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Existence and Asymptotic of Solutions for a p-Laplace Schrödinger Equation with Critical Frequency

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Abstract. We study the Schrödinger equation (Q_ε): $-\varepsilon^{2(p-1)}\Delta_p v + V(x)\,|v|^{p-2}v - |v|^{q-1}v = 0$, $x \in \mathbb{R}^N$, with $v(x) \to 0$ as $|x| \to +\infty$, for the infinite case, as given by Byeon and Wang for a situation of critical frequency, $\{x \in \mathbb{R}^N \,/\, V(x) = \inf V = 0\} \neq \emptyset$. In the semiclassical limit, $\varepsilon \to 0$, the corresponding limit problem is (P): $\Delta_p w + |w|^{q-1}w = 0$, $x \in \Omega$, with $w(x) = 0, x \in \partial\Omega$, where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded strictly star-shaped region related to the potential V. We prove that for (Q_ε) there exists a non-trivial solution with any prescribed L^{q+1}-mass. Applying a Ljusternik-Schnirelman scheme, shows that (Q_ε) and (P) have infinitely many pairs of solutions. Fixed a topological level $k \in \mathbb{N}$, we show that a solution of (Q_ε), $v_{k,\varepsilon}$, sub converges, in $\mathrm{W}^{1,p}(\mathbb{R}^N)$ and up to scaling, to a corresponding solution of (P). We also prove that the energy of each solution, $v_{k,\varepsilon}$ converges to the corresponding energy of the limit problem (P) so that the critical values of the functionals associated, respectively, to (Q_ε) and (P) are topologically equivalent.

1. Introduction

The time-dependent nonlinear Schrödinger equation

$$i\hbar\,\Psi_t(x,t) + \frac{\hbar^2}{2}\Delta\Psi(x,t) - V_0(x)\Psi(x,t) + |\Psi(x,t)|^{q-1}\Psi(x,t) = 0, \tag{1.1}$$

helps to study phenomena like the evolution of Bose-Einstein condensates [12] and the propagation of light through nonlinear optical materials [7]. Here $\hbar$ is the reduced Planck constant. Whenever $\hbar$ is treated as a

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small positive parameter which could tend to zero, a semi-classical state of (1.1) is a standing wave having the form $\Psi(x,t) = v(x)\exp(-iEt/\hbar)$, where v verifies

$$\varepsilon^2 \Delta v(x) - V(x)\, v(x) + |v(x)|^{q-1} v(x) = 0, \tag{1.2}$$

with $\varepsilon^2 = \hbar^2/2$ and $V(x) = V_0(x) - E$.

In this paper, we consider the Dirichlet problem

$$\begin{cases} -\varepsilon^{2(p-1)} \Delta_p v + V(x)|v|^{p-2} v - |v|^{q-1} v = 0, & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} v(x) = 0, \end{cases} \tag{1.3}$$

where $\varepsilon > 0$ and $1 < p < q+1 < p^*$ with $p^* = pN/(N-p)$ if $N > p$ and $p^* = +\infty$ if $N \le p$. Therefore, we generalize (1.2) by replacing the Laplace operator $\Delta = \Delta_2$ with the p-Laplace operator, $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$, which helps to model nonlinear diffusion phenomena.

Let's assume that $\mathcal{Z} = \left\{x \in \mathbb{R}^N \,/\, V(x) = \inf(V)\right\} \neq \emptyset$, and consider the following conditions:

- (V1) $V \in \mathrm{C}(\mathbb{R})$ is non-negative;
- (V2) $V(x) \to +\infty$, as $|x| \to +\infty$;
- (V3) $\inf(V) = 0$;
- (V4) $\mathcal{Z} = \{0\}$ and $V(x) = \exp(-1/a(x))$ if $|x| \le 1$, where, for a bounded strictly star-shaped domain $\Omega \subseteq \mathbb{R}^N$, a is an asymptotically (Ω, b)-quasihomogeneous function (see Section 2.1 for the precise statement).

Let's consider a positive function $g \in \mathrm{C}(]0,+\infty[)$, whose form will be specified later, see (2.3). Then, by using the scaling

$$\begin{aligned} v(x) &= [\varepsilon g(\varepsilon)]^{2(p-1)/(q+1-p)}\, u(g(\varepsilon)x), \\ V_\varepsilon(x) &= \frac{1}{[\varepsilon g(\varepsilon)]^{2(p-1)}} V\left(\frac{x}{g(\varepsilon)}\right), \end{aligned} \tag{1.4}$$

$x \in \mathbb{R}^N$, it's clear that (1.3) is equivalent to

$$\begin{cases} -\Delta_p u(x) + V_\varepsilon(x)|u(x)|^{p-2} u(x) - |u(x)|^{q-1} u(x) = 0, & x \in \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{1.5}$$

Conditions (V1)-(V3) are assumed throughout the document. The coercivity condition (V2) is typical to get compactness of the embedding of E^p_ε, the Sobolev-like space where the solutions of (1.5) are to be found (see the beginning of Section 2.2), into a range of Lebesgue spaces. It is well known that, to obtain the same compactness, (V2) can be replaced by other weaker coercivity properties, see e.g. conditions (V8) and (V9) in [3]. With the help of the mentioned compact embedding, we prove that the functions associated with (1.5) verify the Palais-Smale condition. Then, by a direct method, we first show that for (1.5), there exists a non-trivial solution with any prescribed L^{q+1}-mass; see Theorem 2.2 and Remark 2.3 below. Second, applying a Ljusternik-Schnirelman scheme, shows that (1.5) has infinitely many pairs of solutions, see Theorem 2.4 below.

In the context of Quantum Mechanics, $p = 2$, condition (V3) is usually referred to as a situation of critical frequency because the solutions of (1.5) present concentration phenomena quite different to those of their counterparts in the non-critical setting, $\inf V > 0$, see e.g. [1], [6] and [9] and the references therein.

Grossly speaking, condition (V4) says that $V(x)$ exponentially decreases to zero as x gets closer to $x_0 = 0$ and corresponds to the infinite case, as considered in [6] and [1]. For this situation, in Sections 4 and 5, we prove the asymptotic properties of the solutions as $\varepsilon \to 0$. We show, see Theorem 2.8, that the energy of each solution, $u_{k,\varepsilon}$, $k \in \mathbb{N}$, obtained by the Ljusternik-Schnirelman scheme converges to the corresponding energy of the functions associated with the limit problem of (1.5),

$$\begin{cases} \Delta_p w(x) + |w(x)|^{q-1} w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth, bounded, strictly star-shaped domain related to the potential V via condition (V4) (see Section 2.1). Since the index k represents the topological characteristic of the level set, as captured by Krasnoselskii's genus, the energy asymptotics says that the critical values of the functionals associated, respectively, to (1.5) and (1.6) are topologically equivalent. Finally, we prove asymptotic profiles for the mentioned solutions, see Theorem 2.9. In fact, up to scaling, each of these solutions converges in $\mathrm{W}^{1,p}(\mathbb{R}^N)$ to a function whose restriction to Ω is a solution of (1.6) and shares the energy level.

This paper is organized in the following way. In Section 2.1, we introduce, in a precise way, the setting of the infinite case for a critical frequency situation. Section 2.2, presents the Sobolev-like space E^p_ε, where we find solutions of (1.5) and prove that it is reflexive. The statements of our existence results are also given in Section 2.2. Section 2.3, present the abstract theorems which are key tools in our work. The asymptotic properties are stated in Section 2.4. In Section 2.5 some estimates important, which to deal with the infinite case are introduced, together with some other valuable inequalities. In Section 3.1, we deal with the regularity of the functionals related to the problem (1.5). In Section 3.2, we prove that the functions associated with (1.5) verify the Palais-Smale condition, fundamental property to apply the mentioned abstract tools. The energy asymptotics are proved in Section 4. Finally, the asymptotic profiles are proved in Section 5.

2. Preliminaries, Infinite Case and Main Results

2.1. **Infinite Case Setting.** Let's consider $\Omega \subseteq \mathbb{R}^N$, a smooth bounded domain which is stricly star-shaped, i.e., there exists a ball $B \subseteq \Omega$ such that for every $(x, y) \in B \times \Omega$, the segment $[x, y]$ is contained in Ω. For every $q \geq 1$, Ω is a q-Poincaré domain, i.e., for some $M_q > 0$ and every $u \in \mathrm{C}^1(\Omega)$, $\|u - u_\Omega\|_{\mathrm{L}^q(\Omega)} \leq M_q \left(\int_\Omega |\nabla u(x)|^q \, dx\right)^{1/q}$, where u_Ω denotes the average of u over Ω, [15].

Let's assume that the domain Ω is generated by a positive function $r \in \mathrm{C}(\mathbb{R}^N \setminus \{0\})$ that verifies

$$\begin{array}{ll} x/t \in \partial\Omega, & \text{if } t = r(x), \\ x/t \in \Omega, & \text{if } t > r(x), \\ x/t \in \Omega^c, & \text{if } t < r(x). \end{array} \tag{2.1}$$

Therefore, for every $x \in \mathbb{R}^N \setminus \{0\}$ there exists a unique $s(x) \in \partial\Omega$ such that $x = r(x)\, s(x)$. It also holds

$$\begin{array}{ll} r(x) = 1, & \text{if } x \in \partial\Omega, \\ r(x) > 1, & \text{if } x \in \bar{\Omega}^c, \\ r(x) < 1, & \text{if } x \in \Omega, \end{array} \tag{2.2}$$

and $r\left(x/t\right) = r(x)/t$, for every $x \in \mathbb{R}^N \setminus \{0\}$ and every $t > 0$.

