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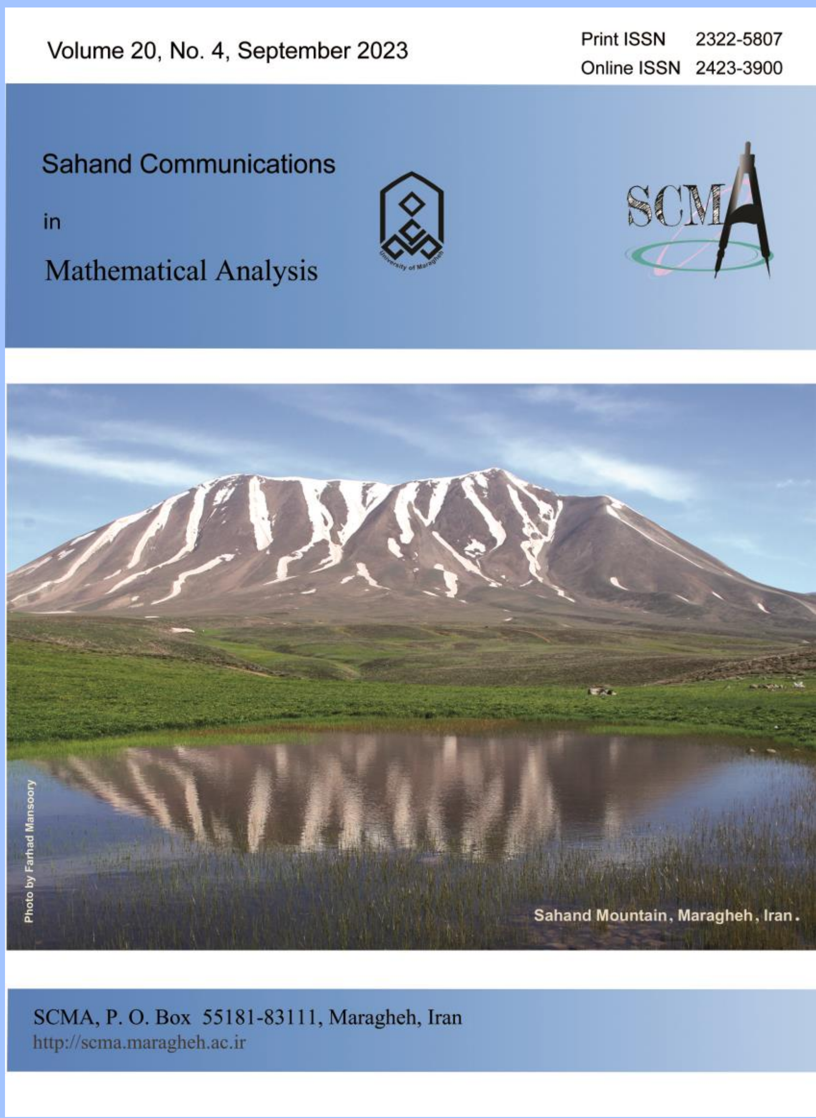

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Abstract. Assuming that Λ is a bounded operator on a Hilbert space H, this study investigate the structure of the g-frames generated by iterations of Λ. Specifically, we provide characterizations of g-frames in the form of $\{\Lambda^n\}_{n=1}^{\infty}$ and describe some conditions under which the sequence $\{\Lambda^n\}_{n=1}^{\infty}$ forms a g-frame for H. Additionally, we verify the properties of the operator Λ when $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H. Moreover, we study the g-Riesz bases and dual g-frames which are generated by iterations. Finally, we discuss the stability of these types of g-frames under some perturbations.

1. Introduction

A frame for a Hilbert space is a generalization of a basis of a vector space to sets that may not be linearly independent. The properties of frames are highly valuable in many fields, including function spaces, signal processing and and broader applications in applied mathematics, computer science and engineering. We refer to [3] for an introduction to frame theory and along with its details and applications. Over time, various types of frames have been introduced. One of them is the g-frame. G-frames are generalized frames, which have been introduced by W. Sun in [8].

In this paper, we focus on a very special class of the g-frames for a Hilbert space, which are generated by iterations of a bounded operator on the underlying Hilbert space. In other words, if Λ is a bounded operator on Hilbert space H, we aim to answer the following questions:

(i) What conditions on the operator Λ are necessary or sufficient to $\{\Lambda^n\}_{n=1}^{\infty}$ constitute a g-frame for H?

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(ii) If $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, then what can be derived about the operator Λ?

We address these questions and explore additional properties of g-frames and g-Riesz bases generated by iterated operators. Some other papers dealing with iterated actions of operators include [1, 2, 6].

The paper is structured as follows. In Section 2, we provide some new results on all kinds of g-frames. In Section 3, we characterize the g-frames which are generated by iterated operators. Specifically, we investigate the properties of operators on a Hilbert space whose iterations generate a g-frame. Furthermore, some other features of these g-frame such as associated frame operators and eigenvalues are verified. Section 4 focuses on to the g-Riesz bases from iterated operators. Eventually, in Section 5, we study the stability of g-frames obtained from iterated operators.

Throughout this paper, H is a separable Hilbert space and $\{H_i\}_{i\in I}$ represent a sequence of separable Hilbert spaces, where the index set I is a subset of $\mathbb{Z}$. Also, $B(H, H_i)$ denotes the set of all bounded linear operators from H into H_i and $B(H)$ is the set of all bounded linear operators on H.

