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Fuzzy μ^*-Open Set and Fuzzy μ^*-Continuous Function

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Abstract. The prime goal of this article is to initiate the notion of fuzzy μ^*-open(closed) sets and fuzzy μ^*-continuous functions and characterize them. These concepts are defined in a fuzzy topological space in presence of a generalized fuzzy topology, which becomes a new tool to study fuzzy topological spaces. It is observed that this class of fuzzy sets fail to form a fuzzy topology but it form a generalized fuzzy topology. Furthermore, the relationship of these fuzzy sets and fuzzy continuity with some existing fuzzy notions are established. Also the notion of fuzzy (τ, μ^*)-open(closed) functions is introduced and their equivalent conditions with fuzzy μ^*-continuous functions are established.

1. Introduction

Based on L.A Zadeh's concept of fuzzy sets [10], C.L. Chang [2] presented the idea of fuzzy topological spaces (fts for short). Also, several topological properties has been generalized successfully by different mathematicians in fuzzy settings. In 2008, G. P. Chetty has extended the concept of generalized topological spaces in fuzzy environment and named it as generalized fuzzy topological spaces ($gfts$, for short) [5]. In 2017 Chakraborty et al. defined fuzzy (μX, μY)-continuous functions between two $gfts$[1]. The idea of μ^*-open [8] and μ^*-continuity [9] in generalized topological space (gts, for short) were initiated and studied by B. Roy et al. in 2015 and by R. K. Tiwari et al. in 2020 respectively. Recently, P. Chettri et al. [3] studied further decomposition of these sets and continuity. Also, P.chettri et al. studied a new type of

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fuzzy sets callled *ps-ro* β-open(closed) fuzzy sets and related continuity in year 2022 [4].

2. Preliminaries

Lists of some notations used in this paper.

I	[0,1]
I^X	Set of all fuzzy sets in X
fts	Fuzzy topological spaces
$gfts$	Generalized fuzzy topological space
Space $(X, \mu X, \tau)$	Triplet where X is a non void set with the generalized fuzzy topology μX and the fuzzy topology τ defined on it.
$i(P)$	Interior a of fuzzy set P
$i_\mu(P)$	μ-interior of a fuzzy set P
$c_\mu(P)$	μ-closure of a fuzzy set P
$cl(P)$	Closure of a fuzzy set P
x_α	Fuzzy point
$\mu^*O(X_\tau)$	Set of all fuzzy μ^*-open sets in a space $(X, \mu X, \tau)$
$\mu^*C(X_\tau)$	Set of all fuzzy μ^*-closed sets in a space $(X, \mu X, \tau)$.

A function from a non void set X into I is termed as fuzzy set in X[10]. The fuzzy sets taking value 0 and 1 $\forall\ x \in X$ are denoted by 0 and 1, respectively. The complement of fuzzy set P is denoted by $1 - P$ and is given as $(1 - P)(t) = 1 - P(t)$, where $t \in X$. For a function f between two sets X to Y, if P and Q are fuzzy sets in X and Y respectively, then $f(P)$ and $f^{-1}(Q)$ are fuzzy sets in Y and X respectively and defined as

$$f(P)(t) = \begin{cases} \sup_{r \in f^{-1}(t)} P(r), & \text{if } f^{-1}(t) \neq \emptyset, \\ 0, & \text{if otherwise,} \end{cases}$$

and $f^{-1}(Q)(t) = Q(f(t))$, $\forall\ t \in X$[2].

In X, a family $\tau \subseteq I^X$ is called a fuzzy topology if $0, 1 \in \tau$ and τ is closed under arbitrary union and finite intersection. The ordered pair (X, τ) is a fuzzy topological space (fts for short). Each element of τ is called a fuzzy open set and its complement as fuzzy closed. In a fts (X, τ), the interior and the closure of a fuzzy set P (denoted by $i(P)$ and $cl(P)$ respectively) are defined by $i(P) = \vee\{V : V \leq P, V \in \tau\}$ and $cl(P) = \wedge\{F : P \leq F,\ 1 - F \in \tau\}$. A function g from fts (X, τ) to (Y, σ) is called fuzzy continuous if $g^{-1}(A) \in \tau\ \forall A \in \sigma$ [2].

A fuzzy set x_α in X termed as fuzzy point takes the value α at x and 0 elsewhere, where $(0 < \alpha \leq 1)$. $x_\alpha \in P$ if $\alpha \leq P(x)$. Clearly, P is the union of all the fuzzy points that belong to A [7]. A function f between fts (X, τ) and (Y, σ) is said to be fuzzy open(fuzzy closed) [6] iff for any fuzzy open(closed) set P in X, $f(P)$ is fuzzy open(closed) set in Y.

