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Hermite-Hadamard-Fejér Inequalities for Preinvex Functions on Fractal Sets

Sikander Mehmood[1*] and Fiza Zafar[2]

Abstract. In this paper, for generalised preinvex functions, new estimates of the Fejér-Hermite-Hadamard inequality on fractional sets $\mathbb{R}^{\rho}$ are given in this study. We demonstrated a fractional integral inequalities based on Fejér-Hermite-Hadamard theory. We establish two new local fractional integral identities for differentiable functions. We construct several novel Fejér-Hermite-Hadamard-type inequalities for generalized convex function in local fractional calculus contexts using these integral identities. We provide a few illustrations to highlight the uses of the obtained findings. Furthermore, we have also given a few examples of new inequalities in use.

1. Introduction

It is important to study the Hermite-Hadamard inequality for convex functions in different fields of science, since it connects the theory of convex functions with integral inequality. In the recent past, many generalizations of the convex functions are developed and researchers have obtained Hermite-Hadamard inequality estimates for the generalized convex functions. Researchers have also shown interest in generalizing this concept to preinvex functions.

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Let F be a convex function such that $F: V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $\kappa_1, \kappa_2 \in V$ with $\kappa_1 < \kappa_2$, then

$$F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\upsilon) d\upsilon \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}, \tag{1.1}$$

is the well-known Hermite-Hadamard inequality for convex functions.

The generalization of inequality (1.1) is given by Fejér [8] as

$$\begin{aligned} F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \int_{\kappa_1}^{\kappa_2} G(\upsilon) d\upsilon &\leq \int_{\kappa_1}^{\kappa_2} G(\upsilon) F(\upsilon) d\upsilon \\ &\leq \frac{F(\kappa_1) + F(\kappa_2)}{2} \int_{\kappa_1}^{\kappa_2} G(\upsilon) d\upsilon, \end{aligned} \tag{1.2}$$

holds, where $G: [\kappa_1; \kappa_2] \rightarrow \mathbb{R}$ is a nonnegative integrable function and it is symmetric about $\upsilon = \frac{\kappa_1 + \kappa_2}{2}$.

The idea of invex sets was given by T. Antczak [2]. As a result, it provided the foundation for defining the preinvex function.

Definition 1.1. A set $V \subseteq \mathbb{R}$ is invex with respect to the map $\Im: V \times V \rightarrow \mathbb{R}$ if for every $\kappa_1, \kappa_2 \in V$ and $s \in [0, 1]$, $\kappa_2 + s\zeta(\kappa_1, \kappa_2) \in V$. The invex set V is also called an $\Im$-connected set.

Remark 1.2. The convex set is always an invex set, but every invex set need not be convex.

The preinvex functions are a class of generalized convex functions. Weir et al. in [16] gave the idea of preinvex functions:

Definition 1.3. Let $V \subseteq \mathbb{R}$ be an invex set and a function $F: V \rightarrow \mathbb{R}$ is said to be preinvex w.r.t. $\Im$ if

$$F(\kappa_2 + s\Im(\kappa_1, \kappa_2)) \leq s\phi(\kappa_1) + (1 - s) F(\kappa_2),$$

$\forall\ \kappa_1, \kappa_2 \in V$ and $s \in [0, 1]$.

In [14], Sun W., defined the generalized preinvex function as:

Definition 1.4. A function $F: V \rightarrow \mathbb{R}^\rho$, where V is a invex subset of $\mathbb{R}$ is called generalized preinvex w.r.t. $\Im$ if

$$F(\kappa_2 + s\Im(\kappa_1, \kappa_2)) \leq s^\rho F(\kappa_1) + (1 - s)^\rho F(\kappa_2),$$

$\forall\ \kappa_1, \kappa_2 \in V$ and $s \in [0, 1]$.

Ordinary calculus, which defines derivatives and integrals of any real or complex order, is extended into fractional calculus. Certain real-world phenomena may be more effectively modelled by these fractional operators, particularly when the dynamics are influenced the system limitations. Fractional derivatives and integrals have many definitions, such as the Riemann-Liouville, Caputo, Hadamard, Riesz, Grünwald-Letnikov, Atangana-Baleanu, Marchaud, etc. Some of the typical characteristics of function differentiation, such as the Leibniz rule, the chain rule, and the semigroup property, to mention a few fail. This is the fact that the majority of them have already been thoroughly researched.

As there are wide applications of fractional calculus and Hermite-Hadamard inequalities in different fields of sciences, researchers are working to extend their work on Hermite-Hadamard inequalities for fractional integrals for the generalized convex functions (see [5, 7, 11–14]). Readers can see more fascinating work on Hermite Hadamard inequalities for Caputo, Riemann-Liouville and Atangana-Baleanu integral operators (refer to [4, 6, 10]).