Definition 1.1. We call a sequence $\{\Lambda_i \in B(H, H_i) : i \in I\}$ a generalized frame, or simply a g-frame, for H with respect to $\{H_i\}_{i\in I}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H. \tag{1.1}$$

We call A and B the lower and upper frame bounds, respectively.

$\{\Lambda_i \in B(H, H_i) : i \in I\}$ is called a λ-tight g-frame if $A = B = \lambda$ and is called a Parseval g-frame if $A = B = 1$. If only the right hand-side of (1.1) holds, we call it a g-Bessel sequence.

For a sequence $\{H_i\}_{i\in I}$ of Hilbert spaces, define

$$\left(\sum_{i\in I} \oplus H_i\right)_{l_2} = \left\{\{f_i\}_{i\in I} \mid f_i \in H_i,\ i \in I \quad \text{and} \quad \sum_{i\in I} \|f_i\|^2 < \infty\right\}.$$

It is easy to show that with pointwise operations and with the inner product defined by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle,$$

$\left(\sum_{i\in I} \oplus H_i\right)_{l_2}$ is a Hilbert space.

If for each $i \in I$, $H_i = H$, then we show the Hilbert space $\left(\sum_{i\in I} \oplus H_i\right)_{l_2}$ by $l^2(H, I)$.

The synthesis operator for a g-Bessel sequence $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is defined as follows:

$$T : \left(\sum_{i\in I} \oplus H_i\right)_{l_2} \to H$$

$$T(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(f_i). \tag{1.2}$$

It is proved in [7], T is well defined, bounded and the adjoint operator of T_Λ is given by

$$T^* : H \to \left(\sum_{i\in I} \oplus H_i\right)_{l_2}$$

$$T^*(f) = \{\Lambda_i f\}_{i\in I}. \tag{1.3}$$

The operator T^* is called the analysis operator of $\{\Lambda_i\}_{i\in I}$. Also, the g-frame operator of $\{\Lambda_i\}_{i\in I}$ is defined as follows:

$$S : H \to H, \qquad Sf = TT^*f = \sum_{i\in I} \Lambda_i^*\Lambda_i f, \tag{1.4}$$

which is a bounded, self-adjoint, positive and invertible operator on H and

$$AI \le S \le BI,$$

where I is identity operator on H (see [7]).

Definition 1.2. Let $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ be g-frames for H with respect to $\{H_i\}_{i\in I}$. Then $\{\Theta_i\}_{i\in I}$ is called a dual g-frame for $\{\Lambda_i\}_{i\in I}$, if it satisfies

$$f = \sum_{i\in I} \Lambda_i^*\Theta_i f, \quad f \in H.$$

Definition 1.3. Consider the sequence $\{\Lambda_i \in B(H, H_i),\ i \in I\}$.

(i) If $\{f : \Lambda_i f = 0,\ i \in I\} = \{0\}$, then we say that $\{\Lambda_i\}_{i\in I}$ is g-complete.

(ii) If $\{\Lambda_i\}_{i\in I}$ is g-complete and there are positive constants A and B such that for any finite subset $J \subseteq I$ and $g_j \in H_j$, $j \in J$,

$$A\sum_{j\in J} \|g_j\|^2 \le \left\|\sum_{j\in J} \Lambda_i^* g_j\right\|^2 \le B\sum_{j\in J} \|g_j\|^2, \tag{1.5}$$

then we say that $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$.

The next proposition is applied in the following of paper.

Proposition 1.4. *If T is an operator on the Hilbert space H such that both T and T^* are bounded below, then T is invertible.*

Proof. See to proof of [5, Corollary 4.9]. □

2. Some New Results on all g-Frames

In this section, we present some new results on all kind of g-frames which are also held in case of ordinary frames.

The following Lemma helps us to find the sequence $\{\varphi_i\}_{i=1}^{\infty}$ with minimal ℓ^2-norm among all sequences representing an element f by a g-frame.

Lemma 2.1. *Let $\{\Lambda_i\}_{i=1}^{\infty}$ be a g-frame for H with respect to $\{H_i\}_{i\in I}$ and with frame operator S. Suppose that $f \in H$ and f has a representation $f = \sum_{i=1}^{\infty} \Lambda_i^*(\varphi_i)$ for some $\{\varphi_i\}_{i=1}^{\infty} \in \left(\sum_{i\in I} \oplus H_i\right)_{l_2}$. Then*

$$\sum_{i=1}^{\infty} \|\varphi_i\|^2 = \sum_{i=1}^{\infty} \left\|\varphi_i - \Lambda_i\left(S^{-1}f\right)\right\|^2 + \sum_{i=1}^{\infty} \left\|\Lambda_i\left(S^{-1}f\right)\right\|^2.$$

Proof. We can write

$$\{\varphi_i\}_{i=1}^{\infty} = \{\varphi_i\}_{i=1}^{\infty} - \left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty} + \left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty}.$$

Since $f = \sum_{i=1}^{\infty} \Lambda_i^*(\varphi_i) = \sum_{i=1}^{\infty} \Lambda_i^*\left(\Lambda_i S^{-1} f\right)$, So

$$\sum_{i=1}^{\infty} \Lambda_i^*(\varphi_i - \Lambda_i S^{-1} f) = 0.$$

That is, $\{\varphi_i\}_{i=1}^{\infty} - \left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty} \in N_T = R_{T^*}^{\perp}$, where T is the synthesis operator of $\{\Lambda_i\}_{i=1}^{\infty}$. Also, $\left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty} \in R_{T^*}$. Therefore

$$\begin{aligned}\sum_{i=1}^{\infty} \|\varphi_i\|^2 &= \left\|\{\varphi_i\}_{i=1}^{\infty} - \left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty} + \left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty}\right\|^2 \\ &= \left\|\{\varphi_i\}_{i=1}^{\infty} - \left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty}\right\|^2 + \left\|\left\{\Lambda_i\left(S^{-1}f\right)\right\}_{i=1}^{\infty}\right\|^2.\end{aligned}$$ □

Now, we obtain an explicit expression for the pseudo-inverse of synthesis operator of a g-frame.