In a nonnull set X, $\mu X \subseteq I^X$ is called a generalized fuzzy topology (in short, gft) if it contains 0 and is closed under arbitrary union. Here, $(X, \mu X)$ is termed as generalized fuzzy topological space (in short, $gfts$) and the members of μX are called the fuzzy μ-open and their complements as fuzzy μ-closed sets. The μ-interior and μ-closure of a fuzzy subset P are denoted by $i_\mu(P)$ and $c_\mu(P)$ respectively and defined similarly as defined in fts [5]. A function $g : (X, \mu X) \to (Y, \mu Y)$ is called fuzzy $(\mu X, \mu Y)$-continuous [1] if $\forall\ A \in \mu Y$, $g^{-1}(A) \in \mu X$.

3. Fuzzy μ^*-Open Set

In this section, we introduce the notion of fuzzy μ^*-open(closed) sets.

Definition 3.1. A fuzzy set P in a space $(X, \mu X, \tau)$ is called fuzzy a μ^*-open set if $P \leq cl(i_\mu(P))$. We call its complement as a fuzzy μ^*-closed set.

Theorem 3.2. *In a space* $(X, \mu X, \tau)$, $A \in \mu^* C(X_\tau)$ *iff* $i(c_\mu(A)) \leq A$.

Proof. Straightforward. □

Theorem 3.3. *In a space* $(X, \mu X, \tau)$, $P \in \mu^* O(X_\tau)$ *iff* $\exists\ Q \in \mu X$ *satisfying* $Q \leq P \leq cl(Q)$.

Proof. Let $P \in \mu^* O(X_\tau)$. Then $P \leq cl(i_\mu(P))$. Taking $Q = i_\mu(P)$, $Q \in \mu X$ and $Q \leq P \leq cl(i_\mu(P)) = cl(Q)$.

Conversely, let $\exists\ Q \in \mu X$ satisfying $Q \leq P \leq cl(Q)$. Now, $Q \leq i_\mu(P)$ and $cl(Q) \leq cl(i_\mu(P))$. So, $P \leq cl(i_\mu(P))$, showing $P \in \mu^* O(X_\tau)$. □

Theorem 3.4. *The collection* $\mu^* O(X_\tau)$ *forms a* gft *in* X.

Proof. Clearly, $0 \in \mu^* O(X_\tau)$. Let $\{A_\delta : \delta \in \Lambda\}$ be the family of μ^*-fuzzy open sets. For each $\delta \in \Lambda$, $U_\delta \leq A_\delta \leq cl(U_\delta)$, where $U_\delta \in \mu X$. Thus, $\vee\{U_\delta : \delta \in \Lambda\} = P$ (say) $\leq \vee\{A_\delta : \delta \in \Lambda\} \leq cl(P)$, where $P \in \mu X$. □

However, $\mu^* O(X_\tau)$ does not form a fuzzy topology in X is shown in the example below:

Example 3.5. Let us take a space $(X, \mu X, \tau)$ with

$$X = \{a, b, c\}, \qquad \mu X = \{0, P, Q, R\}, \qquad \tau = \{0, 1, A, B\},$$

where,

$$A = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0}{c}\right), \qquad P = \left(\frac{0}{a}, \frac{0}{b}, \frac{0.8}{c}\right), \qquad R = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.8}{c}\right).$$
$$B = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.5}{c}\right) \qquad Q = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.5}{c}\right),$$

Here, $D = \left(\frac{r}{a}, \frac{s}{b}, \frac{t}{c}\right)$ means a fuzzy set D in X defined as $D(a) = r$, $D(b) = s$, $D(c) = t$. Here, $\mu^*O(X_\tau) = \{0, P, Q, R, U_1, U_2, U_3\}$, where U_1, U_2, U_3 are fuzzy sets in X satisfying $P \leq U_1 \leq 1-A, Q \leq U_2 \leq 1-B$ and $R \leq U_3 \leq 1-A$. Now, $P \wedge Q = \left(\frac{0}{a}, \frac{0}{b}, \frac{0.5}{c}\right) \notin \mu^*O(X_\tau)$. Hence, $(X, \mu^*O(X_\tau))$ is not a fts.

Theorem 3.6. *In a space $(X, \mu X, \tau)$ if $P \in \mu^*O(X_\tau)$ satisfying $P \leq B \leq cl(P)$, then $B \in \mu^*O(X_\tau)$.*

Proof. Let $P \in \mu^*O(X_\tau)$. So, $\exists\ Q \in \mu X$ satisfying $Q \leq P \leq cl(Q)$. Thus, $Q \leq B$. Also, $cl(P) \leq cl(Q)$ which implies $B \leq cl(Q)$. So, $Q \leq B \leq cl(Q)$. Hence, $B \in \mu^*O(X_\tau)$. □

Theorem 3.7. *In a space $(X, \mu X, \tau)$, if $A \in \mu X$ then $A \in \mu^*O(X_\tau)$.*

Proof. Let $A \in \mu X$. Then, $i_\mu(A) = A$, hence $A \leq cl(i_\mu(A))$. Thus, $A \in \mu^*O(X_\tau)$. □

However, the converse does not hold. In Example 3.5, $E= \left(\frac{0.8}{a}, \frac{0.2}{b}, \frac{1}{c}\right) \in \mu^*O(X_\tau)$ but $E \notin \mu X$.