Let's pick $b : \mathbb{R}^N \to \mathbb{R}$, a continuous Ω-quasi homogeneous function: there exists a function $\beta : [0, +\infty[\to \mathbb{R}$ such that

- b1) $b(x) = \beta(r(x))$, for every $x \in \mathbb{R}^N$;
- b2) β is non-negative and strictly-increasing;
- b3) given $L = \lim\limits_{r \to 0} \beta(cr)/\beta(r)$, it holds $L < 1$ if $c < 1$ and $L > 1$ if $c > 1$.

The function $a \in \mathrm{C}(\mathbb{R}^N)$ that appears in (V4) is asymptotically (Ω, b)-quasihomogeneous function, i.e., a is positive and verifies

$$\frac{a(x)}{b(x)} \to 1, \quad \text{as } |x| \to 0.$$

2.2. **Existence of solutions in a Sobolev-like space.** As usual, $\mathrm{W}^{1,p}(\mathbb{R}^N)$ denotes the Sobolev space of all the functions that, together with their weak derivatives, belong to $\mathrm{L}^p(\mathbb{R}^N)$; it's equipped with the norm given by

$$\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)} = \left[\|\nabla u\|^p_{\mathrm{L}^p(\mathbb{R}^N)} + \|u\|^p_{\mathrm{L}^p(\mathbb{R}^N)}\right]^{1/p}.$$

We denote by E^p_ε the completion of $\mathrm{C}^\infty_0(\mathbb{R}^N)$ in the norm given by

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} [|\nabla u(x)|^p + V_\varepsilon(x)|u(x)|^p] \, dx \right)^{1/p},$$

where V_ε is given in (1.4) and

$$g(\varepsilon) = \frac{1}{\beta^{-1}\left(\dfrac{-1}{\ln(\varepsilon^2)} \right)}. \tag{2.3}$$

It's quite clear that, for $r \in [p, p^*]$, the embeddings $\mathrm{E}^p_\varepsilon \subseteq \mathrm{W}^{1,p}(\mathbb{R}^N) \subseteq \mathrm{L}^r(\mathbb{R}^N)$ are continuous and, therefore, there exists $c_r > 0$ such that, for every $u \in \mathrm{E}^p_\varepsilon$,

$$\|u\|_{\mathrm{L}^r(\mathbb{R}^N)} \leq c_r \|u\|_\varepsilon, \tag{2.4}$$

$$\|u\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)} \leq (1 + c_p^p)^{1/p} \|u\|_\varepsilon. \tag{2.5}$$

Actually, by applying Fréchet-Kolmogorov's theorem (see e.g. [5, Cor.4.27]), it's obtained, for every $r \in [p, p^*[$, that

$$\mathrm{E}^p_\varepsilon \subseteq \mathrm{L}^r(\mathbb{R}^N), \text{ compactly.} \tag{2.6}$$

It is worth to mention that in the proof of (2.6), the coercivity property (V2) compensates the non-boundedness of the domain.

Lemma 2.1. *Let $\varepsilon > 0$. The space E^p_ε is reflexive.*

Proof. Let's consider the Banach space $Y = \mathrm{L}^p_\varepsilon(\mathbb{R}^N) \times \left[\mathrm{L}^p(\mathbb{R}^N)\right]^N$, where

$$\|(u,w)\|_Y = \left(\|u\|^p_{\mathrm{L}^p_\varepsilon(\mathbb{R}^N)} + \|w\|_{[\mathrm{L}^p(\mathbb{R}^N)]^N} \right)^{1/p},$$

$$\|u\|_{\mathrm{L}^p_\varepsilon(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^p d\mu \right)^{1/p}, \quad d\mu = V_\varepsilon(x)dx,$$

$$\|w\|_{[\mathrm{L}^p(\mathbb{R}^N)]^N} = \|(w_1, \dots, w_N)\|_{[\mathrm{L}^p(\mathbb{R}^N)]^N} = \left(\int_{\mathbb{R}^N} |w|^p dx \right)^{1/p}.$$

The reflexivity of the spaces $\mathrm{L}^p(\mathbb{R}^N)$ and $\mathrm{L}^p_\varepsilon(\mathbb{R}^N)$ (see [4, Th. 4.7.15 and Cor. 4.7.16]) implies that of Y. The operator $T : \mathrm{E}^p_\varepsilon \to Y$, given by

$$T(u) = (u, \nabla u),$$

is an isometry. Since E^p_ε is a Banach space, it follows that $T(\mathrm{E}^p_\varepsilon)$ is a closed subspace of Y. Therefore, by [5, Prop. 3.20], $T(\mathrm{E}^p_\varepsilon)$ is also reflexive, so that E^p_ε is reflexive. □

Let's consider the functional $I_\epsilon : \mathrm{E}^p_\varepsilon \to \mathbb{R}$, given by

$$I_\epsilon(u) = \frac{1}{p} \|u\|^p_\varepsilon,$$

and, given $\alpha > 0$, the functional $J_{\epsilon,\alpha}$, which is the restriction of I_ϵ to the manifold $\mathcal{M}_{\epsilon,\alpha} = \left\{u \in \mathrm{E}^p_\varepsilon \,/\, \|u\|_{\mathrm{L}^{q+1}(\mathbb{R}^N)} = \alpha\right\}$. We shall denote $\mathcal{M}_\epsilon = \mathcal{M}_{\epsilon,1}$ and $J_\epsilon = J_{\epsilon,1}$.

Now we can state our first main result.

Theorem 2.2. *Let $\alpha > 0$. There exists $w_* \in \mathcal{M}_{\epsilon,\alpha}$ such that*

$$J_{\epsilon,\alpha}(w_*) = \inf_{w \in \mathcal{M}_{\epsilon,\alpha}} J_{\epsilon,\alpha}(w).$$

Moreover, the function given by

$$u_*(x) = \left[\frac{\alpha^{q+1}}{pc}\right]^{1/(p-q-1)} w_*(x), \quad x \in \mathbb{R}^N, \tag{2.7}$$

with $c = J_{\varepsilon,\alpha}(w_*)$, is a weak solution of (1.5).

Remark 2.3. Let $\alpha > 0$ and u, v and w functions related to each other by

$$v(x) = [\varepsilon g(\varepsilon)]^{2(p-1)/(q+1-p)}\, u(g(\varepsilon)x), \qquad u(x) = \left[\frac{\alpha^{q+1}}{pc}\right]^{1/(p-q-1)} w(x),$$

$x \in \mathbb{R}^N$. If $w \in \mathcal{M}_{\epsilon,\alpha}$, then $u \in \mathcal{M}_{\epsilon,\alpha_1}$ and $v \in \mathcal{M}_{\epsilon,\alpha_2}$, where

$$\alpha_1 = \left[\frac{\alpha^p}{pc}\right]^{1/(p-q-1)}, \qquad \alpha_2 = \left[\varepsilon^{2(p-1)} g^\nu(\varepsilon) \frac{pc}{\alpha^p}\right]^{1/(p-q-1)},$$

with $\nu = 2(p-1) - N[1 - p/(q+1)]$. Therefore, Theorem 2.2 implies that, by choosing an appropriate value of $\alpha > 0$, we can find a non-trivial solution of (1.5) (or (1.3)) with any prescribed L^{q+1}-mass.

Our second main result provides infinitely many pairs of solutions for (1.5), by means of a Ljusternik-Schnirelman scheme. For this we need the concept of genus. Let E be a Banach space. We write

$$\Sigma_E = \left\{A \subseteq E \,/\, A = \overline{A},\, A = -A,\, 0 \notin A\right\}.$$

By $\gamma(A)$, we denote the genus of $A \in \Sigma_E$, i.e., the least natural number k for which there exists an odd function $f \in \mathrm{C}\left(A, \mathbb{R}^k \backslash \{0\}\right)$. If there is no such k, then $\gamma(A) = +\infty$; and, by definition, $\gamma(\emptyset) = 0$. The concept of genus generalizes, [14] the notion of dimension: if $\mathbb{S}^{m-1}$ and $\mathbb{S}^\infty_X$ are the unit-spheres of $\mathbb{R}^m$ and X, an infinite-dimension Banach space, respectively, then $\gamma(\mathbb{S}^{m-1}) = m$ and $\gamma(\mathbb{S}^\infty_X) = +\infty$.

let us fix $\alpha = 1$ and write

$$\Sigma_\varepsilon = \{A \subseteq \mathrm{E}^p_\varepsilon \,/\, A = \bar{A},\ A = -A,\ 0 \notin A\}.$$

It's clear that $\mathcal{M}_\varepsilon \in \Sigma_\varepsilon$. For $k \in \mathbb{N}$, we put

$$\mathcal{A}_{k,\varepsilon} = \{A \in \Sigma_\varepsilon \,/\, A \subseteq \mathcal{M}_\varepsilon \ \wedge\ \gamma(A) \geq k\},$$

$$c_{k,\varepsilon} = \inf_{A\in\mathcal{A}_{k,\varepsilon}} \max_{u\in A} J_\varepsilon(u). \tag{2.8}$$

Theorem 2.4. *Let $k \in \mathbb{N}$.*

i) *$c_{k,\varepsilon}$ is a positive critical value of J_ε which has at least two corresponding critical points.*

ii) *If $w_{k,\varepsilon}$ is a critical point of J_ε such that $J_\varepsilon(w_{k,\varepsilon}) = c_{k,\varepsilon}$, then the function given by*

$$u_{k,\varepsilon}(x) = [pc_{k,\varepsilon}]^{1/(q+1-p)}\, w_{k,\varepsilon}(x), \quad x \in \mathbb{R}^N, \tag{2.9}$$

is a weak solution of (1.5).