Theorem 2.2. *Assume that $\{\Lambda_i\}_{i=1}^{\infty}$ is a g-frame for H with synthesis operator T and frame operator S. Then $T^{\dagger} : H \to \left(\sum_{i\in I} \oplus H_i\right)_{l_2}$ the pseudo-inverse of T is given by*

$$T^{\dagger} f = \{\Lambda_i S^{-1} f\}_{i=1}^{\infty}, \quad f \in H.$$

Proof. By definition of T, if for $f \in H$, $f = \sum_{i=1}^{\infty} \Lambda_i^*(\varphi_i)$, then $T\{\varphi_i\}_{i=1}^{\infty} = f$. By [3, Lemma 2.5.3], the unique solution of minimal norm of the equation $T\{\varphi_i\}_{i=1}^{\infty} = f$ is $\{\varphi_i\}_{i=1}^{\infty} = T^{\dagger} f$. So by Lemma 2.1, we have

$$T^{\dagger} f = \{\Lambda_i S^{-1} f\}_{i=1}^{\infty}. \qquad \square$$

Similar to (ordinary) frames, the optimal frame bounds of a g-frames can be expressed in terms of its synthesis and frame operators and their inverses and pseudo-inverses.

Proposition 2.3. *The optimal frame bounds A, B for a g-frame $\{\Lambda_i\}_{i=1}^{\infty}$ are given by*

$$A = \left\|S^{-1}\right\|^{-1} = \left\|T^{\dagger}\right\|^{-2}, \qquad B = \|S\| = \|T\|^2,$$

where T and S are the synthesis operator and frame operator of $\{\Lambda_i\}_{i=1}^{\infty}$, respectively.

Proof. The optimal upper frame bound is given by

$$\begin{aligned} B &= \sup_{\|f\|=1} \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 \\ &= \sup_{\|f\|=1} \langle Sf, f\rangle \\ &= \|S\| \\ &= \|TT^*\| \\ &= \|T\|^2. \end{aligned}$$

Since the frame operator of the dual frame $\{\Lambda_i S^{-1}\}_{i=1}^{\infty}$ is S^{-1} and

$$B^{-1} I \leq S^{-1} \leq A^{-1} I,$$

so the optimal upper frame bound of $\{\Lambda_i S^{-1}\}_{i=1}^{\infty}$ is A^{-1}. Then, according to what we've just proved, we have $A^{-1} = \left\|S^{-1}\right\|$. Thus, by Theorem 2.2,

$$\begin{aligned} \left\|S^{-1}\right\| &= \sup_{\|f\|=1} \sum_{i=1}^{\infty} \left\|\Lambda_i S^{-1} f\right\|^2 \\ &= \sup_{\|f\|=1} \left\|T^{\dagger} f\right\|^2 \\ &= \|T^{\dagger}\|^2. \end{aligned} \qquad \square$$

3. Properties of g-Frames Generated by Iterated Operators

Consider the sequence $\{\Lambda^n\}_{n=1}^{\infty}$ as a g-frame for H, where $\Lambda : H \to H$ is a bounded operator. We want to investigate the properties of Λ. In other words, we try to find some conditions on Λ such that $\{\Lambda^n\}_{n=1}^{\infty}$ becomes a g-frame. Characterizing these kinds of g-frames is our purpose.

The first question is if such an operator can be found. Next examples show that the answer is positive.

Example 3.1. Consider the orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for H. We define the operator $\Lambda : H \to H$ as below:

$$\Lambda e_i = \frac{1}{2} e_{i+1}, \quad i \in \mathbb{N}.$$

So for each $f \in H$,

$$\begin{aligned} \Lambda f &= \Lambda \left(\sum_{i=1}^{\infty} \langle f, e_i \rangle e_i \right) \\ &= \frac{1}{2} \sum_{i=1}^{\infty} \langle f, e_i \rangle e_{i+1}. \end{aligned}$$

Then for each $n \in \mathbb{N}$, we have

$$\Lambda^n f = \frac{1}{2^n} \sum_{i=1}^{\infty} \langle f, e_i \rangle e_{i+n},$$

and

$$\begin{aligned} \|\Lambda^n f\|^2 &= \frac{1}{2^{2n}} \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2 \\ &= \frac{1}{2^{2n}} \|f\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \|\Lambda^n f\|^2 &= \left(\sum_{n=1}^{\infty} \frac{1}{2^{2n}} \right) \|f\|^2 \\ &= \frac{1}{3} \|f\|^2, \quad f \in H, \end{aligned}$$

that is, $\{\Lambda^n\}_{n=1}^{\infty}$ is a (tight) g-frame for H.