Scientists and engineers have developed a significant interest in the fractal in recent years. A deeper comprehension of the actual models used in research and engineering will be made possible by calculus on fractal sets. One useful approach handling fractal and continuously non-differentiable functions is local fractional calculus. Local fractional calculus serves as the foundation for fractal analysis. It is important to note that fractal analysis presupposes a fundamentally improved way of visualising a fractal set. Numerous fields, including the generation of photographs, small-angle scattering theory, soil mechanics, and the music industry, utilize fractals. Mathematical inequalities are used more frequently as a result of the rising interest in fractal sets among researchers due to its relevance in cryptography and other domains. Fractal image compression is one of the most popular uses of fractal in software engineering. Researchers have used various methods to build several varieties of fractional calculus on fractal sets.

In this paper, we recall $\mathbb{N}$ to be the set of natural numbers, $\mathbb{Z}$ to be the set of integers, $\mathbb{Q}$ be the set of rational numbers and $\mathbb{R}$ be the set of real numbers. A few important concepts for the local fractional calculus were presented by Yang [17]. For $0 < \rho \leq 1$, the elements of different ρ-type set are defined as:

The ρ-type set of integers is:

$$\mathbb{Z}^\rho := \left\{0^\rho, \pm 1^\rho, \pm 2^\rho, \ldots, \pm n^\rho, \ldots\right\}.$$

The ρ-type set of rational numbers is:

$$\mathbb{Q}^\rho := \left\{m^\rho = \left(\frac{\kappa_1}{\kappa_2}\right)^\rho : \kappa_1, \kappa_2 \in \mathbb{Z},\ \kappa_2 \neq 0\right\}.$$

The ρ-type set of irrational numbers is:

$$\mathbb{Q}_c^\rho := \left\{ m^\rho \neq \left(\frac{\kappa_1}{\kappa_2}\right)^\rho : \kappa_1, \kappa_2 \in \mathbb{Z},\ \kappa_2 \neq 0 \right\}.$$

The ρ-type set of real number is $\mathbb{R}^\rho = \mathbb{Q}^\rho \cup \mathbb{Q}_c^\rho$,

The binary operations of addition and multiplication are defined on the ρ-type set $\mathbb{R}^\rho$ and these are defined as: $(\kappa_1^\rho + \kappa_2^\rho) = (\kappa_1 + \kappa_2)^\rho$ and $\kappa_1^\rho \cdot \kappa_2^\rho = (\kappa_1 \cdot \kappa_2)^\rho$ where $\kappa_1^\rho, \kappa_2^\rho \in \mathbb{R}^\rho$

(A) $\mathbb{R}^\rho$ is an abelian group under '+': for $\kappa_1^\rho, \kappa_2^\rho, \kappa_3^\rho \in \mathbb{R}^\rho$
  (i) $\kappa_1^\rho + \kappa_2^\rho \in \mathbb{R}^\rho$;
  (ii) $(\kappa_1^\rho + \kappa_2^\rho) + \kappa_3^\rho = \kappa_1^\rho + (\kappa_2^\rho + \kappa_3^\rho)$;
  (iii) 0^ρ is the identity element of $\mathbb{R}^\rho$, $\kappa_1^\rho + 0^\rho = 0^\rho + \kappa_1^\rho = \kappa_1^\rho$;
  (iv) For all $\kappa_1^\rho \in \mathbb{R}^\rho$ there exists $(-\kappa_1^\rho)^\rho \in \mathbb{R}^\rho$ such that $\kappa_1^\rho + (-\kappa_1)^\rho = (-\kappa_1)^\rho + \kappa_1^\rho = 0^\rho$;
  (v) $\kappa_1^\rho + \kappa_2^\rho = \kappa_2^\rho + \kappa_1^\rho$;

(B) $\mathbb{R}^\rho \backslash \{0^\rho\}$ is an abelian group under '$\cdot$': for $\kappa_1^\rho, \kappa_2^\rho, \kappa_3^\rho \in \mathbb{R}^\rho$
  (i) $\kappa_1^\rho \cdot \kappa_2^\rho \in \mathbb{R}^\rho$;
  (ii) $(\kappa_1^\rho \cdot \kappa_2^\rho) \cdot \kappa_3^\rho = \kappa_1^\rho \cdot (\kappa_2^\rho \cdot \kappa_3^\rho)$;
  (iii) 1^ρ is the identity element of $\mathbb{R}^\rho$, $\kappa_1^\rho \cdot 1^\rho = 1^\rho \cdot \kappa_1^\rho = \kappa_1^\rho$;
  (iv) For all $\kappa_1^\rho \in \mathbb{R}^\rho$ there exists $\left(\frac{1}{\kappa_1}\right)^\rho = \frac{1}{\kappa_1^\rho} \in \mathbb{R}^\rho$ such that $\kappa_1^\rho \cdot \frac{1}{\kappa_1^\rho} = \frac{1}{\kappa_1^\rho} \cdot \kappa_1^\rho = 1^\rho$;
  (v) $\kappa_1^\rho \cdot \kappa_2^\rho = \kappa_2^\rho \cdot \kappa_1^\rho$:

(C) Distributive axiom holds in $\mathbb{R}^\rho$: $\kappa_1^\rho \cdot (\kappa_2^\rho + \kappa_3^\rho) = \kappa_1^\rho \cdot \kappa_2^\rho + \kappa_1^\rho \cdot \kappa_3^\rho$, for $\kappa_1^\rho, \kappa_2^\rho, \kappa_3^\rho \in \mathbb{R}^\rho$

From the above properties, we conclude that $\mathbb{R}^\rho$ is a field.

Definition 1.5. Let $F : \mathbb{R} \to \mathbb{R}^\rho, s \to F(s)$ be a non-differentiable mapping, it is called local fractional at the point $s_{0,}$ if for any $\varepsilon > 0$, there exists $\tau > 0$ satisfying $|F(s) - F(s_0)| < \varepsilon^\rho$ whenever $|s - s_0| < \tau$. Let $F(s)$ be the local continuous function on (κ_1, κ_2) denoted by $F \in C_\rho(\kappa_1, \kappa_2)$.

Definition 1.6. The local fractional derivative of $F(s)$ of order ρ at $s = s_0$ is defined as:

$$\begin{aligned} F^{(\rho)}(s_0) &= {}_{s_0}D_s^\rho F(s) \\ &= \left.\frac{d^\rho F(s)}{ds^\rho}\right|_{s=s_0} \\ &= \lim_{s \to s_0} \frac{\Delta^\rho (F(s) - F(s_0))}{(s - s_0)^\rho}, \end{aligned}$$

where

$$\Delta^\rho (F(s) - F(s_0)) \cong \Gamma(\rho + 1)(F(s) - F(s_0)).$$

If $F^{((m+1)\rho)}(s) = \overbrace{D_s^{\rho} D_s^{\rho} \ldots D_s^{\rho}}^{(m+1)-\text{times}} F(s)$ for any $s \in V \subseteq \mathbb{R}$, then it is denoted by $F \in D_{(m+1)\rho}(V)$, where $m = \mathbb{N} \cup \{0\}$ and $F^{(\rho)}(s) = D_s^{\rho} F(s)$.

Definition 1.7. Let $F \in L_{\rho}[\kappa_1, \kappa_2]$, and let Δ be the partition of $[\kappa_1, \kappa_2]$, where $\Delta = \{a_0, a_1, \ldots, a_M\}$, where $M \in \mathbb{N}$ and it satisfies $\kappa_1 = a_0 < a_1 < \cdots < a_M = \kappa_2$, then ${}_{\kappa_1}J_{\kappa_2}^{(\rho)} F(s)$, the local fractional integral of F of order ρ on the interval $[\kappa_1, \kappa_2]$ is given as:

$$\begin{aligned} {}_{\kappa_1}J_{\kappa_2}^{(\rho)} F(s) &= \frac{1}{\Gamma(\rho+1)} \int_{\kappa_1}^{\kappa_2} F(\upsilon)\,(d\upsilon)^{\rho} \\ &= \frac{1}{\Gamma(\rho+1)} \lim_{\Delta x \to 0} \sum_{i=0}^{M-1} F(\upsilon_i)(\Delta \upsilon_i)^{\rho}, \end{aligned}$$

where $\Delta \upsilon := \max\limits_{0 \le i \le M-1} \Delta \upsilon_i$ and $\Delta \upsilon_i := \upsilon_{i+1} - \upsilon_i$, $i = 0, 1, 2, \ldots, M-1$.

Some more properties of local fractional calculus are given as:

Remark 1.8. It follows that ${}_{\kappa_1}J_{\kappa_2}^{(\rho)} F(s) = 0$ if $\kappa_1 = \kappa_2$ and ${}_{\kappa_1}J_{\kappa_2}^{(\rho)} F(s) = -\,{}_{\kappa_2}J_{\kappa_1}^{(\rho)} F(s)$.