Remark 3.8. In Example 3.5, $A = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0}{c}\right) \in \tau$ but $A \notin \mu^*O(X_\tau)$. Also, the fuzzy set $S = \left(\frac{0.1}{a}, \frac{0}{b}, \frac{0.7}{c}\right) \in \mu^*O(X_\tau)$ but $S \notin \tau$.

Hence, fuzzy μ^*-open $\not\rightleftarrows$ fuzzy open.

Note: However, one sided implication is given by the theorem below:

Theorem 3.9. *In a space $(X, \mu X, \tau)$, if $\tau \leq \mu X$, then every member of τ is a member of $\mu^*O(X_\tau)$.*

Proof. Let $A \in \tau$. Since $\tau \leq \mu X$, by Theorem 3.7, $A \in \mu^*O(X_\tau)$. □

Theorem 3.10. *In a space $(X, \mu X, \tau)$, $A \in \mu^*O(X_\tau)$ iff $cl(Q) = cl(i_\mu(Q))$.*

Proof. Let $Q \in \mu^*O(X_\tau)$. Then, $Q \leq cl(i_\mu(Q))$ which gives $cl(Q) \leq cl(cl(i_\mu(Q))) = cl(i_\mu(Q))$. Also, $cl(i_\mu(Q)) \leq cl(Q)$. Therefore, $cl(Q) = cl(i_\mu(Q))$. Conversely, let $cl(Q) = cl(i_\mu(Q))$. Since, $Q \leq cl(Q)$, $Q \in \mu^*O(X_\tau)$. □

Theorem 3.11. *In a space $(X, \mu X, \tau)$, $Q \in \mu^*O(X_\tau)$ iff for every $x_\alpha \leq Q$, $\exists\ U \in \mu^*O(X_\tau)$ such that $x_\alpha \leq U \leq Q$.*

Proof. Let $x_\alpha \leq Q$, where Q is a fuzzy set in X. By the given condition $\exists\, U_\alpha \in \mu^*O(X_\tau)$ such that $x_\alpha \leq U_\alpha \leq Q$. Then, $Q = \vee\{U_\alpha : x_\alpha \leq Q\}$ and by Theorem 3.4, $Q \in \mu^*O(X_\tau)$. The converse part is trivial. □

Theorem 3.12. *In a space $(X, \mu X, \tau)$, $Q \in \mu^*C(X_\tau)$ iff $i(Q) = i(c_\mu(Q))$.*

Proof. Let $Q \in \mu^*C(X_\tau)$. Then, $i(c_\mu(Q)) \leq Q$ and $i(c_\mu(Q)) \leq i(Q)$. Also, $Q \leq c_\mu(Q)$ and hence, $i(Q) = i(c_\mu(Q))$. Conversely, let $i(Q) = i(c_\mu(Q))$. As, $i(Q) \leq Q$, the result follows.

□

Theorem 3.13. *In a space $(X, \mu X, \tau)$, if $A \in \mu^*C(X_\tau)$ then it can be expressed as $A = U \wedge V$ for some $U \in \tau$ and $i(V) = i(c_\mu(V))$.*

Proof. Let $A \in \mu^*C(X_\tau)$. By Theorem 3.12, $i(A) = i(c_\mu(A))$. As, $A = 1 \wedge A$, choosing $U = 1$ and $V = A$, the result follows. □

4. Fuzzy μ^*-Continuous Function

Definition 4.1. A function g between a space $(X, \mu X, \tau)$ and fts (Y, σ) is a fuzzy μ^*-continuous function if $g^{-1}(V) \in \mu^*O(X_\tau)\ \forall\ V \in \sigma$.

Now, we shall find the relationship of fuzzy μ^*-continuity with the well known notion of fuzzy continuity.

Example 4.2. Let us consider a space $(X, \mu X, \tau)$ where

$$X = \{a, b, c\}, \qquad \mu X = \{0, 1, P, Q, R\}, \qquad \tau = \{0, 1, A, Q, C\},$$

and

$$P = \left(\frac{0}{a}, \frac{0}{b}, \frac{0.8}{c}\right), \qquad Q = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.5}{c}\right), \qquad R = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.8}{c}\right).$$

Also, $A = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0}{c}\right)$, $C = \left(\frac{0.1}{a}, \frac{0}{b}, \frac{0}{c}\right)$. Here,

$$\mu^*O(X_\tau) = \{0, 1, P, Q, R, U_1, U_2, U_3\},$$

where U_1, U_2, U_3 satisfying $P \leq U_1 \leq 1 - A, Q \leq U_2 \leq 1 - Q, R \leq U_3 \leq 1 - A$. Let us consider a fts (Y, σ) where $Y = \{m, n, t\}$, $\sigma = \{0, 1, A_1, B_1\}$ and $A_1 = \left(\frac{0.1}{m}, \frac{0}{n}, \frac{0}{t}\right)$, $B_1 = \left(\frac{0.1}{m}, \frac{0}{n}, \frac{0.5}{t}\right)$. Let us define a function $g : X \to Y$ by $g(a) = m$, $g(b) = n$ and $g(c) = n$. Clearly, g is fuzzy continuous. As, $B_1 \in \sigma$ but $g^{-1}(B_1) = \left(\frac{0.1}{a}, \frac{0}{b}, \frac{0}{c}\right) \notin \mu^*O(X_\tau)$, g is not fuzzy μ^*-continuous.