Example 3.2. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for H. Now, consider the operator $\Lambda : H \to H$ as below:

$$\Lambda e_1 = \frac{1}{3} e_2, \qquad \Lambda e_2 = \frac{1}{3} e_1, \qquad \Lambda e_i = \frac{1}{2} e_{i+1}, \quad i \geq 3.$$

Then for each $n \in \mathbb{N}$ and $f \in H$,

$$\begin{aligned}\Lambda^n f &= \Lambda^n \left(\sum_{i=1}^{\infty} \langle f, e_i \rangle e_i \right) \\ &= \frac{1}{3^n} \left(\langle f, e_1 \rangle e_2 + \langle f, e_2 \rangle e_1 \right) + \frac{1}{2^n} \sum_{i=3}^{\infty} \langle f, e_i \rangle e_{i+n}.\end{aligned}$$

So

$$\frac{1}{9^n} \|f\|^2 \leq \|\Lambda^n f\|^2 \leq \frac{1}{4^n} \|f\|^2, \quad f \in H$$

which implies

$$\begin{aligned}\frac{1}{8} \|f\|^2 = \left(\sum_{n=1}^{\infty} \frac{1}{9^n} \right) \|f\|^2 &\leq \sum_{n=1}^{\infty} \|\Lambda^n f\|^2 \\ &\leq \left(\sum_{n=1}^{\infty} \frac{1}{4^n} \right) \|f\|^2 \\ &= \frac{1}{3} \|f\|^2, \quad f \in H.\end{aligned}$$

So $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H.

The next step is verifying the features of the operator $\Lambda : H \to H$. First, consider the following examples. These examples show that the norm of Λ has no any effect on $\{\Lambda^n\}_{n=1}^{\infty}$ to be a g-frame for H.

Example 3.3. Assume that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for H and $\alpha > 1$. Define the operator $\Lambda : H \to H$ by

$$\Lambda e_i = \alpha e_{i+1}, \quad i \in \mathbb{N}.$$

So for each $f \in H$,

$$\|\Lambda f\|^2 = \alpha^2 \|f\|^2,$$

and

$$\|\Lambda^n f\|^2 = \alpha^{2n} \|f\|^2.$$

Then

$$\sum_{n=1}^{\infty} \|\Lambda^n f\|^2 = \left(\sum_{n=1}^{\infty} \alpha^{2n} \right) \|f\|^2 \to \infty, \quad f \in H.$$

Example 3.4. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for H and $\alpha > 1$. Define the operator $\Lambda : H \to H$ by

$$\Lambda e_1 = \alpha e_2, \qquad \Lambda e_i = \frac{1}{\alpha} e_{i+1}, \quad i \geq 2.$$

So for each $f \in H$ and $n \in \mathbb{N}$, we obtain

$$\Lambda^n f = \langle f, e_1 \rangle \frac{1}{\alpha^{n-2}} e_{n+1} + \sum_{i=2}^{\infty} \langle f, e_i \rangle \frac{1}{\alpha^n} e_{i+n},$$

and

$$\frac{1}{\alpha^{2n}} \|f\|^2 \leq \|\Lambda^n f\|^2 \leq \frac{1}{\alpha^{2(n-2)}} \|f\|^2.$$

Then

$$\|\Lambda\| \geq \alpha > 1,$$

and $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H.

According to the preceding examples, we conjecture that the assumption $\|\Lambda\| < 1$ is a sufficient (but not necessary) condition on the operator Λ to $\{\Lambda^n\}_{n=1}^{\infty}$ be a g-frame for H. In the following proposition, we give a sufficient condition for Λ, so $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame.

Proposition 3.5. *Let Λ be an operator on H such that for each $f \in H$,*

$$\alpha\|f\| \leq \|\Lambda f\| \leq \beta\|f\|,$$

where $\alpha, \beta \in (0,1)$. Then $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H with bounds $\frac{\alpha^2}{1-\alpha^2}$ and $\frac{\beta^2}{1-\beta^2}$.

Proof. For each $f \in H$ and $n \in \mathbb{N}$, we have

$$\alpha^n\|f\| \leq \|\Lambda^n f\| \leq \beta^n\|f\|,$$

So

$$\left(\sum_{n=1}^{\infty} \alpha^{2n}\right) \|f\|^2 \leq \sum_{n=1}^{\infty} \|\Lambda^n f\|^2 \leq \left(\sum_{n=1}^{\infty} \beta^{2n}\right) \|f\|^2.$$

Hence $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H with bounds $\frac{\alpha^2}{1-\alpha^2}$ and $\frac{\beta^2}{1-\beta^2}$. □

The next theorem indicates an interval for the range of $\|\Lambda\|$.

Theorem 3.6. *Let $\Lambda \in B(H)$ be such that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H with bounds A and B. Then*

$$\sqrt{\frac{A}{1+B}}\|f\| \leq \|\Lambda f\| \leq \sqrt{\frac{B}{1+A}}\|f\|, \quad f \in H.$$

Proof. By assumption, we have

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} \|\Lambda^n f\|^2 \leq B\|f\|^2, \quad f \in H. \tag{3.1}$$

Considering Λf instead of f in (3.1), we obtain

$$A\|\Lambda f\|^2 \leq \sum_{n=1}^{\infty} \|\Lambda^{n+1} f\|^2 \leq B\|\Lambda f\|^2, \quad f \in H. \tag{3.2}$$

By taking away (3.2) from (3.1), we have

$$\frac{A}{1+B}\|f\|^2 \leq \|\Lambda f\|^2 \leq \frac{B}{1+A}\|f\|^2, \quad f \in H. \tag{3.3}$$

So the proof is complete. □

Similar to previous theorem, we have another main result.