(a) If $F(s) = G^{(\rho)}(s) \in L_{\rho}[\kappa_1, \kappa_2]$, then we have

$${}_{\kappa_1}J_{\kappa_2}^{(\rho)} F(s) = G(\kappa_2) - G(\kappa_1).$$

(b) Let $F(s)$, $G(s) \in D_{\rho}[\kappa_1, \kappa_2]$ and $F^{(\rho)}(s)$, $G^{(\rho)}(s) \in C_{\rho}[\kappa_1, \kappa_2]$, then, we have

$${}_{\kappa_1}J_{\kappa_2}^{(\rho)} F(s) G^{(\rho)}(s) = F(s)G(s)\big|_{\kappa_1}^{\kappa_2} - {}_{\kappa_1}J_{\kappa_2}^{(\rho)} F^{(\rho)}(s) G(s).$$

(c) $\frac{d^{\rho}}{dx^{\rho}} x^{m\rho} = \frac{\Gamma(1+m\rho)}{\Gamma(1+(m-1)\rho)} x^{(m-1)\rho}$, $m \in \mathbb{R}$

(d) $\frac{1}{\Gamma(\rho+1)} \int_{\kappa_1}^{\kappa_2} x^{m\rho}\,(dx)^{\rho} = \frac{\Gamma(1+m\rho)}{\Gamma(1+(m+1)\rho)} \left(\kappa_2^{(1+m)\rho} - \kappa_1^{(1+m)\rho}\right)$, $m \in \mathbb{R}$.

In [13], Sun W. developed the local fractional integral inequalities.

Theorem 1.9. *Let $V \subseteq \mathbb{R}$ be an invex subset w.r.t $\Im$ where $\Im$ be a function such that $\Im : V \times V \to \mathbb{R}$. Suppose that $F : V \to \mathbb{R}^{\rho}$ $(\rho \in (0,1])$ such that $F \in D_{\rho}(V)$ and $F^{(\rho)} \in C_{\rho}[\kappa_1, \kappa_1 + \Im(\kappa_2, \kappa_1)]$, for every $\kappa_1, \kappa_2 \in V$ and $\kappa_1 < \kappa_1 + \Im(\kappa_2, \kappa_1)$. If $\left|F^{(\rho)}\right|^q$ be a generalized preinvex function on V, then the following inequality holds:*

$$\begin{aligned} &\left|(\lambda-1)^{\rho} F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2, \kappa_1)\right) - \lambda^{\rho} \frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2, \kappa_1))}{2^{\rho}} \right. \\ &\quad \left. + \frac{\Gamma(1+\rho)}{\Im^{\rho}(\kappa_2, \kappa_1)}\, {}_{\kappa_1}J_{\kappa_1+\Im(\kappa_2,\kappa_1)}^{\rho} F(x) \right| \end{aligned}$$

$$\leq \frac{\Im^{\rho}\left(\kappa_2,\kappa_1\right)}{2^{\rho}}\left[\frac{\Gamma\left(1+p\rho\right)}{2^{\rho}\Gamma\left(1+\left(1+p\right)\rho\right)}\left(\lambda^{(1+p)\rho}+\left(1-\lambda\right)^{(1+p)\rho}\right)\right]^{\frac{1}{p}}$$
$$\times\left(\frac{\Gamma\left(1+\rho\right)}{4^{\rho}\Gamma\left(1+2\rho\right)}\right)^{\frac{1}{q}}\left[\left|F^{\rho}\left(\kappa_2\right)\right|^{q}+3^{\rho}\left|F^{\rho}\left(\kappa_1\right)\right|^{q}\right]^{\frac{1}{q}}$$
$$+\left[3^{\rho}\left|F^{\rho}\left(\kappa_2\right)\right|^{q}+\left|F^{\rho}\left(\kappa_1\right)\right|^{q}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $p,q>1$

Fejér type Hermite-Hadamard inequalities for a generalized h-convex functions is proposed by Luo C. [11] as:

Theorem 1.10. *Let* $F:[\kappa_1;\kappa_2]\to\mathbb{R}^{\rho}$ *be generalized* h*-convex and let* $G:[\kappa_1;\kappa_2]\to\mathbb{R}^{\rho}$, $G>0^{\rho}$ *be symmetric w.r.t.* $\frac{\kappa_1+\kappa_2}{2}$. *If* $F(x)$, $G(x)\in J_x^{(\rho)}[\kappa_1;\kappa_2]$, *then, we have*

$$\frac{{}_{\kappa_1}J_{\kappa_2}^{(\rho)}F\left(s\right)G\left(s\right)}{\left(\kappa_2-\kappa_1\right)^{\rho}}\leq\left[F(\kappa_1)+F(\kappa_2)\right]\ {}_0J_1^{(\rho)}h(s)G(sa+(1-s)\kappa_2).$$

In this paper, we have developed some new Hermite-Hadamard-Fejér identities for preinvex functions for fractal sets. Then, we give error bounds for both the left and right-hand sides of Hermite-Hadamard-Fejér inequalities. We also give some applications of the new inequalities.