By using Lagrange multipliers, it's not difficult to see that formulas (2.7) and (2.9) produce weak solutions of (1.5) departing from critical points of $J_{\varepsilon,\alpha}$ and J_ε. For example, if we assume that w is a critical point of $J_{\varepsilon,\alpha}$, then $L'_\lambda(w) = 0$ and $\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^N)} = \alpha$, where $\lambda \in \mathbb{R}$ is the Lagrange multiplier and $L_\lambda = I_\varepsilon - \lambda\Phi$ with $\Phi : \mathrm{E}^p_\varepsilon \to \mathbb{R}$, the function that defines the manifold, given by

$$\Phi(u) = \|u\|^{q+1}_{\mathrm{L}^{q+1}(\mathbb{R}^N)} - \alpha^{q+1}. \tag{2.10}$$

Therefore, see Proposition 3.2 below, for every $h \in \mathrm{E}^p_\varepsilon$,

$$\int_{\mathbb{R}^N} \left[|\nabla w|^{p-2}\nabla w \nabla h + V_\varepsilon\, |w|^{p-2} w\, h\right] dx - \lambda(q+1)\int_{\mathbb{R}^N} |w|^{q-1} w h dx = 0. \tag{2.11}$$

By choosing $h = w$, we get $\lambda(q+1) = pc/\alpha^{q+1}$ with $c = J_{\varepsilon,\alpha}(w)$, so that w is a weak solution of

$$-\Delta_p w(x) + V_\varepsilon(x)|w(x)|^{p-2}w(x) - \frac{pc}{\alpha^{q+1}}|w(x)|^{q-1}w(x) = 0, \quad x \in \mathbb{R}^N,$$

whence the function $u = \left[\alpha^{q+1}/(pc)\right]^{1/(p-q-1)} w$ is a weak solution of (1.5).

2.3. **Abstract Tools.** Let E be a Banach space, $I \in \mathrm{C}^1(E)$ and $M = G^{-1}(\{0\}) \subseteq E$, the manifold determined by a functional $G \in \mathrm{C}^{1,\eta}(E)$, $\eta > 0$, which verifies

$$\forall u \in M : G'(u) \neq 0.$$

Recall that $G \in \mathrm{C}^{1,\eta}(E)$ means that $G' : E \to E'$ is of class $\mathrm{C}^{0,\eta}$.

A sequence $(u_n)_{n\in\mathbb{N}} \subseteq M$ is said to be a Palais-Smale sequence, or simply a (PS) sequence, for the functional I iff $(I(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $I|'_M(u_n) \to 0$ in E', as $n \to +\infty$. If for some $m \in \mathbb{R}$, it holds $I(u_n) \to m$, as $n \to +\infty$, we say that $(u_n)_{n\in\mathbb{N}} \subseteq M$ is a $(PS)_m$ sequence.

We say that the functional I verifies the condition (PS) (or $(PS)_m$) on M iff every (PS) (or $(PS)_m$) sequence has a converging subsequence. If

$(PS)_m$ holds, the critical level $K_m = \{u \in M \,/\, I|'_M(u) = 0 \wedge I(u) = m\}$ is compact.

We shall prove Theorem 2.2 by applying the following result, see [2, Theorem 7.12 and Remark 7.13] and [8].

Theorem 2.5. *Suppose that $I|_M$ is bounded from below and satisfies $(PS)_\mu$, where $\mu = \inf_{w\in M} I(w)$. Then, there exists $w_0 \in M$ such that*

$$I(w_0) = \mu, \qquad (I|_M)'(w_0) = 0.$$

To prove Theorem 2.4, we shall use Theorem 2.7 below, see e.g. [13]. In Section 2.2, Krasnoselskii's genus was introduced; it verifies the following properties whose proof can be found in [14].

Proposition 2.6. *Let $A, B \in \Sigma_E$.*

i) *If $f \in \mathrm{C}(A, B)$ is odd, then $\gamma(A) \le \gamma(B)$.*
ii) *If $A \subseteq B$, then $\gamma(A) \le \gamma(B)$.*
iii) *If A is compact, then $\gamma(A) < +\infty$.*

Now we can present our second abstract tool, a Ljusternik-Schnirelman scheme.

Theorem 2.7. *Let $\tilde{M} \in \Sigma_E$ be a C^1 manifold and assume that I is even. Suppose that $I|_{\tilde{M}}$ verifies the (PS) condition. For each $k \in \mathbb{N}$, let*

$$C_k(I) = \inf_{A\in\mathcal{A}_k(\tilde{M})} \max_{u\in A} I(u),$$

where $\mathcal{A}_k(\tilde{M}) = \{A \in \Sigma_E \,/\, A \subseteq M \;\wedge\; \gamma(A) \ge k\}$.

i) *If $C_k(I) \in \mathbb{R}$, then $C_k(I)$ is a critical value for $I|_{\tilde{M}}$.*
ii) *If $c \equiv C_k(I) = \cdots = C_{k+m}(I)$, then $\gamma(K_c) \ge m + 1$. In particular, if $m > 1$, K_c, contains infinitely many elements.*

Then, in the context of Theorems 2.5 and 2.7, we shall prove that the following objects verify the needed conditions:

$$E = \mathrm{E}^p_\varepsilon, \quad M = \tilde{M} = \mathcal{M}_\varepsilon, \quad I = I_\varepsilon, \quad I|_M = I|_{\tilde{M}} = J_{\varepsilon,\alpha}, \quad G = \Phi.$$

2.4. **Asymptotic behaviour of the solutions.** Now we deal with the infinite case as given in [6] and [1], so that condition (V4), detailed in Section 2.1, is assumed to hold.

Let's first mention that Theorem 2.7 can be applied to (1.6), the limit problem of (1.5). For this, let's consider the functional $J : \mathcal{M} \subseteq \mathrm{W}^{1,p}_0(\Omega) \to \mathbb{R}$, given by

$$\begin{aligned} J(u) &= \frac{1}{p}\,\|u\|^p_{\mathrm{W}^{1,p}_0(\Omega)} \\ &= \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx, \end{aligned}$$

where $\mathcal{M} = \left\{ u \in \mathrm{W}_0^{1,p}(\Omega) \,/\, \|u\|_{\mathrm{L}^{q+1}(\Omega)} = 1 \right\}$. Let's write, for $k \in \mathbb{N}$,

$$\Sigma = \left\{ A \subseteq \mathrm{W}_0^{1,p}(\Omega)\,/ A = \bar{A},\ A = -A,\ 0 \notin A \right\},$$
$$\mathcal{A}_k = \left\{ A \in \Sigma / A \subseteq \mathcal{M} \wedge \gamma(A) \geq k \right\},$$
$$c_k = \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u). \tag{2.12}$$

Let $k \in \mathbb{N}$. As in Theorem 2.4, the following points are true.

i) c_k is a positive critical value of J which has at least two corresponding critical points.
ii) If w_k is a critical point of J such that $J(w_k) = c_k$, then the function given by $u_k(x) = [pc_k]^{1/(q+1-p)}\, w_k(x)$, $x \in \Omega$, is a weak solution of (1.6).

In the context of Theorem 2.7, the Ljusternik-Schnirelman device is applied to problems (1.5) and (1.6) to obtain the solutions $u_{k,\varepsilon}$ and u_k, $k \in \mathbb{N}$, respectively, the index k represents the topological characteristic of the level set, as captured by Krasnoselskii's genus. Therefore, the following result on the asymptotic energies implies that the critical values of J_ε and J are topologically equivalent.

Theorem 2.8. *Let $k \in \mathbb{N}$. Then, $c_{k,\varepsilon} \to c_k$, as $\varepsilon \to 0$.*

Grossly speaking, our last main result, that we will present states that, as $\varepsilon \to 0$ and up to scaling, each function $u_{k,\varepsilon}$ sub converges in $\mathrm{W}^{1,p}(\mathbb{R}^N)$ to a solution of (1.6) that shares the energy level. Let's recall [6] that a family of functions $(f_\varepsilon)_{\varepsilon>0}$ is said to sub converge in a space X, as $\varepsilon \to 0$, iff every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero, has a subsequence $(\varepsilon_{n_i})_{i\in\mathbb{N}}$ such that $\left(f_{\varepsilon_{n_i}}\right)_{i\in\mathbb{N}}$ converges in X, as $i \to +\infty$.

Theorem 2.9. *Let $k \in \mathbb{N}$. Then, as $\varepsilon \to 0, the family\, (u_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{W}^{1,p}(\mathbb{R}^N)$ to some $\phi_k \in \mathrm{W}^{1,p}(\mathbb{R}^N)$ such that its restriction to Ω is a solution of (1.6) and verifies $J\left(\hat{\phi}_k\Big|_\Omega\right) = c_k$, where*

$$\hat{\phi}_k(x) = [pc_k]^{1/(q+1-p)}\, \phi_k(x), \quad x \in \mathbb{R}^N.$$

2.5. **Preliminaries and Some Useful Results.** Let's review some properties which come from (2.1), b1), b2) and b3).