Theorem 3.7. *Suppose that Λ is a bounded operator on H such that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H with bounds A and B. Then for each $i \in \mathbb{N}$,*

$$\left(\frac{A}{1+B}\right)^i \|f\|^2 \leq \|\Lambda^i f\|^2 \leq \left(\frac{B}{1+A}\right)^i \|f\|^2, \quad f \in H.$$

Proof. For each $f \in H$, we have

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} \|\Lambda^n f\|^2 \leq B\|f\|^2. \tag{3.4}$$

Let $i > 1$, by putting $\Lambda^i f$ and $\Lambda^{i-1} f$ instead of f in (3.4), we get

$$\frac{A}{1+A}\|\Lambda^{i-1} f\|^2 \leq \|\Lambda^i f\|^2 \leq \frac{B}{1+B}\|\Lambda^{i-1} f\|^2, \quad f \in H. \tag{3.5}$$

By repeating the above inequality and Theorem 3.6, we have

$$\left(\frac{A}{1+B}\right)^i \|f\|^2 \leq \|\Lambda^i f\|^2 \leq \left(\frac{B}{1+A}\right)^i \|f\|^2, \quad f \in H. \qquad □$$

Corollary 3.8. *If Λ is a bounded operator on H such that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, then for each $n \in \mathbb{N}$, the operator Λ^n has closed range.*

Proof. By Theorem 3.7, for each $n \in \mathbb{N}$, Λ^n is bounded below and so it has closed range. □

Corollary 3.9. *Let Λ be a bounded operator on H so that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H. If Λ is self-adjoint, then Λ is invertible.*

Proof. By Theorem 3.7 and Proposition 1.4, Λ is invertible. □

Proposition 3.10. *Let $\Lambda \in B(H)$ and $\{\Lambda^n\}_{n=1}^{\infty}$ be a tight g-frame for H. Then $\|\Lambda\| < 1$.*

Proof. If $\{\Lambda^n\}_{n=1}^{\infty}$ is a A-tight g-frame for H, then by Theorem 3.6, we have

$$\|\Lambda f\|^2 = \frac{A}{1+A}\|f\|^2, \quad f \in H.$$

So $\|\Lambda\| < 1$. □

Using [5, Theorem 2.38], we can estimate the spectral radius of Λ.

Proposition 3.11. *Suppose that Λ is a bounded operator on H such that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H with bounds A and B. Then the spectral radius $r(\Lambda)$ of Λ satisfies*

$$\sqrt{\frac{A}{1+B}} \leq r(\Lambda) \leq \sqrt{\frac{B}{1+A}} .$$

Proof. By Theorem 3.7, for each $n \in \mathbb{N}$, we have

$$\left(\frac{A}{1+B}\right)^{\frac{n}{2}} \|f\| \leq \|\Lambda^n f\| \leq \left(\frac{B}{1+A}\right)^{\frac{n}{2}} \|f\|, \quad f \in H.$$

So

$$\left(\frac{A}{1+B}\right)^{\frac{n}{2}} \leq \|\Lambda^n\| \leq \left(\frac{B}{1+A}\right)^{\frac{n}{2}},$$

which implies

$$\left(\frac{A}{1+B}\right)^{\frac{1}{2}} \leq \lim_{n\to\infty} \|\Lambda^n\|^{\frac{1}{n}} \leq \left(\frac{B}{1+A}\right)^{\frac{1}{2}} .$$

Therefore

$$\sqrt{\frac{A}{1+B}} \leq r(\Lambda) \leq \sqrt{\frac{B}{1+A}}. \qquad \square$$

Proposition 3.12. *If $\Lambda \in B(H)$ and $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, then*

$$\|\Lambda\|_b = \inf_{\|f\|=1} \|\Lambda f\| < 1.$$

Proof. Suppose that $\|\Lambda\|_b = \inf_{\|f\|=1} \|\Lambda f\| \geq 1$. For each $f \in H$, we have

$$(\|\Lambda\|_b^2)^n \|f\|^2 \leq \|\Lambda^n f\|^2.$$

Since $\sum_{n=1}^{\infty} \left(\|\Lambda\|_b^2\right)^n$ is divergent, so $\{\Lambda^n\}_{n=1}^{\infty}$ can not be a g-frame for H. This contradiction shows that

$$\|\Lambda\|_b = \inf_{\|f\|=1} \|\Lambda f\| < 1. \qquad \square$$

Proposition 3.13. *If $\Lambda \in B(H)$ is invertible, $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H and S be the frame operators of $\{\Lambda^n\}_{n=1}^{\infty}$, then $\|S\| > 1$.*

Proof. For each $g \in H$, let $f = \Lambda^{-1} g$, then

$$\begin{aligned}\sum_{n=1}^{\infty} \left\|\Lambda^n \left(\Lambda^{-1} g\right)\right\|^2 &= \|g\|^2 + \sum_{n=2}^{\infty} \left\|\Lambda^{n-1} g\right\|^2 \\ &> \|g\|^2.\end{aligned}$$

So $A > 1$, where A is the optimal lower frame bound. Proposition 2.3 implies that $\left\|S^{-1}\right\| < 1$. So $\|S\| > 1$. $\square$

One of the other properties of Λ is as below:

Proposition 3.14. *If $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, where Λ is a bounded operator on H and $\|\Lambda\| < 1$, then*

$$\lim_{n\to\infty} \Lambda^n = 0.$$

Proof. Since $\lim_{n\to\infty} \|\Lambda^n\| \leq \lim_{n\to\infty} \|\Lambda\|^n = 0$, the proof is complete. □

Under some conditions, we can find an explicit expression for frame operator.