Example 4.3. Let us consider a space $(X, \mu X, \tau)$ where

$$X = \{a, b, c\}, \qquad \mu X = \{0, 1, P, Q, R, S\}, \qquad \tau = \{0, 1, A, B, C\}.$$

where

$$P = \left(\tfrac{0.1}{a}, \tfrac{0.4}{b}, \tfrac{0}{c}\right), \qquad R = \left(\tfrac{0.2}{a}, \tfrac{0.4}{b}, \tfrac{0.5}{c}\right),$$

$$Q = \left(\tfrac{0.2}{a}, \tfrac{0}{b}, \tfrac{0.5}{c}\right), \qquad S = \left(\tfrac{0.1}{a}, \tfrac{0}{b}, \tfrac{0}{c}\right)$$

and

$$A = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0}{c}\right), \qquad B = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.5}{c}\right), \qquad C = \left(\frac{0.1}{a}, \frac{0}{b}, \frac{0}{c}\right).$$

Here, $\mu^*O(X_\tau)=\{0,1,P,Q,R,S,U_1,U_2,U_3, U_4\}$ where U_1,U_2,U_3,U_4 are fuzzy sets satisfying $P \leq U_1 \leq 1-B, Q \leq U_2 \leq 1-B, R \leq U_3 \leq 1-B, S \leq U_4 \leq 1-B$. Let us consider a fts (Y,σ) where $Y = \{m,n,t\}$, $\sigma = \{0,1,A_1,B_1\}$ and

$$A_1 = \left(\frac{0.1}{m}, \frac{0}{n}, \frac{0}{t}\right), \qquad B_1 = \left(\frac{0.1}{m}, \frac{0}{n}, \frac{0.5}{t}\right).$$

We define a function $g : X \to Y$, where $X = \{a,b,c\}$ and $Y = \{m,n,t\}$ by $g(a) = m$, $g(b) = t$ and $g(c) = n$. Clearly, g is fuzzy μ^*-continuous. Now, $B_1 \in \sigma$ but $g^{-1}(B_1) = \left(\frac{0.1}{a}, \frac{0.5}{b}, \frac{0}{c}\right) \notin \tau$. Hence, the function is not fuzzy continuous.

Hence, from Example 4.2 and Example 4.3 we have:

fuzzy μ^*-continuity $\underset{\not\longrightarrow}{\not\longleftarrow}$ fuzzy continuity.

Now if we consider $\mu Y = \{O,1,P_1\}$ where $P_1 = \left(\frac{0}{m}, \frac{0}{n}, \frac{0.8}{t}\right)$, then we see that $g^{-1}(P_1) = \left(\frac{0}{a}, \frac{0.8}{b}, \frac{0}{c}\right) \notin \mu X$. So g is not fuzzy $(\mu X, \mu Y)$-continuous.

Theorem 4.4. *Let $(X,\mu X,\tau)$ be a space and (Y,σ) be a fts such that $\tau \leq \mu X$, then every fuzzy continuous function g from X to Y is fuzzy μ^*-continuous.*

Proof. Let $V \in \sigma$ then $g^{-1}(V) \in \tau$. Since $\tau \leq \mu X$, using Theorem 3.7, $g^{-1}(V) \in \mu^*O(X_\tau)$. Hence, g is a fuzzy μ^*-continuous. □

Theorem 4.5. *A fuzzy (μ,σ)-continuous function between a space $(X,\mu X,\tau)$ and a fts (Y,σ) is fuzzy μ^*-continuous.*

Proof. Straightforward. □

However, the converse does not hold as members of $\mu^*O(X_\tau)$ need not be fuzzy μ-open.

Example 4.6. Let us consider two spaces $(X,\mu X,\tau)$ and $(Y,\mu Y,\sigma)$ where

$$X = \{a,b,c,d,e\}, \qquad \mu X = \{0,1,P,Q,R,S\}, \qquad \tau = \{0,1,A,B,C,D\},$$

and

$$Y = \{m,n,t\}, \qquad \mu Y = \{0,1,P_1,Q_1,R_1\}, \qquad \sigma = \{0,1,A_1,B_1\}.$$