The functions g and V_ε are given in (1.4) and (2.3) so that, by (V4),

$$V_\varepsilon(x) = \frac{1}{[\varepsilon\, g(\varepsilon)]^{2(p-1)}} \exp\left(-\frac{1}{a\,(x/g(\varepsilon))}\right), \quad |x| \leq g(\varepsilon).$$

In [6] the following properties are stated. First we have that $g(\varepsilon) \to +\infty$, as $\varepsilon \to 0$. Second, there exists $\gamma > 0$ such that $\beta(r)/r^\gamma \to 0$, as $r \to 0$, and $g(\varepsilon)/|\ln(\varepsilon)|^{1/\gamma} \to 0$, as $\varepsilon \to 0$. Third, for every $\tau > 0$,

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1}{g^2(\varepsilon)} \exp\left(\frac{\tau}{\beta\left(1/g(\varepsilon)\right)}\right) &= \lim_{\varepsilon\to 0} \frac{1}{[\varepsilon^\tau g(\varepsilon)]^2} \\ &= +\infty.\end{aligned}$$

As a consequence of condition b3), the following results are true. Proposition 2.10 is stated in [6] for compact subsets of Ω. Proposition 2.11 is given as in [1, Prop. 2.9].

Proposition 2.10. *For every measurable set $B \subseteq \Omega$,*

$$\|V_\varepsilon\|_{\mathrm{L}^\infty(B)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Proposition 2.11. *There exists $D \in]0,1[$ such that for all $d > 1$,*

$$\lim_{\varepsilon\to 0} \min_{x\in R_{\varepsilon,D,d}} V_\varepsilon(x) = +\infty,$$

where $R_{\varepsilon,D,d} = \left\{x \in \mathbb{R}^N / |x| \le Dg(\varepsilon) \wedge r(x) \ge d\right\}$.

Remark 2.12. As a consequence of Proposition 2.10, we have that

$$\forall \mu > 0, \exists \check{\varepsilon} = \check{\varepsilon}(\mu) > 0 : \ \varepsilon \in]0, \check{\varepsilon}[\quad \Rightarrow \quad \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} < \mu. \tag{2.13}$$

By (2.2), in the context of Proposition 2.11, $R_{\varepsilon,D,d}$ is the set of points that belong to the closed ball centered at zero and of radius $Dg(\varepsilon)$ but which are outside the expanded star $\Omega_d = \{x \in \mathbb{R}^N \,/\, r(x) < d\}$.

Given an open set $\omega \subseteq \mathbb{R}^N$, we will always identify a function $f \in \mathrm{W}_0^{1,p}(\omega)$ with its extension by zero: $\overline{f}(x) = f(x)$ if $x \in \omega$ and $\overline{f}(x) = 0$ if $x \in \omega^c$. We have the following result.

Proposition 2.13. *Let $\varepsilon > 0$. Then the embedding $\mathrm{W}_0^{1,p}(\Omega) \subseteq \mathrm{E}_\varepsilon^p$ is continuous. On $\mathrm{W}_0^{1,p}(\Omega)$ the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\mathrm{W}_0^{1,p}(\Omega)}$ are equivalent.*

Proof. We have, for every $u \in \mathrm{W}_0^{1,p}(\Omega)$, that

$$\|u\|_{\mathrm{W}_0^{1,p}(\Omega)} \le \|u\|_\varepsilon \le C_{\Omega,\varepsilon} \|u\|_{\mathrm{W}_0^{1,p}(\Omega)}, \tag{2.14}$$

where

$$C_{\Omega,\varepsilon} = \left(1 + C_\Omega^p \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}\right)^{1/p} > 0. \tag{2.15}$$

with $C_\Omega^p > 0$ the constant appearing in Poincaré's inequality, [5, Cor. 9.19]. □

Remark 2.14. For future reference, let's mention that Proposition 2.13, in particular point (2.14), is still true if we replace Ω by any $U \subseteq \mathbb{R}^N$ open and bounded.

Let's recall that the inverse Hölder's inequality is given in the following way, [16, Th.13.6]. Let $r \in]0,1[$, $f \in \mathrm{L}^r(\mathbb{R}^N)$ and $g \in \mathrm{L}^{r'}(\mathbb{R}^N)$. Then,

$$\int_{\mathbb{R}^N} f(x)g(x)\,dx \geq \left(\int_{\mathbb{R}^N} f(x)^r\,dx\right)^{1/r} \left(\int_{\mathbb{R}^N} g(x)^{r'}\,dx\right)^{1/r'}. \tag{2.16}$$

To end this section, we introduce a number of useful inequalities, [10]. Let $x, y \in \mathbb{R}^N$. Then,

$$2^{2-\theta}|y-x|^{\theta-1} \geq \left||y|^{\theta-2}y - |x|^{\theta-2}x\right|, \qquad \text{if } 1 \leq \theta \leq 2; \tag{2.17}$$

$$2^{2-\theta}|y-x|^{\theta} \leq (|y|^{\theta-2}y - |x|^{\theta-2}x)\cdot(y-x), \quad \text{if } \theta \geq 2; \tag{2.18}$$

$$|y-x|^2 \leq \frac{(|y|^{\theta-2}y - |x|^{\theta-2}x)\cdot(y-x)}{(\theta-1)(1+|y|^2+|x|^2)^{(\theta-2)/2}}, \qquad \text{if } 1 < \theta < 2; \tag{2.19}$$

$$|x-y|^{\theta} \leq 2^{\theta/2}\left(|x|^2+|y|^2\right)^{\theta/2}, \qquad \text{if } \theta > 0. \tag{2.20}$$

In (2.18) and (2.19), the dot represents the inner-product on $\mathbb{R}^N$. The inequality (2.20) follows easily from the parallelogram identity in $\mathbb{R}^N$.

3. Proof of the Existence Results

In this section we prove our existence results, Theorems 2.2 and 2.4, by verifying the conditions of Theorems 2.5 and 2.7, respectively.

3.1. **Regularity of the manifold and the energy functional.**

Let's first show that the manifold $\mathcal{M}_\varepsilon$ verifies the conditions required in Theorems 2.5 and 2.7. Observe that the closedness of $\mathcal{M}_\varepsilon$ comes from point (2.4). The symmetry of $\mathcal{M}_\varepsilon$ and $0 \notin \mathcal{M}_\varepsilon$ is clear.

Proposition 3.1. *The manifold $\mathcal{M}_\varepsilon$ is of class $\mathrm{C}^{1,q}$.*

Proof. The proof that Φ, given in (2.10), is Fréchet-differentiable is quite standard and uses the continuity of the embedding $\mathrm{E}^p_\varepsilon \subseteq \mathrm{L}^{q+1}(\mathbb{R}^N)$. We have, for $u, h \in \mathrm{E}^p_\varepsilon$, that

$$\langle \Phi'(u), h\rangle = (q+1)\int_{\mathbb{R}^N} |u(x)|^{q-1}u(x)h(x)dx.$$

Let's prove that $\Phi \in \mathrm{C}^{1,q}$, i.e., that $\Phi' : \mathrm{E}^p_\varepsilon \to (\mathrm{E}^p_\varepsilon)'$ is of class $\mathrm{C}^{0,q}$:

$$\exists \beta > 0, \forall u, v \in \mathrm{E}^p_\varepsilon : \quad \|\Phi'(u) - \Phi'(v)\| \leq \beta\,\|u-v\|^q_\varepsilon. \tag{3.1}$$

Let's assume that $1 < p < 1+q < \min\{p^*, 2\}$ so that, by (2.17), it holds

$$\forall x, y \in \mathbb{R}^N : \quad \left||y|^{q-1}y - |x|^{q-1}x\right| \leq 2^{1-q}|y-x|^q. \tag{3.2}$$

Let $u, v, w \in \mathrm{E}^p_\varepsilon$. By (3.2) and Hölder inequality with $P = (q+1)/q$ and $P' = q+1$, we get

$$\frac{1}{q+1}|\Phi'(u)w - \Phi'(v)w|$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}^N} \left[|u(x)|^{q-1} u(x) - |v(x)|^{q-1} v(x) \right] w(x) dx \right| \\
&\leq \left[\int_{\mathbb{R}^N} \left| |u|^{q-1} u - |v|^{q-1} v \right|^{(q+1)/q} dx \right]^{q/(q+1)} \|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^N)} \\
&\leq 2^{1-q} \left[\int_{\mathbb{R}^N} |u(x) - v(x)|^{q+1} \, dx \right]^{q/(q+1)} \|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^N)} \\
&\leq 2^{1-q} c_{q+1}^{q+1} \|u - v\|_\varepsilon^q \|w\|_\varepsilon ,
\end{aligned}
$$

which, by the arbitrariness of w, u and v, implies (3.1) with $\beta = (q+1)2^{1-q}\, c_{q+1}^{q+1}$, where c_{q+1} comes from (2.4). The case of $2 \leq 1 + q < p^*$ is handled in a similar way. □

Having in mind Theorems 2.5 and 2.7, let's now verify that the energy functional, I_ϵ, is of class C^1.

Proposition 3.2. *The functional I_ε is Fréchet-differentiable and, for every $u, h \in \mathrm{E}_\varepsilon^p$,*

$$
\langle I_\varepsilon'(u), h \rangle = \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \nabla h + V_\varepsilon(x) \, |u|^{p-2} u h \right] dx. \tag{3.3}
$$

Proof. Let $u, h \in \mathrm{E}_\varepsilon^p$. By direct computation,

$$
\begin{aligned}
\partial_h I_\epsilon(u) &= \frac{d}{dt} J_\varepsilon(u + th) \bigg|_{t=0} \\
&= \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \cdot \nabla h + V_\varepsilon(x) \, |u|^{p-2} u h \right] dx,
\end{aligned}
$$

so that the directional derivative exists and coincides with the right side of (3.3).