Proposition 3.15. *Suppose that Λ is a bounded operator on H such that $\|\Lambda\| < 1$, Λ is a normal operator and $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H. Then the frame operator of $\{\Lambda^n\}_{n=1}^{\infty}$ is given by*

$$S = (I - \Lambda^*\Lambda)^{-1} - I,$$

where I is the identity operator on H.

Proof. By $\|\Lambda\| < 1$, we have

$$\|\Lambda^*\Lambda\| = \|I - (I - \Lambda^*\Lambda)\| < 1,$$

where I is the identity operator on H. Then by [3, Theorem 2.2.3], $I - \Lambda^*\Lambda$ is invertible and

$$(I - \Lambda^*\Lambda)^{-1} = \sum_{n=0}^{\infty} (\Lambda^*\Lambda)^n. \tag{3.6}$$

If S is the frame operator of $\{\Lambda^n\}_{n=1}^{\infty}$, then for each $f \in H$,

$$\begin{aligned} Sf &= \sum_{n=1}^{\infty} (\Lambda^n)^*\Lambda^n f \\ &= \sum_{n=1}^{\infty} (\Lambda^*\Lambda)^n f. \end{aligned}$$

By (3.6), for each $f \in H$,

$$\begin{aligned} (I - \Lambda^*\Lambda)^{-1}(f) &= \left(\sum_{n=0}^{\infty} (\Lambda^*\Lambda)^n\right)(f) \\ &= \sum_{n=0}^{\infty} (\Lambda^*\Lambda)^n(f) \\ &= I(f) + S(f). \end{aligned}$$

Therefore

$$S = (I - \Lambda^*\Lambda)^{-1} - I.$$

□

Corollary 3.16. *If Λ is a bounded self-adjoint operator on H such that $\|\Lambda\| < 1$ and $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, then the frame operator of $\{\Lambda^n\}_{n=1}^{\infty}$ is given by*

$$S = (I - \Lambda^2)^{-1} - I,$$

where I is the identity operator on H.

We can obtain new g-frames, which are generated by iterations, from an existing one.

Proposition 3.17. *Suppose that Λ is a bounded operator on H such that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H. Then for each real number $r > 1$, there exists an operator Γ_r on H such that $\|\Gamma_r\| = \frac{1}{r}$ and $\{\Gamma_r^n\}_{n=1}^{\infty}$ is a g-frame for H.*

Proof. By Theorem 3.6, there exists a constant $\alpha > 0$ such that

$$\alpha\|f\| \leq \|\Lambda f\| \leq \|\Lambda\|\|f\|, \quad f \in H.$$

For each $r > 1$, let

$$\Gamma_r f = \frac{\Lambda f}{r\|\Lambda\|}, \quad f \in H.$$

So

$$\frac{\alpha}{r\|\Lambda\|}\|f\| \leq \|\Gamma_r f\| \leq \frac{1}{r}\|f\|, \quad f \in H.$$

Therefore by Proposition 3.5, the proof is complete. □

For a g-frame $\{\Lambda^n\}_{n=1}^{\infty}$, we can find a dual g-frame which is obtained by iterations of an operator.

Proposition 3.18. *Let $\Lambda \in B(H)$ and $\{\Lambda^n\}_{n=1}^{\infty}$ be a g-frame for H. Then there exist an operator $\Theta \in B(H)$ and $\rho > 0$, so that $\{\rho\Theta^n\}_{n=1}^{\infty}$ is a dual g-frame for $\{\Lambda^n\}_{n=1}^{\infty}$.*

Proof. By Theorem 3.7, for each $n \in \mathbb{N}$, Λ^n is bounded below and by [3, Lemma 2.4.1], $(\Lambda^n)^*$ is onto. So for each $n \in \mathbb{N}$, the pseudo-inverse operator $((\Lambda^n)^*)^{\dagger}$ of $(\Lambda^n)^*$ exists and

$$(\Lambda^n)^*((\Lambda^n)^*)^{\dagger} f = f, \quad f \in H,$$

specially, we have

$$\Lambda^*(\Lambda^*)^{\dagger} f = f, \quad f \in H. \tag{3.7}$$

So, for each $f \in H$,

$$\frac{1}{\|\Lambda^*\|}\|f\| \leq \left\|(\Lambda^*)^{\dagger} f\right\|,$$

and by (3.7), for each $n \in \mathbb{N}$, we obtain

$$(\Lambda^*)^n \left((\Lambda^*)^{\dagger}\right)^n f = f, \quad f \in H. \tag{3.8}$$

In case of $\|\Lambda\| < 1$, we put

$$\Theta = \frac{1}{2\|(\Lambda^*)^\dagger\|}(\Lambda^*)^\dagger,$$

then $\Theta \in B(H)$ and

$$\frac{1}{2\|(\Lambda^*)^\dagger\|\|\Lambda^*\|}\|f\| \leq \|\Theta f\| \leq \frac{1}{2}\|f\|, \quad f \in H.$$

By Proposition 3.5, $\{\Theta^n\}_{n=1}^\infty$ is a g-frame for H. Let $q = \frac{1}{2\|(\Lambda^*)^\dagger\|}$, since $0 < q < 1$, so $\sum_{n=1}^\infty q^n = \frac{q}{1-q}$. Assuming $\rho = \left(\frac{q}{1-q}\right)^{-1}$, the sequence $\{\rho\Theta^n\}_{n=1}^\infty$ is a g-frame for H and for each $f \in H$, we have

$$\begin{aligned}\sum_{n=1}^\infty (\Lambda^n)^*(\rho\Theta^n)f &= \rho\sum_{n=1}^\infty (\Lambda^n)^*\left(\frac{1}{2\|(\Lambda^*)^\dagger\|}(\Lambda^*)^\dagger\right)^n f \\ &= \rho\sum_{n=1}^\infty (\Lambda^*)^n\left(\frac{1}{2\|(\Lambda^*)^\dagger\|}(\Lambda^*)^\dagger\right)^n f.\end{aligned}$$

Therefore by (3.8),

$$\sum_{n=1}^\infty (\Lambda^n)^*(\rho\Theta^n)f = \rho\sum_{n=1}^\infty \frac{1}{2^n\|(\Lambda^*)^\dagger\|^n} f = f, \quad f \in H.$$

Hence $\{\rho\Theta^n\}_{n=1}^\infty$ is a dual g-frame for $\{\Lambda^n\}_{n=1}^\infty$.