Here,

$$P = \left(\tfrac{0}{a}, \tfrac{0}{b}, \tfrac{0}{c}, \tfrac{0}{d}, \tfrac{0.7}{e}\right), \qquad R = \left(\tfrac{0.2}{a}, \tfrac{0.2}{b}, \tfrac{0}{c}, \tfrac{0}{d}, \tfrac{0.7}{e}\right),$$

$$Q = \left(\tfrac{0.2}{a}, \tfrac{0.2}{b}, \tfrac{0}{c}, \tfrac{0}{d}, \tfrac{0.6}{e}\right), \qquad S = \left(\tfrac{0.5}{a}, \tfrac{0.2}{b}, \tfrac{0.8}{c}, \tfrac{0.9}{d}, \tfrac{0.8}{e}\right),$$

and

$$A = \left(\tfrac{0.2}{a}, \tfrac{0}{b}, \tfrac{0.3}{c}, \tfrac{0}{d}, \tfrac{0}{e}\right), \qquad C = \left(\tfrac{0.5}{a}, \tfrac{1}{b}, \tfrac{0.3}{c}, \tfrac{0}{d}, \tfrac{0}{e}\right),$$

$$B = \left(\tfrac{0.5}{a}, \tfrac{1}{b}, \tfrac{0.2}{c}, \tfrac{0}{d}, \tfrac{0}{e}\right), \qquad D = \left(\tfrac{0.2}{a}, \tfrac{0}{b}, \tfrac{0.2}{c}, \tfrac{0}{d}, \tfrac{0}{e}\right).$$

Also,

$$P_1 = \left(\frac{0}{m}, \frac{0}{n}, \frac{0.7}{t}\right) \qquad Q_1 = \left(\frac{0.2}{m}, \frac{0}{n}, \frac{0.6}{t}\right), \qquad R_1 = \left(\frac{0.2}{m}, \frac{0}{n}, \frac{0.7}{t}\right),$$

and

$$A_1 = \left(\frac{0.2}{m}, \frac{0}{n}, \frac{0.3}{t}\right), \qquad B_1 = \left(\frac{0.3}{m}, \frac{1}{n}, \frac{0.5}{t}\right).$$

Now, $\mu^*O(X_\tau) = \{0, 1, P, Q, R, S, U_1, U_2, U_3, U_4\}$ where the fuzzy sets $U_i, i = 1, 2, 3, 4$ are such that $P \leq U_1 \leq 1 - C, Q \leq U_2 \leq 1 - C, R \leq U_3 \leq 1, S \leq U_4 \leq 1 - A$. Let us define a function $g : X \to Y$ by $g(a) = m = g(b)$, $g(c) = n = g(d)$ and $g(e) = t$. Clearly, g is fuzzy $(\mu X, \mu Y)$-continuous. Now, $A_1 \in \sigma$ but $g^{-1}(A_1) = \left(\tfrac{0.2}{a}, \tfrac{0.2}{b}, \tfrac{0}{c}, \tfrac{0}{d}, \tfrac{0.3}{e}\right) \notin \mu^*O(X_\tau)$. Hence, g is not fuzzy μ^*-continuous.

Hence, from Example 4.2 and Example 4.6 we have:

fuzzy μ^*-continuity $\underset{\not\longrightarrow}{\not\longleftarrow}$ fuzzy $(\mu X, \mu Y)$-continuity.

All results from above three examples can be seen at a time in the following table:

Examples	fuzzy continuous	fuzzy μ^*-continuous	$(\mu X,\ \mu Y)$-continuous)
Example 4.2	✓	×	
Example 4.3	×	✓	×
Example 4.6		×	✓

Theorem 4.7. *The following results are equivalent for a function g between a space $(X, \mu X, \tau)$ and a fts (Y, σ)*

(1) *g is fuzzy μ^*-continuous.*
(2) *for each x_α in X and each $A \in \sigma$ containing $g(x_\alpha)$, $\exists\ U \in \mu^*O(X_\tau)$ containing x_α such that $g(U) \leq A$.*
(3) *$g^{-1}(F) \in \mu^*C(X_\tau)$ of each F such that $1_Y - F \in \sigma$.*
(4) *$i(c_\mu(g^{-1}(F))) \leq g^{-1}(cl(F))$ for any fuzzy subset F of Y.*
(5) *$g(i(c_\mu(N))) \leq cl(g(N))$ for any fuzzy subset N of X.*

Proof. (1) $\Rightarrow$ (2): For any x_α in X and $A \in \tau$ with $g(x_\alpha) \leq A$, $g^{-1}(A) = U(say) \in \mu^*O(X_\tau)$ we have $x_\alpha \leq U$ and $g(U) \leq A$.

(2) $\Rightarrow$ (3): For any fuzzy closed set F in Y, $1-F=V(say)\in\sigma$. Let $x_\alpha\in g^{-1}(V)$. Then $g(x_\alpha)\in V$ and by the given condition $\exists$ $U_{x_\alpha}\in\mu^*O(X_\tau)$ containing x_α and $g(U_{x_\alpha})\leq V$. Clearly, $x_\alpha\in U_{x_\alpha}\leq g^{-1}(V)$ gives $\vee_{x_\alpha\in g^{-1}(V)}\{x_\alpha\}\leq\vee_{x_\alpha\in g^{-1}(V)}\{U_{x_\alpha}\}\leq\vee_{x_\alpha\in g^{-1}(V)}(g^{-1}(V))$. So, $g^{-1}(V)=\vee_{x_\alpha\in g^{-1}(V)}\{U_{x_\alpha}\}\in\mu^*O(X_\tau)$. Now, $g^{-1}(V)=g^{-1}(1-F)=1-g^{-1}(F)$. Hence, $g^{-1}(V)\in\mu^*C(X_\tau)$.