The linear functional $\Psi : \mathrm{E}_\varepsilon^p \to \mathbb{R}$, given by

$$
\Psi(w) = \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \cdot \nabla w + V_\varepsilon(x) \, |u|^{p-2} u w \right] dx,
$$

is continuous so that I_ε is differentiable at u. In fact, since $p' = p/(p-1)$, given $w \in \mathrm{E}_\varepsilon^p$, we have, by Hölder and triangle inequalities, that

$$
\begin{aligned}
|\Psi(w)| &\leq \int_{\mathbb{R}^N} \left| |\nabla u|^{p-2} \nabla u \cdot \nabla w + (V_\varepsilon^{1/p'} |u|^{p-2} u)(V_\varepsilon^{1/p} w) \right| dx \\
&\leq \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^N)}^{p-1} \|\nabla w\|_{\mathrm{L}^p(\mathbb{R}^N)} \\
&\quad + \left(\int_{\mathbb{R}^N} V_\epsilon \, |u|^p dx \right)^{(p-1)/p} \left(\int_{\mathbb{R}^N} V_\epsilon \, |w|^p dx \right)^{1/p} \\
&\leq \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^N)}^{p-1} \|w\|_\varepsilon + \|u\|_\varepsilon^{p-1} \|w\|_\varepsilon \leq c \, \|w\|_\varepsilon ,
\end{aligned}
$$

for any constant $c > \|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^N)}^{p-1} + \|u\|_\varepsilon^{p-1}$.

Let's recall that for $z > 0$ and $\mu \in \mathbb{R}$ such that $z + \mu > 0$ it holds

$$(z + \mu)^{p/2} = z^{p/2} + \frac{p}{2} z^{(p-2)/2}\mu + g(\mu),$$

with $g(\mu) = o(\mu)$. With this, it is not difficult to show that $\Gamma(h)/\left\|h\right\|_\varepsilon \to 0$, as $\left\|h\right\|_\varepsilon \to 0$, where $\Gamma(h) = I_\epsilon(u + h) - I_\epsilon(u) - \Psi(h)$, so that $\Gamma(h) = o(h)$. Since u and h were chosen arbitrarily, we have proved that I_ε is Fréchet-differentiable and that (3.3) holds. □

Proposition 3.3. *The functional I_ε is of class C^1.*

Proof. We have to prove that I'_ε is continuous. I'_ϵ is continuous at $u_0 \in \mathrm{E}^p_\varepsilon$ iff given any $\mu > 0$ there exists $\delta > 0$ such that if $u \in \mathrm{E}^p_\varepsilon$ verifies $\left\|u - u_0\right\|_\varepsilon < \delta$, then

$$\forall v \in \mathrm{E}^p_\varepsilon : \quad \left|\langle I'_\epsilon(u) - I'_\epsilon(u_0), v\rangle\right| \le \mu \left\|v\right\|_\varepsilon . \tag{3.4}$$

Let's assume that $1 < p \le 2$. Let $u_0 \in \mathrm{E}^p_\varepsilon$ and $\mu > 0$. Take $0 < \delta < \left(2^{p-3}\mu\right)^{1/(p-1)}$. Then, for $u, v \in \mathrm{E}^p_\varepsilon$ with $\left\|u - u_0\right\|_\varepsilon < \delta$, we get, by using (2.17) and Hölder's inequality, that

$$\begin{aligned}
\left|\langle I'_\varepsilon(u_0) - I'_\varepsilon(u), v\rangle\right| &\le \int_{\mathbb{R}^N} \left||\nabla u_0|^{p-2}\nabla u_0 - |\nabla u|^{p-2}\nabla u\right| |\nabla v| dx \\
&\quad + \int_{\mathbb{R}^N} V_\varepsilon(x) \left||u_0|^{p-2}u_0 - |u|^{p-2}u\right| |v| dx \\
&\le 2^{2-p}\left(\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^{p-1}|\nabla v|\, dx + \int_{\mathbb{R}^N} V_\varepsilon(x)|u_0 - u|^{p-1}|v|\, dx\right) \\
&\le 2^{2-p}\left[\left(\int_{\mathbb{R}^N} |\nabla u_0 - \nabla u|^p\, dx\right)^{\frac{p-1}{p}} \left\|\nabla v\right\|_{\mathrm{L}^p(\mathbb{R}^N)}\right. \\
&\quad \left. + \left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_0 - u|^p\, dx\right)^{\frac{p-1}{p}} \left\|V_\varepsilon^{1/p} v\right\|_{\mathrm{L}^p(\mathbb{R}^N)}\right] \\
&\le 2^{2-p}\left(\left\|\nabla u_0 - \nabla u\right\|^{p-1}_{\mathrm{L}^p(\mathbb{R}^N)} + \left\|V_\varepsilon^{1/p}(u_0 - u)\right\|^{p-1}_{\mathrm{L}^p(\mathbb{R}^N)}\right) \left\|v\right\|_\varepsilon \\
&\le 2^{3-p}\left\|u_0 - u\right\|^{p-1}_\varepsilon \left\|v\right\|_\varepsilon \le 2^{3-p}\delta^{p-1}\left\|v\right\|_\varepsilon \le \mu\left\|v\right\|_\varepsilon .
\end{aligned}$$

We conclude (3.4) by the arbitrariness of u_0, μ and v. The case of $p > 2$ is worked out in a similar way. □

3.2. **Palais-Smale condition.** In this section we prove that the energy functional verifies the Palais-Smale Condition.

Theorem 3.4. *The functional I_ε verifies (PS) on $\mathcal{M}_\varepsilon$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{E}^p_\varepsilon$ be such that, for some $D > 0$,

$$\forall n \in \mathbb{N} : \quad 0 \le I_\varepsilon(u_n) \le D; \tag{3.5}$$

$$I'_\varepsilon(u_n) \to 0, \quad \text{as } n \to +\infty. \tag{3.6}$$

From (3.5) we get that $\|u_n\|_\varepsilon \leq (pD)^{1/p}$, for every $n \in \mathbb{N}$, so that, by Lemma 2.1 and [5, Th.3.8], there exist a subsequence $(u_m)_{m\in\mathbb{N}} = (u_{n_m})_{m\in\mathbb{N}} \subseteq \mathrm{E}^p_\varepsilon$ and some $u \in \mathrm{E}^p_\varepsilon$ such that $u_n \rightharpoonup u$, as $m \to +\infty$, i.e.,

$$\forall \eta \in (\mathrm{E}^p_\varepsilon)': \quad \langle \eta, u_m - u\rangle \to 0, \quad \text{as } m \to +\infty. \tag{3.7}$$

Now let's assume that $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_\varepsilon$. Since $\mathcal{M}_\varepsilon$ is closed in E^p_ε, the limit function u also belongs to $\mathcal{M}_\varepsilon$.

By (3.6) and (3.7), it follows that

$$\langle I'_\varepsilon(u_m) - I'_\varepsilon(u), u_m - u\rangle \to 0, \quad \text{as} \quad m \to +\infty. \tag{3.8}$$

Using (3.3) we get

$$\begin{aligned}\langle I'_\varepsilon(u_m) - I'_\varepsilon(u), u_m - u\rangle &= \int_{\mathbb{R}^N} \left(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_m - \nabla u)\,dx \\ &\quad + \int_{\mathbb{R}^N} V_\varepsilon(x)\left(|u_m|^{p-2}u_m - |u|^{p-2}u\right)\cdot(u_m - u)\,dx.\end{aligned} \tag{3.9}$$

i) Let's assume that $p \geq 2$. Then, from (3.9) and (2.18), we get

$$\begin{aligned}\langle I'_\varepsilon(u_m) - I'_\varepsilon(u), u_m - u\rangle &\geq 2^{2-p}\int_{\mathbb{R}^N} \left[|\nabla u_m - \nabla u|^p + V_\varepsilon(x)|u_m - u|^p\right] dx \\ &= 2^{2-p}\,\|u_m - u\|^p_\varepsilon\,.\end{aligned} \tag{3.10}$$

ii) Let's assume that $1 < p < 2$. Then, by (3.9), (2.19), (2.16) and (2.20) with $r = p/2$, $r' = p/(p-2)$ and $\theta = p$, it follows that

$$\begin{aligned}&\langle I'_\varepsilon(u_m) - I'_\varepsilon(u), u_m - u\rangle \\ &\quad\geq (p-1)\int_{\mathbb{R}^N} |\nabla u_m - \nabla u|^2\left(1 + |\nabla u_m|^2 + |\nabla u|^2\right)^{\frac{p-2}{2}}\,dx \\ &\qquad + (p-1)\int_{\mathbb{R}^N} V_\varepsilon(x)|u_m - u|^2\left(1 + |u_m|^2 + |u|^2\right)^{\frac{p-2}{2}}\,dx \\ &\quad\geq (p-1)\left(\int_{\mathbb{R}^N} |\nabla u_m - \nabla u|^p\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^N}\left(|\nabla u_m|^2 + |\nabla u|^2\right)^{\frac{p}{2}}\right)^{\frac{p-2}{p}} \\ &\qquad + (p-1)\left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_m - u|^p\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^N} V_\varepsilon(x)\left(|u_m|^2 + |u|^2\right)^{\frac{p}{2}}\right)^{\frac{p-2}{p}} \\ &\quad\geq \frac{p-1}{2^{p/2}}\left(\int_{\mathbb{R}^N} |\nabla u_m - \nabla u|^p\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^N} |\nabla u_m - \nabla u|^p\right)^{\frac{p-2}{p}}\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&+\frac{p-1}{2^{p/2}}\left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_m-u|^p\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^N} V_\varepsilon(x)|u_m-u|^p\right)^{\frac{p-2}{p}} \\
&\geq \frac{p-1}{2^{p/2}}\int_{\mathbb{R}^N}\left[|\nabla u_m-\nabla u|^p+V_\varepsilon(x)|u_m-u|^p\right]dx \\
&=\frac{p-1}{2^{p/2}}\|u_m-u\|_\varepsilon^p .
\end{aligned}
$$

Points (3.10) and (3.11), together with (3.8), imply that $(u_m)_{m\in\mathbb{N}}$ converges to u in E^p_ε. □

4. Energy Asymptotics

In this setion we prove energy asymptotics: Theorem 2.8 states that, given $k\in\mathbb{N}$, $c_{k,\varepsilon}\to c_k$, as $\varepsilon\to 0$. Our first step is the following result.