2. Main Results

In the main section, we let

$$|G|_{\infty}=\sup_{s\in[\kappa_1,\ \kappa_1+\Im(\kappa_2,\kappa_1)]}|G(s)|,$$

such that $G:[\kappa_1;\kappa_1+\Im\left(\kappa_2,\kappa_1\right)]\to\mathbb{R}$ is a continuous function and $F^{(\rho)}$is the derivative of F with respect to variable s. The collection of all real valued and Riemann integrable functions on the interval $[\kappa_1,\kappa_1+\Im\left(\kappa_2,\kappa_1\right)]$ is denoted by $L\left[\kappa_1,\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right]$.

Lemma 2.1. *For an open invex subset* V *of* $\mathbb{R}$, *there is a function* $\Im$ *such that* $\Im:V\times V\to\mathbb{R}^{\rho}$. *Let* F *be a differentiable function such that* $F:V\to\mathbb{R}^{\rho}$ *where* $F^{(\rho)}\in L\left[\kappa_1,\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right]$ *and* $\Im\left(\kappa_2,\kappa_1\right)>0$. *Let* G *be an integrable function such that* $G:\left[\kappa_1,\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right]\to[0,\infty)$, *then* $\forall\ \kappa_1,\kappa_2\in V$, *we have the following result:*

(2.1)
$$F\left(\kappa_1+\frac{1}{2}\Im\left(\kappa_2,\kappa_1\right)\right)\ {}_{\kappa_1}J_{(\kappa_1+\Im(\kappa_2,\kappa_1))}^{(\rho)}G(\upsilon)\ -\ {}_{\kappa_1}J_{(\kappa_1+\Im(\kappa_2,\kappa_1))}^{(\rho)}\left(FG\right)\left(\upsilon\right)$$
$$=\frac{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{2\rho}}{\Gamma(1+\rho)}\int_0^1 h(s)F^{(\rho)}(\kappa_1+s\Im\left(\kappa_2,\kappa_1\right))\left(ds\right)^{\rho},$$

where

$$h(s)=\begin{cases} \frac{1}{\Gamma(1+\rho)}\int_0^s G(\kappa_1+w\Im(\kappa_2,\kappa_1))\,(dw)^\rho, & s\in[0,\frac{1}{2}),\\ -\frac{1}{\Gamma(1+\rho)}\int_s^1 G(\kappa_1+w\Im(\kappa_2,\kappa_1))\,(dw)^\rho, & s\in[\frac{1}{2},1]. \end{cases}$$

Proof. Consider

$$\begin{aligned}
&\frac{1}{\Gamma(1+\rho)}\int_0^1 h(s)F^{(\rho)}(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\\
&=\frac{1}{\Gamma(1+\rho)}\int_0^{\frac{1}{2}}\left(\frac{1}{\Gamma(1+\rho)}\int_0^s G(\kappa_1+w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\right)\\
&\quad\times F^{(\rho)}(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\\
&\quad-\frac{1}{\Gamma(1+\rho)}\int_{\frac{1}{2}}^1\left(\frac{1}{\Gamma(1+\rho)}\int_1^s G(\kappa_1+w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\right)\\
&\quad\times F^{(\rho)}(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\\
&=J_1+J_2.
\end{aligned}$$

From the first integral J_1, we obtain

$$\begin{aligned}
J_1&=\frac{1}{\Gamma(1+\rho)}\int_0^{\frac{1}{2}}\left(\frac{1}{\Gamma(1+\rho)}\int_0^s G(\kappa_1+w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\right)\\
&\quad\times F^{(\rho)}(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\\
&=\frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)}\\
&\quad\times\int_0^s G(\kappa_1+w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\,F(\kappa_1+s\Im(\kappa_2,\kappa_1)\Big|_0^{\frac{1}{2}}\\
&\quad-\frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)}\\
&\quad\times\int_0^{\frac{1}{2}} G(\kappa_1+s\Im(\kappa_2,\kappa_1))F(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\\
&=\frac{F\left(\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)\right)}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)}\int_0^{\frac{1}{2}} G(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\\
&\quad-\frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)}\\
&\quad\times\int_0^{\frac{1}{2}} F(\kappa_1+s\Im(\kappa_2,\kappa_1))G(\kappa_1+s\Im(\kappa_2,\kappa_1))\,(ds)^\rho.
\end{aligned}\tag{2.2}$$

By substituting $\upsilon = \kappa_1 + s\zeta(\kappa_2, \kappa_1)$ in (2.2)

$$J_1 = \frac{F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2, \kappa_1)\right)}{(\Im(\kappa_2, \kappa_1))^{2\rho}} \frac{1}{\Gamma(\rho+1)} \int_{\kappa_1}^{\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)} G(\upsilon)\,(d\upsilon)^\rho \tag{2.3}$$
$$- \frac{1}{(\Im(\kappa_2, \kappa_1))^{2\rho}} \frac{1}{\Gamma(\rho+1)} \int_{\kappa_1}^{\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)} F(\upsilon)G(\upsilon)\,(d\upsilon)^\rho$$
$$= \frac{F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2, \kappa_1)\right)}{(\Im(\kappa_2, \kappa_1))^{2\rho}} \ {}_{\kappa_1}J^{(\rho)}_{\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)} G(\upsilon)$$
$$- \frac{1}{(\Im(\kappa_2, \kappa_1))^{2\rho}} \ {}_{\kappa_1}J^{(\rho)}_{\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)} (FG)(\upsilon).$$