(3) $\Rightarrow$ (1): Straightforward.

(3) $\Rightarrow$ (4): For the fuzzy set $F\leq Y$, $g^{-1}(cl(F))\in\mu^*C(X_\tau)$. Hence, $i(c_\mu(g^{-1}(F)))\leq i(c_\mu(g^{-1}(cl(F))))\leq g^{-1}(cl(F))$. So,

$$i(c_\mu(g^{-1}(F)))\leq g^{-1}(cl(F)).$$

(4) $\Rightarrow$ (5): For $N\leq X$, $g(N)\leq Y$ and

$$i(c_\mu(g^{-1}(g(N))))\leq g^{-1}(cl(g(N))).$$

Also, $i(c_\mu(N))\leq g^{-1}(cl(g(N)))$. So, $g(i(c_\mu(N)))\leq cl(g(N)$.

(5) $\Rightarrow$ (3): For $S\leq Y$, $g^{-1}(S)\leq X$ and $g(i(c_\mu(g^{-1}(S))))\leq cl(g(g^{-1}(S)))\leq cl(S)=S$. Hence, $i(c_\mu(g^{-1}(S)))\leq g^{-1}(S)$, showing $g^{-1}(S)\in\mu^*C(X_\tau)$. □

Theorem 4.8. *Let us consider two fts (Y,σ), (Z,ρ) and $(X,\mu X,\tau)$ be a space. If $g:Y\to Z$ is fuzzy continuous and $h:X\to Y$ is fuzzy μ^*-continuous then $g\circ h:X\to Z$ is fuzzy μ^*-continuous.*

Proof. For any $P\in\rho$, we have $(g\circ h)^{-1}(P)=h^{-1}(g^{-1}(P))$. Now, $(g^{-1}(P))$ being fuzzy open in Y, $h^{-1}(g^{-1}(P))$ is fuzzy μ^*-open in X. Hence, $(g\circ h):X\to Z$ is a fuzzy μ^*-continuous. □

Theorem 4.9. *Let $(X,\mu X,\tau)$ be a space and (X_i,τ_i)(where $i=1,2$) be fts. Consider the projection functions $p_i:X_1\times X_2\to X_i$ $(i=1,2)$ of $X_1\times X_2$ in X_i. If $g:X\to X_1\times X_2$ is fuzzy μ^*-continuous, then $p_i\circ g$ is also fuzzy μ^*-continuous.*

Proof. Proof: The projection function being fuzzy continuous, by Theorem 4.8 the result follows. □

Theorem 4.10. *Let $(X_i,\mu X_i,\tau_i)$ for $i=1,2$ be two spaces. Let $f:X_1\to X_2$ be a function. The sufficient condition for f to be fuzzy μ^*-continuous is the graph $g:X_1\to X_1\times X_2$ of f is fuzzy μ^*-continuous.*

Proof. For any fuzzy open set $\lambda_1\times\lambda_2$ in $X_1\times X_2$, $g^{-1}(\lambda_1\times\lambda_2)=\lambda_1$, which is fuzzy μ^*-open. Let us define a projection function $p_2:X_1\times X_2\to X_2$. Clearly p_2 is a fuzzy continuous function. Hence, by Theorem 4.8, $p_2\circ g$ is fuzzy μ^*-continuous function. Now, $(p_2\circ g)(a)=p_2(g(a))=p_2(a,f(a))=f(a)$, $\forall\ a\in X_1$. So, we have $(p_2\circ g)=f$. Thus, f is a fuzzy μ^*-continuous function. □

5. Fuzzy (τ, μ^*)-Closed (Open) Function

Definition 5.1. A function g between a fts (X, τ) and a space $(Y, \mu Y, \sigma)$ is called fuzzy (τ, μ^*)-closed(open) if for any fuzzy closed(open) set P, $g(P)$ is fuzzy μ^*-closed(open) in Y.

Now we shall see that this (τ, μ^*)-open(closed) map is independent of the existing fuzzy open (closed) map by the following examples.