Proposition 4.1. *Let $k\in\mathbb{N}$. Then, for every $\varepsilon>0$, $\mathcal{A}_k\subseteq\mathcal{A}_{k,\varepsilon}$, and $c_{k,\varepsilon}\leq c_k C_{\Omega,\varepsilon}$, where $C_{\Omega,\varepsilon}>0$ is given in* (2.15)*. Moreover,*

$$\limsup_{\varepsilon\to 0} c_{k,\varepsilon}\leq c_k. \tag{4.1}$$

Proof. Let $\varepsilon>0$. Since, by Proposition 2.13, the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\mathrm{W}_0^{1,p}(\Omega)}$ are equivalent on $\mathrm{W}_0^{1,p}(\Omega)$, it follows that $\mathcal{A}_k\subseteq\mathcal{A}_{k,\varepsilon}$. By the points (2.8), (2.12) and (2.14), we get

$$
\begin{aligned}
c_{k,\varepsilon} &= \inf_{A\in\mathcal{A}_{k,\varepsilon}}\max_{u\in A} J_\varepsilon(u) \\
&\leq \inf_{A\in\mathcal{A}_k}\max_{u\in A} J_\varepsilon(u) \\
&\leq C_{\Omega,\varepsilon}\inf_{A\in\mathcal{A}_k}\max_{u\in A} J(u) \\
&= C_{\Omega,\varepsilon} c_k.
\end{aligned} \tag{4.2}
$$

Proposition 2.10 implies that $\|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}\to 0$, as $\varepsilon\to 0$. Therefore, (4.1) follows from (2.14) and (4.2). □

In what follows we shall need an auxiliary problem:

$$
\begin{cases}
\Delta_p w(x)+|w(x)|^{q-1}w(x)=0, & x\in\Omega^\delta, \\
w(x)=0, & x\in\partial\Omega^\delta,
\end{cases} \tag{4.3}
$$

where, for $\delta>0$, $\Omega^\delta=\{x\in\mathbb{R}^N\,/\,\mathrm{dist}(\Omega,x)<\delta\}$ is an expanded star. Theorem 2.7 also deals with problem (4.3) via the the functional $J^\delta:\mathcal{M}^\delta\subseteq\mathrm{W}_0^{1,p}(\Omega^\delta)\to\mathbb{R}$, given by

$$
\begin{aligned}
J^\delta(u) &= \frac{1}{p}\|u\|^p_{\mathrm{W}_0^{1,p}(\Omega^\delta)} \\
&= \frac{1}{p}\int_{\Omega^\delta}|\nabla u(x)|^p dx,
\end{aligned}
$$

where, $\mathcal{M}^\delta = \left\{u \in \mathrm{W}_0^{1,p}(\Omega^\delta) \;/\; \|u\|_{\mathrm{L}^{q+1}(\Omega^\delta)} = 1\right\}$. We write, for $k \in \mathbb{N}$,

$$\Sigma^\delta = \{A \subseteq \mathrm{W}_0^{1,p}(\Omega^\delta) \;/\; A = \bar{A},\ A = -A,\ 0 \notin A\},$$
$$\mathcal{A}_k^\delta = \left\{A \in \Sigma^\delta \;/\; A \subseteq \mathcal{M}^\delta \ \wedge\ \gamma(A) \geq k\right\}$$
$$c_k^\delta = \inf_{A \in \mathcal{A}_k^\delta} \max_{u \in A} J^\delta(u).$$

Then, given $k \in \mathbb{N}$, the following points hold.

i) c_k^δ is a positive critical value of J^δ which has at least two corresponding critical points.

ii) If w_k^δ is a critical point of J^δ such that $J(w_k^\delta) = c_k^\delta$, then the function given by $u_k^\delta(x) = \left[p c_k^\delta\right]^{1/(q+1-p)} w_k^\delta(x),\ x \in \Omega^\delta$, is a weak solution of (4.3).

Proposition 4.2. *Let $k \in \mathbb{N}$ and $\sigma > 0$. There exist $\delta_0, \varepsilon_2 > 0$ such that*

$$c_k^\delta \leq \frac{\sigma}{4} + \left[\frac{1}{1-\delta}\left(1 + \frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\right]^p \left(c_{k,\varepsilon} + \frac{\sigma}{4}\right),$$

for every $\delta \in \,]0, \delta_0[$ and every $\varepsilon \in \,]0, \varepsilon_2[$.

Proof. We shall walk thru several steps.

i) Let $\varepsilon > 0$ and $\delta \in]0, 1\,[$. By (2.8) there exists $A_\sigma(\varepsilon) \in \mathcal{A}_{k,\varepsilon}$ such that

$$\max_{u \in A_\sigma(\varepsilon)} J_\varepsilon(u) \leq c_{k,\varepsilon} + \frac{\sigma}{4}. \tag{4.4}$$

In (2.13) we choose

$$\mu = \frac{\sigma^p}{4^p C_\Omega^p c_k^p}, \tag{4.5}$$

and $\varepsilon_0 = \breve{\varepsilon}(\mu) = \varepsilon_0(\sigma, k) > 0$. From now on, let's assume that $\varepsilon \in]0, \varepsilon_0[$. Then, as in the proof of Proposition 4.1, we get, using (4.5), that

$$\begin{aligned} c_{k,\varepsilon}^p &\leq c_k^p + C_\Omega^p \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} c_k^p \\ &\leq c_k^p + \frac{C_\Omega^p c_k^p \sigma^p}{4^p C_\Omega^p c_k^p} \\ &= c_k^p + \frac{\sigma^p}{4^p} \\ &\leq \left(c_k + \frac{\sigma}{4}\right)^p \\ &\equiv b_{k,\sigma}^p. \end{aligned} \tag{4.6}$$

Then, points (4.4) and (4.6) imply that, for every $v \in A_\sigma(\varepsilon)$,

$$J_\varepsilon(v) \le b_{k,\sigma},$$

$$\int_{\mathbb{R}^N} |\nabla v(x)|^p dx \le p\, b_{k,\sigma}, \tag{4.7}$$

$$\int_{\mathbb{R}^N} V_\varepsilon(x) \cdot |v(x)|^p dx \le p\, b_{k,\sigma}. \tag{4.8}$$

ii) For $\rho > 0$, let's denote $V_{\rho,\varepsilon} = \inf\{V_\varepsilon(x) \,/\, x \in \mathbb{R}^N \backslash \Omega^\rho\}$. By Proposition 2.11 and condition (V2), we get

$$V_{\delta,\varepsilon} \to +\infty, \quad \text{as } \varepsilon \to 0. \tag{4.9}$$

From (4.8) we get, for every $v \in A_\sigma(\varepsilon)$,

$$\begin{aligned} \|v\|^p_{\mathrm{L}^p(\mathbb{R}^N \backslash \Omega^\delta)} &= \int_{\mathbb{R}^N \backslash \Omega^\delta} |v(x)|^p dx \\ &\le \frac{p\, b_{k,\sigma}}{V_{\delta,\varepsilon}}. \end{aligned} \tag{4.10}$$

On the other hand, by (4.7) and Sobolev-Gagliardo-Nirenberg theorem, [5], it follows, for every $v \in A_\sigma(\varepsilon)$, that

$$\begin{aligned} \|v\|_{\mathrm{L}^{p^*}(\mathbb{R}^N \backslash \Omega)} &\le \|v\|_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \\ &\le \theta \|\nabla v\|_{\mathrm{L}^p(\mathbb{R}^N)} \\ &\le \theta\, (p\, b_{k,\sigma})^{1/p}, \end{aligned} \tag{4.11}$$

where $\theta = \theta_{p,N} > 0$. Now, since $1 < p < q+1 < p^*$ we choose $\beta \in]0,1[$ such that $1/(q+1) = (1-\beta)/p + \beta/p^*$. Hence by (4.10), (4.11) and the interpolation inequality for L^p-spaces, [5, pg.93], it follows, for $v \in A_\sigma(\varepsilon)$, that

$$\begin{aligned} \|v\|_{\mathrm{L}^{q+1}(\mathbb{R}^N \backslash \Omega^\delta)} &\le \|v\|^{1-\beta}_{\mathrm{L}^p(\mathbb{R}^N \backslash \Omega^\delta)} \|v\|^\beta_{\mathrm{L}^{p^*}(\mathbb{R}^N \backslash \Omega^\delta)} \\ &\le \left(\frac{p\, b_{k,\sigma}}{V_{\delta,\varepsilon}}\right)^{(1-\beta)/p} \theta^\beta\, (p\, b_{k,\sigma})^{\beta/p} \\ &= \frac{\theta^\beta\, (p\, b_{k,\sigma})^{1/p}}{V_{\delta,\varepsilon}^{(1-\beta)/p}}, \end{aligned}$$