From the second integral J_2, we obtain

(2.4)

$$J_2 = \frac{1}{\Gamma(1+\rho)} \int_{\frac{1}{2}}^{1} \left(\frac{1}{\Gamma(1+\rho)} \int_{1}^{s} G(\kappa_1 + w\Im(\kappa_2, \kappa_1))\,(dw)^\rho \right)$$
$$\times F^{(\rho)}(\kappa_1 + s\Im(\kappa_2, \kappa_1))\,(ds)^\rho$$
$$= \frac{1}{(\Im(\kappa_2, \kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)}$$
$$\times \int_{1}^{s} G(\kappa_1 + w\Im(\kappa_2, \kappa_1))\,(dw)^\rho\, F(\kappa_1 + s\Im(\kappa_2, \kappa_1)\Big|_{\frac{1}{2}}^{1}$$
$$- \frac{1}{(\Im(\kappa_2, \kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)}$$
$$\times \int_{\frac{1}{2}}^{1} F(\kappa_1 + s\Im(\kappa_2, \kappa_1))G(\kappa_1 + s\Im(\kappa_2, \kappa_1))\,(ds)^\rho$$
$$= -\frac{F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2, \kappa_1)\right)}{(\Im(\kappa_2, \kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)} \int_{1}^{\frac{1}{2}} G(\kappa_1 + s\Im(\kappa_2, \kappa_1))\,(ds)^\rho$$
$$- \frac{1}{(\Im(\kappa_2, \kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)}$$
$$\times \int_{\frac{1}{2}}^{1} F(\kappa_1 + s\Im(\kappa_2, \kappa_1))G(\kappa_1 + s\Im(\kappa_2, \kappa_1))\,(ds)^\rho.$$

By substituting $\upsilon = \kappa_1 + s\zeta(\kappa_2, \kappa_1)$ in (2.4)

$$J_2 = -\frac{F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2, \kappa_1)\right)}{(\Im(\kappa_2, \kappa_1))^{2\rho}} \frac{1}{\Gamma(1+\rho)} \int_{\kappa_1 + \Im(\kappa_2,\kappa_1)}^{\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)} G(\upsilon)d\upsilon \tag{2.5}$$
$$- \frac{1}{(\Im(\kappa_2, \kappa_1))^{2\rho}} \frac{1}{\Gamma(1+\rho)} \int_{\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)}^{\kappa_1 + \Im(\kappa_2,\kappa_1)} F(\upsilon)G(\upsilon)ds$$

$$= \frac{F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)\right)}{(\Im(\kappa_2,\kappa_1))^{2\rho}} {}_{\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)}J^{(\rho)}_{\kappa_1+\Im(\kappa_2,\kappa_1)} G(\upsilon)$$
$$- \frac{1}{(\Im(\kappa_2,\kappa_1))^{2\rho}} {}_{\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)}J_{\kappa_1+\Im(\kappa_2,\kappa_1)} (FG)(\upsilon).$$

By adding (2.3)and (2.5) , we get the required result (2.1). □

Lemma 2.2. *For an open invex subset V of $\mathbb{R}$, there is a function $\Im$ such that $\Im : V \times V \to \mathbb{R}^{\rho}$. Let F be a differentiable function such that $F : V \to \mathbb{R}^{\rho}$ where $F^{(\rho)} \in L[\kappa_1, \kappa_1 + \Im(\kappa_2,\kappa_1)]$ and $\Im(\kappa_2,\kappa_1) > 0$. Let G be an integrable function such that $G : [\kappa_1, \kappa_1 + \Im(\kappa_2,\kappa_1)] \to [0,\infty)$, then $\forall\ \kappa_1, \kappa_2 \in V$, we have the following result:*

$$\left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2,\kappa_1))}{2}\right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} (FG)(\upsilon) = \frac{(\Im(\kappa_2,\kappa_1))^{\rho}}{2\Gamma(1+\rho)} \int_0^1 k(s) F^{(\rho)}(\kappa_1 + s\Im(\kappa_2,\kappa_1))\, (ds)^{\rho}, \tag{2.6}$$

where

$$k(s) = \frac{1}{\Gamma(1+\rho)} \int_0^s G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^{\rho} - \frac{1}{\Gamma(1+\rho)} \int_s^1 G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^{\rho}, \quad s \in [0,1].$$