Example 5.2. Let us consider a space $(X, \mu X, \tau)$ where $X = \{a, b, c\}$, $\mu X = \{0, 1, P, Q, R\}$, $\tau = \{0, 1, A, Q, C\}$ and $P = \left(\frac{0}{a}, \frac{0}{b}, \frac{0.8}{c}\right)$, $Q = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.5}{c}\right)$, $R = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0.8}{c}\right)$. Also, $A = \left(\frac{0.2}{a}, \frac{0}{b}, \frac{0}{c}\right)$, $C = \left(\frac{0.1}{a}, \frac{0}{b}, \frac{0}{c}\right)$. Here, $\mu^*O(X_\tau)$ $=\{0, 1, P, Q, R, U_1, U_2, U_3\}$ where U_1, U_2, U_3 satisfy $P \leq U_1 \leq 1 - A, Q \leq U_2 \leq 1 - Q, R \leq U_3 \leq 1 - A$. Let us consider another space $(Y, \mu Y, \sigma)$ where $Y = \{m, n\}$, $\sigma = \{0, 1, A_1, B_1, C_1\}$ and $A_1 = \left(\frac{0.2}{m}, \frac{0}{n}\right)$, $B_1 = \left(\frac{0.1}{m}, \frac{0}{n}\right)$, and $C_1 = \left(\frac{0.2}{m}, \frac{0.5}{n}\right)$. $\mu Y = \{0, 1, P_1\}$ where, $P_1 = \left(\frac{0.7}{m}, \frac{0.5}{n}\right)$. Let us define a function $f : X \to Y$ by $f(a) = m$, $f(b) = f(c) = n$. Clearly, f is a fuzzy open map. But $U_1 = \left(\frac{0.6}{a}, \frac{0.5}{b}, \frac{0.8}{c}\right) \in \mu^*O(X_\tau)$ and $f(U_1) \notin \mu^*O(Y_\sigma)$. Hence, f is not a fuzzy (τ, μ^*)-open map.

Example 5.3. Considering the space $(X, \mu X, \tau)$ and the function f same as in Example 5.2 and taking $(Y, \mu Y, \sigma)$ as $\mu Y = \{0, 1, Q_1\}$ where, $Q_1 = \left(\frac{0.1}{m}, \frac{0}{n}\right)$ and $\sigma = \{0, 1, \left(\frac{0.2}{m}, \frac{0}{n}\right), \left(\frac{0.2}{m}, \frac{0.5}{n}\right)\}$ we get $\mu^*O(Y_\sigma) = \{0, 1, Q_1 \leq V_1 \leq 1 - C\}$. Clearly, $f(C) = \left(\frac{0.1}{m}, \frac{0}{n}\right)$ is not fuzzy open in Y hence f is not fuzzy open, but f is (τ, μ^*)-open map.

Hence, from Example 5.2 and Example 5.3 we have:

fuzzy open map $\begin{array}{c}\not\longleftarrow\\ \not\longrightarrow\end{array}$ fuzzy(τ, μ^*)-open map.

Similarly, we can show,

fuzzy closed map. $\begin{array}{c}\not\longleftarrow\\ \not\longrightarrow\end{array}$ fuzzy(τ, μ^*)-closed map

Theorem 5.4. *The following results are equivalent in a space $(Y, \mu Y, \sigma)$ and in a fts (X, τ) with a function $g : X \to Y$.*

(1) *g is a fuzzy (τ, μ^*)-open function.*
(2) *$g(i(N)) \leq i_{\mu^*}(g(N))$, for all fuzzy sets N in X.*
(3) *$g^{-1}(c_{\mu^*}(Q)) \leq cl(g^{-1}(Q))$, for all fuzzy sets Q in Y.*
(4) *$i(g^{-1}(Q)) \leq g^{-1}(i_{\mu^*}(Q))$, for each fuzzy set Q in Y*

Proof. (1) $\Rightarrow$ (2): Let g be fuzzy (τ, μ^*)-open function. Let N be a fuzzy set then $g(i(N)) \leq g((N))$, $g(i(N)) \in \mu^*O(Y_\sigma)$ so, $g(i(N)) = i_{\mu^*}(g(i(N))) \leq i_{\mu^*}(g(N))$.

(2) $\Rightarrow$ (1): Let $N \in \tau$ then $N = i(N)$ which gives $g(N) = g(i(N)) \leq i_{\mu^*}(g(N)) \leq g(N)$. Hence, $g(N) = i_{\mu^*}(g(N))$ which shows $g(N) \in \mu^*O(Y_\sigma)$.

(2) $\Rightarrow$ (3): For any fuzzy set Q in Y, $g^{-1}(1_Y - Q) = N$(say) is a fuzzy set in X. We have, $i(g^{-1}(1_Y - Q)) \leq g^{-1}(i^*_\mu(1_Y - Q))$. So, $i(1_X - g^{-1}(Q)) \leq g^{-1}(1_Y - c_{\mu^*}(Q)) = 1_X - cl(g^{-1}(Q)) \leq 1_X - g^{-1}(c_{\mu^*}(Q))$ which gives $g^{-1}(c_{\mu^*}(Q)) \leq cl(g^{-1}(Q))$.

(3) $\Rightarrow$ (4): Let S be any fuzzy set in Y and let $R = 1_Y - S$. We have, $g^{-1}(c_{\mu^*}(R)) \leq cl(g^{-1}(R))$ which gives $g^{-1}(c_{\mu^*}(1_Y - S)) \leq cl(g^{-1}(1_Y - S)) \Rightarrow g^{-1}(1_Y - i_{\mu^*}(S)) \leq cl(1_X - g^{-1}(S))$ which gives $1_X - g^{-1}(i_{\mu^*}(S)) \leq 1_X - i(g^{-1}(S))$, hence $i(g^{-1}(S)) \leq g^{-1}(i_{\mu^*}(S))$.