which, by (4.9), implies that

$$\max_{v \in A_\sigma(\varepsilon)} \|v\|_{\mathrm{L}^{q+1}(\mathbb{R}^N \backslash \Omega^\delta)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Therefore, given $s > 0$, there exists $\varepsilon_1 = \varepsilon_1(\delta, s; \sigma, k) \in]0, \varepsilon_0[$ such that, for every $\varepsilon \in]0, \varepsilon_1[$, $\max\limits_{v \in A_\sigma(\varepsilon)} \|v\|_{\mathrm{L}^{q+1}(\mathbb{R}^N \backslash \Omega^\delta)} < \delta^s$. In particular, for $s=1$ and $\hat{\varepsilon}_1=\varepsilon_1(\delta,1;\sigma,k)\in\,]0,\varepsilon_0[$, we get, for every $\varepsilon\in\,]0,\hat{\varepsilon}_1[$,

$$\forall v\in A_\sigma(\varepsilon):\quad \|v\|_{\mathrm{L}^{q+1}(\Omega^\delta)}\ge 1-\delta. \tag{4.12}$$

iii) Let's denote $\Lambda^\delta=\Omega^\delta\backslash\overline{\Omega^{\delta/2}}$ and pick a cut-off function $\phi_\delta\in\mathrm{C}_0^\infty\left(\mathbb{R}^N\right)$ such that, for some $r>1$, $\phi_\delta(x)=1$ if $x\in\Omega^{\delta/2}$, $\phi_\delta(x)=0$ if $x\in\mathbb{R}^N\backslash\Omega^\delta$, $\phi_\delta(x)\in]0,1[$ if $x\in\Lambda^\delta$ and $|\nabla\phi_\delta(x)|\le 1/\delta^r$ if $x\in\Lambda^\delta$. Let's prove that $\Gamma_\delta:A_\sigma(\varepsilon)\subseteq\mathcal{M}_\varepsilon\to\mathcal{M}^\delta$, given by

$$\Gamma_\delta[u]=\frac{u\phi_\delta}{\|u\phi_\delta\|_{\mathrm{L}^{q+1}(\Omega^\delta)}},$$

is well-defined and Lipschitz continuous. From now on we assume that $\varepsilon\in]0,\tilde{\varepsilon}_1[$, where $\tilde{\varepsilon}_1=\min\{\hat{\varepsilon}_1,\varepsilon_1(\delta/2,1;\sigma,k)\}$.

a) By (4.12) we have, for $v\in A_\sigma(\varepsilon)$, that

$$\begin{aligned}1&\ge\|v\phi_\delta\|^{q+1}_{\mathrm{L}^{q+1}(\Omega^\delta)}\\&=\int_{\Omega^{\delta/2}}|v(x)|^{q+1}dx+\int_{\Lambda^\delta}|\phi_\delta(x)v(x)|^{q+1}\,dx\\&\ge\int_{\Omega^{\delta/2}}|v(x)|^{q+1}dx\\&\ge(1-\delta/2)^{q+1}\\&>(1-\delta)^{q+1},\end{aligned}\tag{4.13}$$

so that Γ_δ is well defined.

b) Let $u,v\in A_\sigma(\varepsilon)\subseteq\mathcal{M}_\varepsilon$. Then, by (4.13),

$$\begin{aligned}&\|\Gamma_\delta[u]-\Gamma_\delta[v]\|_{\mathrm{W}_0^1(\Omega^\delta)}\\&\quad=\left\|\frac{\nabla(u\phi_\delta)}{\|u\phi_\delta\|_{\mathrm{L}^{q+1}(\Omega^\delta)}}-\frac{\nabla(v\phi_\delta)}{\|v\phi_\delta\|_{\mathrm{L}^{q+1}(\Omega^\delta)}}\right\|_{\mathrm{L}^p(\Omega^\delta)}\\&\quad\le\frac{1}{1-\delta}\|\nabla\left(\phi_\delta(u-v)\right)\|_{\mathrm{L}^p(\Omega^\delta)}\\&\quad\le\frac{1}{1-\delta}\left[\|\phi_\delta\nabla(u-v)\|_{\mathrm{L}^p(\Omega^\delta)}+\|(u-v)\nabla\phi_\delta\|_{\mathrm{L}^p(\Omega^\delta)}\right].\end{aligned}\tag{4.14}$$

By Remark 2.14 with $U=\Omega^\delta$, we get

$$\begin{aligned}\|\phi_\delta\nabla(u-v)\|_{\mathrm{L}^p(\Omega^\delta)}&=\left(\int_{\Omega^\delta}\phi_\delta^p(x)|\nabla(u-v)(x)|^p dx\right)^{1/p}\\&\le\|u-v\|_{\mathrm{W}_0^1(\Omega^\delta)}\le\|u-v\|_\varepsilon.\end{aligned}\tag{4.15}$$

On the other hand, by the defining properties of ϕ_δ, we have that

$$\begin{aligned}\|(u-v)\nabla\phi_\delta\|_{\mathrm{L}^p(\Omega^\delta)} &= \left(\int_{G^\delta} |u(x)-v(x)|^p\,|\nabla\phi_\delta(x)|^p\,dx\right)^{1/p}\\ &\leq \frac{1}{\delta^r \inf\limits_{y\in G^\delta} V_\varepsilon(y)}\left(\int_{G^\delta} V_\varepsilon(x)|u(x)-v(x)|^p dx\right)^{1/p}\\ &\leq \frac{\|u-v\|_\varepsilon}{\delta^r V_{\delta/p,\varepsilon}},\end{aligned}$$

which, together with (4.14) and (4.15), imply that

$$\|\Gamma_\delta[u]-\Gamma_\delta[v]\|_{\mathrm{W}_0^1(\Omega^\delta)} \leq \frac{1}{1-\delta}\left(1+\frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\|u-v\|_\varepsilon.$$

Since u and v were chosen arbitrarily, we have proved that Γ_δ is Lipschitz continuous.

iv) Since the operator Γ_δ is odd and continuous, point i) in Proposition 2.6 implies that

$$\Gamma_\delta\left[A_\sigma(\varepsilon)\right] \in \mathcal{A}_k^\delta, c_k^\delta \leq \max_{v\in\Gamma_\delta[A_\sigma(\varepsilon)]} J^\delta(v).$$

We choose $u \in A_\sigma(\varepsilon)$ such that $\bar{v} = \Gamma_\delta[u]$ verifies

$$c_k^\delta \leq \max_{v\in\Gamma_\delta[A_\sigma(\varepsilon)]} J^\delta(v) \leq J^\delta(\bar{v}) + \frac{\sigma}{4}. \tag{4.16}$$

Now we claim that there exists some $w \in A_\sigma(\varepsilon)$ such that

$$J^\delta(\bar{v}) \leq \left[\frac{1}{1-\delta}\left(1+\frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\right]^p J_\varepsilon(w). \tag{4.17}$$

Then, points (4.4), (4.16) and (4.17) allow us to conclude:

$$\begin{aligned}c_k^\delta &\leq J^\delta(\bar{v}) + \frac{\sigma}{4}\\ &\leq \frac{\sigma}{4} + \left[\frac{1}{1-\delta}\left(1+\frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\right]^p \max_{u\in A_\sigma(\varepsilon)} J_\varepsilon(u)\\ &\leq \frac{\sigma}{4} + \left[\frac{1}{1-\delta}\left(1+\frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\right]^p \left(c_{k,\varepsilon} + \frac{\sigma}{4}\right).\end{aligned}$$

v) To finish, let us prove the claim (4.17). We shall prove that the choice $w = u$ works well. Working as in points iii)-b), we prove that, for every $\tilde{u} \in A_\sigma(\varepsilon)$,

$$\|\Gamma_\delta[\tilde{u}]\|_{\mathrm{W}_0^1(\Omega^\delta)} \leq \frac{1}{1-\delta}\left(1+\frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\|\tilde{u}\|_\varepsilon.$$

By choosing $\tilde{u} = u$ in the last inequality, we get

$$\begin{aligned}\|\overline{v}\|_{\mathrm{W}_0^{1,p}(\Omega^\delta)} &= \|\Gamma_\delta[u]\|_{\mathrm{W}_0^{1,p}(\Omega^\delta)} \\ &\le \frac{1}{1-\delta}\left(1 + \frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\|u\|_\varepsilon,\end{aligned}$$

whence the estimate (4.17) follows. ☐

Before proving Theorem 2.8, let's observe that, choosing $\delta > 0$ small enough and using (4.9), we get

$$\left[\frac{1}{1-\delta}\left(1 + \frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\right]^p \to (1-\delta)^{-p} \approx 1, \quad \text{as } \varepsilon \to 0. \tag{4.18}$$

Proof of Theorem 2.8. By adapting Lemmas 3.3 and 3.4 of [9], we get, for $k \in \mathbb{N}$, that

$$\forall \delta > 0 : \quad c_k^\delta \le c_k;$$

$$\forall \sigma > 0, \exists \delta_\sigma > 0, \forall \delta \in]0, \delta_\sigma[: \quad c_k \le c_k^\delta + \sigma. \tag{4.19}$$

Let $\sigma > 0$ such that $\sigma << 1$ and take δ_σ from (4.19). We take $\delta_0 = \delta_0(\sigma)$ and $\varepsilon_2 = \varepsilon_2(\sigma)$ from Proposition 4.2. We also choose $\delta_* > 0$ such that

$$\forall \delta \in]0, \delta_*[: \quad (1-\delta)^p < 1 + \sigma. \tag{4.20}$$

Let's take $\tilde{\delta}_\sigma = \{\delta_\sigma, \delta_0, \delta_*\}$. Then, by (4.19), (4.20) and Proposition 4.1, we have that

$$\begin{aligned} c_k &\le c_k^\delta + \sigma \\ &\le \frac{5}{4}\sigma + \left[\frac{1}{1-\delta}\left(1 + \frac{1}{\delta^r V_{\delta/2,\varepsilon}}\right)\right]^p \left(c_{k,\varepsilon} + \frac{\sigma}{4}\right) \\ &\le (5 + [\dots]^p)\frac{\sigma}{4} + [\dots]^p c_k \cdot C_{\Omega,\varepsilon}. \end{aligned}$$

The last inequality, together with (4.20), (4.18), (2.15), Proposition 2.10 and the arbitrariness of σ, proves that $c_{k,\varepsilon} \to c_k$, $\varepsilon \to 0$. ☐

5. Asymptotic Profiles

In this section, we prove Theorem 2.9. Given $k \in \mathbb{N}$, it states that, as $\varepsilon \to 0$, the family $(u_{k,\varepsilon})_{\varepsilon>0}$ sub converges in $\mathrm{W}^{1,p}(\mathbb{R}^N)$ to some $\phi_k \in \mathrm{W}^{1,p}(\mathbb{R}^N)$ such that its restriction to Ω is a solution of (1.6) and verifies $J\left(\hat{\phi}_k\Big|_\Omega\right) = c_k$, where $\hat{\phi}_k(x) = [pc_k]^{1/(q+1-p)}\,\phi_k(x)$, $x \in \mathbb{R}^N$.