Proof. Let us consider

(2.7)
$$\frac{1}{\Gamma(1+\rho)} \int_0^1 \gamma(s) F^{(\rho)}(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^{\rho}$$
$$= \frac{1}{\Gamma(1+\rho)} \int_0^1 \left[\frac{1}{\Gamma(1+\rho)} \int_0^s G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^{\rho} - \frac{1}{\Gamma(1+\rho)} \int_s^1 G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^{\rho}\right] F^{(\rho)}(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^{\rho}$$
$$= J_1 + J_2.$$

From the first integral J_1, we obtain

$$J_1 = \frac{1}{\Gamma(\rho+1)} \int_0^1 \left(\frac{1}{\Gamma(1+\rho)} \int_0^s G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^{\rho}\right) \times F^{(\rho)}(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^{\rho} \tag{2.8}$$
$$= \frac{1}{(\Im(\kappa_2,\kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)}$$

$$
\begin{aligned}
&\times \int_0^s G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\, F(\kappa_1 + s\Im(\kappa_2,\kappa_1)\Big|_0^1 \\
&- \frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)} \\
&\times \int_0^1 G(\kappa_1 + s\Im(\kappa_2,\kappa_1))F(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^\rho \\
= {}& \frac{F(\kappa_1 + \Im(\kappa_2,\kappa_1)}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)}\int_0^1 G(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^\rho \\
&- \frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)} \\
&\times \int_0^1 F(\kappa_1 + s\Im(\kappa_2,\kappa_1))G(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^\rho\,.
\end{aligned}
$$

By substituting $\upsilon = \kappa_1 + s\zeta(\kappa_2,\kappa_1)$ in (2.8)

(2.9)

$$
\begin{aligned}
J_1 = {}& \frac{1}{(\Im(\kappa_2,\kappa_1))^{2\rho}}\frac{1}{\Gamma(1+\rho)}F(\kappa_1 + \Im(\kappa_2,\kappa_1) \\
&\times \left[\int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} G(\upsilon)\,(d\upsilon)^\rho - \int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} F(\upsilon)G(\upsilon)\,(d\upsilon)^\rho\right] \\
= {}& \frac{F(\kappa_1 + \Im(\kappa_2,\kappa_1)}{(\Im(\kappa_2,\kappa_1))^{2\rho}}\ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}G(\upsilon) \\
&- \frac{1}{(\Im(\kappa_2,\kappa_1))^{2\rho}}\ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}(FG)(\upsilon).
\end{aligned}
$$

From the second integral J_2, we obtain

$$
\begin{aligned}
(2.10)\quad J_2 = {}& \frac{1}{\Gamma(1+\rho)}\int_0^1 \left(\frac{-1}{\Gamma(1+\rho)}\int_s^1 G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\right) \\
&\times F^{(\rho)}(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^\rho \\
= {}& \frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)} \\
&\times (-1)\int_s^1 G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho\, F(\kappa_1 + s\Im(\kappa_2,\kappa_1)\Big|_0^1 \\
&- \frac{1}{(\Im(\kappa_2,\kappa_1))^\rho}\frac{1}{\Gamma(1+\rho)} \\
&\times \int_0^1 (F(\kappa_1 + s\Im(\kappa_2,\kappa_1))G(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^\rho
\end{aligned}
$$

$$= \frac{F(\kappa_1)}{(\Im(\kappa_2,\kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)} \int_0^1 G(\kappa_1 + s\Im(\kappa_2,\kappa_1))(ds)^{\rho}$$
$$- \frac{1}{(\Im(\kappa_2,\kappa_1))^{\rho}} \frac{1}{\Gamma(1+\rho)}$$
$$\times \int_0^1 F(\kappa_1 + s\Im(\kappa_2,\kappa_1)) G(\kappa_1 + s\Im(\kappa_2,\kappa_1))(ds)^{\rho}.$$

By substituting $\upsilon = \kappa_1 + s\zeta(\kappa_2,\kappa_1)$ in (2.10),

(2.11)

$$J_2 = \frac{1}{(\Im(\kappa_2,\kappa_1))^{2\rho}} \frac{1}{\Gamma(1+\rho)}$$
$$\times \left[F(\kappa_1) \int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} G(\upsilon)(d\upsilon)^{\rho} - \int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} F(\upsilon)G(\upsilon)(d\upsilon)^{\rho} \right]$$
$$= \frac{F(\kappa_1)}{(\Im(\kappa_2,\kappa_1))^{2\rho}} \ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon)$$
$$- \frac{1}{(\Im(\kappa_2,\kappa_1))^{2\rho}} \ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} (FG)(\upsilon).$$