(4) $\Rightarrow$ (2): Let U be any fuzzy set n X. Let $S = g(U)$, then we have $i(U) \leq i(g^{-1}(g(U))) \leq g^{-1}(i_{\mu^*}(g(U)))$, so $g(i(U)) \leq g(g^{-1}(i_{\mu^*}(g(U)))) \leq i_{\mu^*}(g(U))$. Hence, the result follows. □

Theorem 5.5. *Let (X, τ) be a fts and $(Y, \mu Y, \sigma)$ be a space. The function $g : X \to Y$ is fuzzy (τ, μ^*)-closed iff for any fuzzy set S in X, $c_{\mu^*}(g(S)) \leq g(cl(S))$.*

Proof. Let g be fuzzy (τ, μ^*)-closed and S be any fuzzy set in X. Then $g(cl(S)) \in \mu^*C(Y_\sigma)$. Now, $c_{\mu^*}(g(S)) \leq c_{\mu^*}g(cl(S))$, which gives $c_{\mu^*}(g(S)) \leq g(cl(S))$ as $g(cl(S)) \in \mu^*C(Y_\sigma)$.
Conversely, let S be a fuzzy closed set in X. Then $c_{\mu^*}(g(S)) \leq g(cl(S)) = g(S) \leq c_{\mu^*}g(S)$, which shows that $g(S) \in \mu^*C(Y_\sigma)$. Hence, g is (τ, μ^*)-closed. □

Theorem 5.6. *The following results are equivalent for a function g from fts (X, τ) to a space $(Y, \mu Y, \sigma)$.*

(1) *g is a (τ, μ^*)-closed function.*
(2) *$\exists\ V \in \mu^*O(Y_\sigma)$ with $B \leq V$ and $g^{-1}(V) \leq U$ for each $B \leq Y$, $U \in \tau$ such that $g^{-1}(B) \leq U$.*

Proof. (1) $\Rightarrow$ (2): Let f be a (τ, μ^*)-closed function. Let B be a fuzzy set in Y and $U \in \tau$ such that $g^{-1}(B) \leq U$. Let $V = 1_Y - g(1_X - U)$. Clearly, $V \in \mu^*O(Y_\sigma)$. Since f is a (τ, μ^*)-closed function, $g(1 - U) \in \mu^*C(Y_\sigma)$. Now, $1_X - U \leq 1_X - g^{-1}(B) = g^{-1}(1_Y - B)$, which gives $g(1_X - U) \leq (1_Y - B)$. Hence, $B \leq 1_Y - g(1_X - U) = V$, further $g^{-1}(V) = g^{-1}(1_Y - g(1_X - U)) = 1_X - g^{-1}(g(1_X - U)) \leq 1_X - (1_X - U) = U$. Therefore, $g^{-1}(V) \leq U$.

(2) $\Rightarrow$ (1): Let g satisfies the given condition. Let P be any fuzzy closed set in X, then $B = (1_X - P) \in \tau$. Now, $g^{-1}(1_Y - g(P)) = 1_X - g^{-1}(g(P)) \leq 1_X - P = B$. By hypothesis there exists a $V \in \mu^*O(Y_\sigma)$ such that $1_Y - g(P) \leq V$ and $g^{-1}(V) \leq B = 1_X - P$. Hence, $1_Y - V \leq g(P)$. Also $P \leq 1_X - g^{-1}(V) = g^{-1}(1_Y - V)$. So, $g(P) \leq g(g^{-1}(1_Y - V)) \leq 1_Y - V$. Thus, $g(P) = 1_Y - V \in \mu^*C(Y_\sigma)$. Hence, g is a (τ, μ^*)-closed function. □

Theorem 5.7. *The following results are equivalent for a bijective function g from a fts (X, τ) to a space $(Y, \mu Y, \sigma)$.*

(1) $g^{-1} : Y \to X$ *is fuzzy* μ^**-continuous.*
(2) g *is a fuzzy* (τ, μ^*)*-open function.*
(3) g *is a fuzzy* (τ, μ^*)*-closed function.*

Proof. (1) $\Rightarrow$ (2): Let $A \in \tau$ and for bijective function g, $g^{-1} : Y \to X$ is a fuzzy μ^*-continuous. Then $(g^{-1})^{-1}(A) = g(A) \in \mu^*O(Y_\sigma)$. Hence, g is fuzzy (τ, μ^*)-open function.

(2) $\Rightarrow$ (3): Consider a fuzzy closed set V in X and let $P = 1_X - V$. g being a fuzzy (τ, μ^*) open bijective function, $g(P) = g(1_X - P) = 1_Y - g(V) \in \mu^*C(Y_\sigma)$. So, g is a fuzzy (τ, μ^*)-closed function.

(3) $\Rightarrow$ (1): Let g be a fuzzy (τ, μ^*)- closed bijective function and S be a fuzzy closed set in X, then $g(S) \in \mu^*C(Y_\sigma)$. Also, $g(S) = (g^{-1})^{-1}(S)$. Thus, $g^{-1} : Y \to X$ is fuzzy μ^*-continuous. □

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