Now, suppose $\|\Lambda\| \geq 1$. By (3.8), for each $n \in \mathbb{N}$,

$$\frac{1}{\|\Lambda^*\|^n\|(\Lambda^*)^\dagger\|^n} \leq 1.$$

In this case, put

$$\Theta = \frac{1}{2\|\Lambda^*\|\|(\Lambda^*)^\dagger\|}(\Lambda^*)^\dagger,$$

so $\Theta \in B(H)$ and

$$\|\Theta f\| \leq \frac{1}{2\|\Lambda^*\|}\|f\| \leq \frac{1}{2}\|f\|, \quad f \in H.$$

Let $q = \frac{1}{2\|\Lambda^*\|\|(\Lambda^*)^\dagger\|}$, then $0 < q < 1$ and the rest of proof is similar to previous case. □

Next result states that if Λ is invertible, then the iterated dual g-frame of $\{\Lambda^n\}_{n=1}^\infty$ is unique.

Proposition 3.19. *Let $\Lambda, \Gamma \in B(H)$, $\{\Lambda^n\}_{n=1}^\infty$ be any g-frame for H and $\{\Gamma^n\}_{n=1}^\infty$ be a dual g-frame of $\{\Lambda^n\}_{n=1}^\infty$. Then $\Lambda^*\Gamma = \frac{1}{2}I$, where I is the identity operator on H. Moreover, if Λ is invertible, then $\Gamma = \frac{1}{2}(\Lambda^*)^{-1}$.*

Proof. For each $f \in H$, we have

$$\begin{aligned} f &= \sum_{n=1}^{\infty} (\Lambda^n)^* \Gamma^n f \\ &= \Lambda^* \Gamma f + \sum_{n=2}^{\infty} (\Lambda^n)^* \Gamma^n f \\ &= \Lambda^* \Gamma f + \Lambda^* \sum_{n=2}^{\infty} \left(\Lambda^{n-1}\right)^* \Gamma^{n-1} (\Gamma f) \\ &= \Lambda^* \left(\Gamma f + \sum_{n=2}^{\infty} \left(\Lambda^{n-1}\right)^* \Gamma^{n-1} (\Gamma f) \right) \\ &= \Lambda^* \left(\Gamma f + \Gamma f \right) \\ &= 2\Lambda^* \Gamma f. \end{aligned}$$

So $\Lambda^* \Gamma = \frac{1}{2} I$. The rest of proof is obvious. □

Under some conditions, the precise formula of frame operator of $\{\Lambda^n\}_{n=1}^{\infty}$ is given as below:

Proposition 3.20. *If $\Lambda \in B(H)$ is invertible, $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H and S is the frame operator of $\{\Lambda^n\}_{n=1}^{\infty}$, Then*

$$S = \frac{1}{2} \Lambda^* \Lambda.$$

Proof. Considering the canonical dual g-frame of $\{\Lambda^n\}_{n=1}^{\infty}$ in Proposition 3.19, we have $\Gamma = \Lambda S^{-1} = \frac{1}{2} (\Lambda^*)^{-1}$. Then

$$S = \frac{1}{2} \Lambda^* \Lambda.$$ □

Corollary 3.21. *Let $\Lambda \in B(H)$ be invertible. If $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, then for each $f \in H$*

$$T^{\dagger} f = \left\{ \frac{1}{2} \Lambda^{n-1} (\Lambda^*)^{-1} f \right\}_{i=1}^{\infty}.$$

Proof. By Theorem 2.2 and Proposition 3.20, it is obvious.

In the following, we verify the eigenvalues of Λ. □

Proposition 3.22. *If $\Lambda \in B(H)$, $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H and λ is any eigenvalue of Λ, then $0 < |\lambda| < 1$.*

Proof. Let λ be an eigenvalue of Λ. Then there exists a $f \neq 0$ such that $\Lambda f = \lambda f$. Therefore

$$A \|f\|^2 \leq \sum_{n=1}^{\infty} \|\Lambda^n f\|^2 = \sum_{n=1}^{\infty} (|\lambda^2|^n) \|f\|^2$$

$$\leq B\|f\|^2.$$

Since $0 < \sum_{n=1}^{\infty}(|\lambda^2|^n)$ is convergent, so $0 < |\lambda| < 1$. □

4. G-Riesz Bases in the Form of Iterated Operators

In this section, we study the properties of g-Riesz bases generated by iterations of a bounded operator on H. The main result is the following.

