Theta}\right) + \frac{1}{2(\Theta+1)} - \frac{1}{2(\Theta-1)} - \frac{\Theta}{\Theta^2+1}, \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \epsilon\left(\frac{1}{2(\Theta+1)} - \frac{1}{2(\Theta-1)} - \frac{1}{\Theta}\right) + \frac{2}{\Theta} + \frac{1}{2(\Theta-1)} - \frac{1}{2(\Theta+1)} - \frac{\Theta}{\Theta^2+1}. \end{cases}$$

By using the inverse Laplace transform, we deduce

$$\begin{cases} {}^1y^\epsilon(t) = \epsilon\left(1+\sinh(\frac{t^\iota}{\iota})\right) - \sinh(\frac{t^\iota}{\iota})) - \cos(\frac{t^\iota}{\iota})), \\ {}^2y^\epsilon(t) = (2-\epsilon)\left(1+\sinh(\frac{t^\iota}{\iota}))\right) - \sinh(\frac{t^\iota}{\iota})) - \cos(\frac{t^\iota}{\iota})). \end{cases}$$

In case I, there is no solution [1].

Case III: If $y(t)$ is (ι^2)-differentiable and $y^{(\iota)}(t)$ is (ι_1)-differentiable, then

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = \epsilon\left(\frac{1}{2(\Theta+1)} - \frac{1}{2(\Theta-1)} + \frac{1}{\Theta}\right) + \frac{1}{2(\Theta-1)} - \frac{1}{2(\Theta+1)} - \frac{\Theta}{\Theta^2+1}, \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = \epsilon\left(\frac{1}{2(\Theta-1)} - \frac{1}{2(\Theta+1)} - \frac{1}{\Theta}\right) + \frac{2}{\Theta} + \frac{1}{2(\Theta+1)} - \frac{1}{2(\Theta-1)} - \frac{\Theta}{\Theta^2+1}. \end{cases}$$

By using the inverse Laplace transform, we deduce

$$\begin{cases} {}^1y^\epsilon(t) = \epsilon(1 - \sinh(\frac{t^\iota}{\iota})) + \sinh(\frac{t^\iota}{\iota}) - \cos(\frac{t^\iota}{\iota}), \\ {}^2y^\epsilon(t) = (2-\epsilon)(1 - \sinh(\frac{t^\iota}{\iota})) + \sinh(\frac{t^\iota}{\iota}) - \cos(\frac{t^\iota}{\iota}). \end{cases}$$

In this case, since $y(t)$ is (ι^2)-differentiable and $y^{(\iota)}(t)$ is (ι^1)-differentiable, the solution is acceptable for $t \in (0, \ln(1+\sqrt{2}))$ [1].

Case IV: If $y(t)$ and $y^{(\iota)}(t)$ are (ι^2)-differentiable, then

$$\begin{cases} \mathcal{L}_\iota[{}^1y^\epsilon(t)] = (\epsilon - 1)\left(\frac{\Theta}{\Theta^2+1} - \frac{1}{\Theta^2+1}\right) + \epsilon\left(\frac{1}{\Theta} - \frac{\Theta}{\Theta^2+1}\right), \\ \mathcal{L}_\iota[{}^2y^\epsilon(t)] = (1 - \epsilon)\left(\frac{\Theta}{\Theta^2+1} - \frac{1}{\Theta^2+1}\right) + (2-\epsilon)\left(\frac{1}{\Theta} - \frac{\Theta}{\Theta^2+1}\right). \end{cases}$$

By using the inverse Laplace transform, we deduce

$$\begin{cases} {}^1y^\epsilon(t) = \epsilon(1 - \sin(\frac{t^\iota}{\iota})) + \sin(\frac{t^\iota}{\iota}) - \cos(\frac{t^\iota}{\iota}), \\ {}^2y^\epsilon(t) = (2-\epsilon)(1 - \sin(\frac{t^\iota}{\iota})) + \sin(\frac{t^\iota}{\iota}) - \cos(\frac{t^\iota}{\iota}). \end{cases}$$

In this case, the solution is acceptable for $t \in \left(0, \frac{\pi}{2}\right)$ [1].

6. Conclusion

This research aims to develop and prove some results regarding fuzzy conformable differentiability of order $1 < \iota \leq 2$. It also aims to establish the relationship between a fuzzy function conformable Laplace transforms. This study uses the fuzzy conformable Laplace transform method to solve fuzzy conformable differential equations of order $0 < \iota \leq 2$ (FDEs) under generalized conformable differentiability. The efficiency of the suggested strategy is demonstrated by a numerical example.

We will solve fractional fuzzy conformable partial differential equations and use the conformable Laplace method to solve a large class of Fuzzy Fractional differential equations FDEs in future studies.

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