Lemma 5.1. *Let $k \in \mathbb{N}$. Then, there exists $\hat{\phi}_k \in \mathrm{E}^p_\varepsilon$ such that $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges to $\hat{\phi}_k$ pointwise and weakly both in E^p_ε and $\mathrm{W}^{1,p}(\mathbb{R}^N)$, as $\varepsilon \to 0$.*

Proof. By Theorem 2.8 and point (2.5), given $\sigma > 0$, there exists $\varepsilon_{\sigma,1} > 0$ such that, for every $\varepsilon \in]0, \varepsilon_{\sigma,1}[$,

$$pc_{k,\varepsilon} \leq pc_k + \sigma \equiv B_{k,\sigma}$$

$$(1 + c_p^p)^{-1/p} \|w_{k,\varepsilon}\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)} \leq \|w_{k,\varepsilon}\|_\varepsilon^p \leq B_{k,\sigma}. \tag{5.1}$$

Therefore, by [5, Th.3.18&4.9] and Theorem 2.1, there exists $\hat{\phi}_k \in \mathrm{E}_\varepsilon^p \subseteq \mathrm{W}^{1,p}(\mathbb{R}^N)$ toward which $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges pointwise and weakly both in E_ε^p and $\mathrm{W}^{1,p}(\mathbb{R}^N)$. □

Lemma 5.2. *Let $k \in \mathbb{N}$. The function ϕ_k is a weak solution of the limit problem* (1.6) *and* $J\left(\hat{\phi}_k\Big|_\Omega\right) = c_k$.

Proof. i) Let $\sigma > 0$ and $\varepsilon_{\sigma,1} > 0$ as in Lemma 5.1, and let $\varepsilon \in]0, \varepsilon_{\sigma,1}[$. Since $w_{k,\varepsilon} \in \mathcal{M}_\varepsilon$ is a critical point of J_ε we have, by (2.11), that, for every $\eta \in \mathrm{C}_0^\infty\left(\mathbb{R}^N\right)$,

$$\int_{\mathbb{R}^N} \left[|\nabla w_{k,\varepsilon}|^{p-2}\nabla w_{k,\varepsilon}\nabla\eta + V_\varepsilon\, |w_{k,\varepsilon}|^{p-2} w_{k,\varepsilon}\eta\right] dx = pc_{k,\varepsilon} \int_{\mathbb{R}^N} |w_{k,\varepsilon}|^{q-1} w_{k,\varepsilon}\eta dx. \tag{5.2}$$

ii) By [5, Th.9.9], we have that

$$\|w_{k,\varepsilon}\|^p_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \leq c_p^p \int_{\mathbb{R}^N} |\nabla w_{k,\varepsilon}(x)|^p\, dx \leq c_p^p B_{k,\sigma}. \tag{5.3}$$

Let $0 < \delta < 1$. By Hölder's inequality and (5.3), we get

$$\begin{aligned} \|w_{k,\varepsilon}\|^p_{\mathrm{L}^p(\Omega^\delta)} &\leq \left|\Omega^\delta\right|^{p/N} \|w_{k,\varepsilon}\|^p_{\mathrm{L}^{p^*}(\Omega^\delta)} \\ &\leq \left|\Omega^\delta\right|^{p/N} \|w_{k,\varepsilon}\|^p_{\mathrm{L}^{p^*}(\mathbb{R}^N)} \\ &\leq c_p^p \left|\Omega^\delta\right|^{p/N} B_{k,\sigma}. \end{aligned} \tag{5.4}$$

By (4.9), there exists $\varepsilon_{\sigma,2} \in]\,0, \varepsilon_{\sigma,1}[$ such that, for all $\varepsilon \in]0, \varepsilon_{\sigma,2}[$, it holds $V_{\delta,\varepsilon}^{-1} < 1$ and, thanks to (5.1),

$$\|\hat{w}_{k,\varepsilon}\|^p_{\mathrm{L}^p(\mathbb{R}^N\setminus\Omega^\delta)} \leq \int_{\mathbb{R}^N\setminus\Omega^\delta} \frac{V_\varepsilon(x)}{V_{\delta,\varepsilon}} |\hat{w}_{k,\varepsilon}(x)|^p\, dx \leq B_{k,\sigma}. \tag{5.5}$$

iii) Let $\eta \in \mathrm{C}_0^\infty(\Omega)$ and $\varepsilon \in]0, \varepsilon_{\sigma,2}[$. By (5.4), (5.5) and Proposition 2.10, we get

$$\left|\int_\Omega V_\varepsilon(x)\hat{w}_{k,\varepsilon}\eta(x)dx\right| \leq \|w_{k,\varepsilon}\|_{\mathrm{L}^p(\Omega)} \|\eta\|_{\mathrm{L}^{p'}(\Omega)} \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}$$

$$\leq \left[B_{k,\sigma}\left(1 + C_N^p \left|\Omega^\delta\right|^{p/N}\right)\right]^{1/p} \|\eta\|_{\mathrm{L}^{p'}(\Omega)} \|V_\varepsilon\|_{\mathrm{L}^\infty(\Omega)}$$

and

$$\int_\Omega V_\varepsilon(x) w_{k,\varepsilon} \eta(x) dx \to 0, \quad \text{as } \varepsilon \to 0. \tag{5.6}$$

iv) By (2.6), $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{L}^{q+1}(\mathbb{R}^N)$ to $\hat{\phi}_k$. Therefore, by (5.2), (5.6), Theorem 2.8 and the arbitrariness of η, we get, for every $\eta \in \mathrm{C}_0^\infty(\Omega)$, that

$$\int_\Omega \nabla\hat{\phi}_k \nabla\eta dx = pc_k \int_\Omega \left|\hat{\phi}_k\right|^{q-1} \hat{\phi}_k \eta \, dx. \tag{5.7}$$

Now let us pick $(\psi_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}_0^\infty(\Omega)$ converging in $\mathrm{L}^{q+1}(\Omega)$ to $\hat{\phi}_k|_\Omega$. Then, by replacing $\eta = \psi_n$ in (5.7) and letting $n \to +\infty$, we get, by Lemma 5.1, that $c_k = J\left(\hat{\phi}_k\Big|_\Omega\right)$.

v) Given any $\delta, \beta > 0$, we put $\Gamma_{\delta,\beta} = \left\{x \in \mathbb{R}^N / \Omega^\delta : |\hat{u}_k(x)| \geq \beta\right\}$. By Reduction to Absurdity, we prove that $|\Gamma_{\delta,\beta}| = 0$. Therefore,

$$\hat{u}_k(x) = 0 \ \text{ for a.e. } x \in \mathbb{R}^N \backslash \Omega, \tag{5.8}$$

which, by Proposition 9.18 in [5], implies that $\hat{u}_k|_\Omega \in \mathrm{W}_0^1(\Omega)$. We conclude by this and (5.7).

□

Proof of Theorem 2.9. By (2.6), Lemma 5.1 and point (5.8), it follows that

$$\|w_{k,\varepsilon}\|_{\mathrm{L}^p(\mathbb{R}^N)} \to \left\|\hat{\phi}_k\right\|_{\mathrm{L}^p(\mathbb{R}^N)}, \quad \text{as } \varepsilon \to 0. \tag{5.9}$$

By (4.1) and (5.9), we have that

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} |\nabla w_{k,\varepsilon}|^p \, dx &\leq p \limsup_{\varepsilon\to 0} c_{k,\varepsilon} \\ &\leq pc_k \\ &= \int_{\mathbb{R}^N} \left|\nabla\hat{\phi}_k\right|^p dx. \end{aligned} \tag{5.10}$$

From (5.9) and (5.10) it follows that

$$\limsup_{\varepsilon\to 0} \|\hat{w}_{k,\varepsilon}\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)} \leq \|\hat{u}_k\|_{\mathrm{W}^{1,p}(\mathbb{R}^N)},$$

so that, by [5, Prop.3.32], we have that $(w_{k,\varepsilon})_{\varepsilon>0}$ subconverges in $\mathrm{W}^{1,p}(\mathbb{R}^N)$ to $\hat{\phi}_k$, as $\varepsilon \to 0$. We conclude by Lemma 5.2. □

To finish, let us mention that, working like in this paper, the asymptotic results on energy and profiles obtained, respectively, in [9] and [11] for the flat case and finite case (cases also introduced in [6]) should also hold for the p-version we dealt with.

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