Theorem 4.1. *Let Λ be a bounded operator on H such that $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H. If $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-Riesz basis for H, then Λ is invertible.*

Proof. Let $\{\Lambda^n\}_{n=1}^{\infty}$ be a g-Riesz basis for H. Putting $\mathbb{J} = \{1\} \subseteq \mathbb{N}$ in Definition 1.3, we obtain that

$$\mathcal{A}\|g\|^2 \leq \|\Lambda^* g\|^2 \leq \mathcal{B}\|g\|^2, \quad g \in H,$$

where $\mathcal{A}$ and $\mathcal{B}$ are the constants in Definition 1.3. So Λ^* is one-to-one. Since $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H, so by Theorem 3.6, Λ is bounded below. Therefore by [3, Lemma 2.4.1], Λ^* is onto. Hence Λ is invertible. □

The following example shows that the converse of above theorem is not true always.

Example 4.2. Assume that the operator $\Lambda : H \to H$ is defined by

$$\Lambda f = \frac{1}{2}f, \quad f \in H.$$

It is obvious that Λ is bounded and $\|\Lambda\| = \frac{1}{2}$. Also, Λ is self-adjoint and invertible operator. By Proposition 3.5, $\{\Lambda^n\}_{n=1}^{\infty}$ is a g-frame for H. Now, we show that $\{\Lambda^n\}_{n=1}^{\infty}$ in not a g-Riesz basis for H. Let e be a fix element in H and put

$$g_1 = \frac{1}{2}e, \qquad g_2 = -e, \qquad g_n = 0, \quad n \geq 3.$$

It is clear that $\{g_n\}_{n=1}^{\infty} \in l^2(H, \mathbb{N})$ and

$$\begin{aligned}\sum_{n=1}^{\infty}\Lambda^n g_n &= \Lambda g_1 + \Lambda^2 g_2 + \sum_{n=3}^{\infty}\Lambda^n g_n \\ &= \frac{1}{4}e - \frac{1}{4}e + 0 \\ &= 0.\end{aligned}$$

So by [10, Theorem 2.8], $\{\Lambda^n\}_{n=1}^{\infty}$ is not a g-Riesz basis for H.

5. Stability of g-Frames from Iterated Operator

The stability of g-frames and their duals has been investigated by W. Sun in [9]. In this section, we study the stability of g-frames obtained from iteration. In the following, one case of perturbations of these g-frames is stated.

Proposition 5.1. *Let $\Lambda, \Gamma \in B(H)$ and $\{\Lambda^n\}_{n=1}^{\infty}$ be a g-frame for H with bounds A and B. Assume that there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that* $\max\left\{\sqrt{\lambda_1} + \sqrt{\frac{\mu}{A(1-\mu)}}, \lambda_2, \mu\right\} < 1$, *and the following condition is satisfied,*

$$\|(\Lambda^n - \Gamma^n)f\|^2 \leq \lambda_1\|\Lambda^n f\|^2 + \lambda_2\|\Gamma^n f\|^2 + \mu^n\|f\|^2, \quad f \in H,\ n \geq 1.$$

Then $\{\Gamma^n\}_{n=1}^{\infty}$ is a g-frame for H with bounds

$$A\left(1 - \frac{\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\frac{\mu}{A(1-\mu)}}}{1 + \sqrt{\lambda_2}}\right)^2,$$

$$B\left(1 + \frac{\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\frac{\mu}{A(1-\mu)}}}{1 - \sqrt{\lambda_2}}\right)^2.$$

Proof. For each $f \in H$, we have

$$\begin{aligned}
&\left(\sum_{n=1}^{\infty}\|(\Lambda^n - \Gamma^n)f\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\lambda_1\sum_{n=1}^{\infty}\|\Lambda^n f\|^2 + \lambda_2\sum_{n=1}^{\infty}\|\Gamma^n f\|^2 + \left(\sum_{n=1}^{\infty}\mu^n\right)\|f\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \sqrt{\lambda_1}\left(\sum_{n=1}^{\infty}\|\Lambda^n f\|^2\right)^{\frac{1}{2}} + \sqrt{\lambda_2}\left(\sum_{n=1}^{\infty}\|\Gamma^n f\|^2\right)^{\frac{1}{2}} \\
&\qquad + \sqrt{\frac{\mu}{1-\mu}}\|f\|.
\end{aligned}$$

So by [9, Theorem 3.1], the proof is complete. □

Next result is another case of stability for generating g-frames via iterated operators.

Theorem 5.2. *Let $\Lambda, \Gamma \in B(H)$ and $\{\Lambda^n\}_{n=1}^{\infty}$ be a g-frame for H with bounds A and B. Suppose that there exist positive constants α, β such that $\beta < 1 - \|\Lambda\|$ and*

$$\alpha\|f\| \leq \|(\Lambda - \Gamma)f\| \leq \beta\|f\|, \quad f \in H.$$

Then $\{\Gamma^n\}_{n=1}^{\infty}$ *is a g-frame for* H.

Proof. By Theorem 3.7, we can write

$$\sqrt{\frac{A}{1+B}}\|f\| \leq \|\Lambda f\| \leq \|\Lambda\|\|f\|, \quad f \in H.$$

So for each $f \in H$,

$$\|\Gamma f\| \leq \|\Lambda f - \Gamma f\| + \|\Lambda f\| \leq (\beta + \|\Lambda\|)\|f\|,$$

also

$$\|\Gamma f\| \geq |\|\Lambda f - \Gamma f\| - \|\Lambda f\|| \geq \left|\alpha - \sqrt{\frac{A}{1+B}}\right| \|f\|.$$

Let's suppose without loss of generality that $\alpha \neq \sqrt{\frac{A}{1+B}}$, (otherwise we just take a smaller below bound for Λ). Therefore Proposition 3.5 implies that $\{\Gamma^n\}_{n=1}^{\infty}$ is a g-frame for H. □

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