By adding (2.9) and (2.11), we get the required result (2.6). □

Theorem 2.3. *For an open invex subset V of $\mathbb{R}$, there is a function $\Im$ such that $\Im : V \times V \to \mathbb{R}^{\rho}$. Let F be a differentiable function such that $F : V \to \mathbb{R}^{\rho}$ where $F^{(\rho)} \in L[\kappa_1, \kappa_1 + \Im(\kappa_2,\kappa_1)]$ and $\Im(\kappa_2,\kappa_1) > 0$. Let $G : [\kappa_1, \kappa_1 + \Im(\kappa_2,\kappa_1)] \to [0,\infty)$ be a function symmetric to $\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)$ and it is also an integrable function. If $\left|F^{(\rho)}\right|$ is a preinvex function on V then for every $\kappa_1, \kappa_2 \in V$, we have the following result:*

$$\left| F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)\right) \ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} (FG)(\upsilon) \right|$$
$$\leq \frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)} (\Im(\kappa_2,\kappa_1))^{2\rho} |G|_{\infty} \left[1 - 2\left(\frac{1}{2}\right)^{3\rho} \right]$$
$$\times \left(\left|F^{(\rho)}(\kappa_1)\right| + \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right| \right). \tag{2.12}$$

Proof. Applying modulus on both sides of (2.1),

(2.13)

$$\left| \frac{F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)\right)}{(\Im(\kappa_2,\kappa_1))^{2\rho}} \ {}_{\kappa_1}J^{(\rho)}_{\kappa_1+\Im(\kappa_2,\kappa_1)} G(\upsilon) \right.$$

$$
\begin{aligned}
&-\frac{1}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{2\rho}}\ {}_{\kappa_1}J^{(\rho)}_{\kappa_1+\Im(\kappa_2,\kappa_1)}\left(FG\right)\left(\kappa_1\right)\Bigg|\\
&=\Bigg|\frac{1}{\Gamma(1+\rho)}\int_0^{1/2}\left(\frac{1}{\Gamma(1+\rho)}\int_0^{s}G(\kappa_1+w\Im\left(\kappa_2,\kappa_1\right))\left(dw\right)^{\rho}\right)\\
&\times F^{(\rho)}(\kappa_1+s\Im\left(\kappa_2,\kappa_1\right))\left(ds\right)^{\rho}\\
&+\frac{1}{\Gamma(1+\rho)}\int_{1/2}^{1}\left(-\frac{1}{\Gamma(1+\rho)}\int_s^{1}G(\kappa_1+w\Im\left(\kappa_2,\kappa_1\right))\left(dw\right)^{\rho}\right)\\
&\times F^{(\rho)}(\kappa_1+s\Im\left(\kappa_2,\kappa_1\right))\left(ds\right)^{\rho}\Bigg|.
\end{aligned}
$$

From preinvexity of $\left|F^{(\rho)}\right|$ on V and Lemma 2.1, we have

$$
\begin{aligned}
&\Bigg|\frac{F\left(\kappa_1+\frac{1}{2}\Im\left(\kappa_2,\kappa_1\right)\right)}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{2\rho}}\quad aI^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}G(v)\\
&\quad-\frac{1}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{2\rho}}\quad aI^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}\left(FG\right)\left(\kappa_1\right)\Bigg|\\
&\le\frac{1}{\Gamma(1+\rho)}\int_0^{1/2}\left(\frac{1}{\Gamma(1+\rho)}\int_0^{s}\left|G(\kappa_1+w\Im\left(\kappa_2,\kappa_1\right))\right|\left(dw\right)^{\rho}\right)\\
&\times\left[(1-s)^{\rho}\left|F^{(\rho)}\left(\kappa_1\right)\right|+s^{\rho}\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|\right]\left(ds\right)^{\rho}\\
&+\frac{1}{\Gamma(1+\rho)}\int_{1/2}^{1}\left(\frac{1}{\Gamma(1+\rho)}\int_s^{1}\left|G(\kappa_1+w\Im\left(\kappa_2,\kappa_1\right))\right|\left(dw\right)^{\rho}\right)\\
&\times\left[(1-s)^{\rho}\left|F^{(\rho)}\left(\kappa_1\right)\right|+s^{\rho}\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|\right]\left(ds\right)^{\rho}\\
&=J_1+J_2
\end{aligned}
\tag{2.14}
$$

By changing the integration order in the first term of (2.14), we have

$$
\begin{aligned}
J_1&=\frac{1}{\Gamma(\rho+1)}\int_0^{1/2}\left|G(\kappa_1+w\Im\left(\kappa_2,\kappa_1\right))\right|\frac{1}{\Gamma(1+\rho)}\\
&\times\int_w^{1/2}\left[(1-s)^{\rho}\left|F^{(\rho)}\left(\kappa_1\right)\right|+s^{\rho}\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|\right]\left(ds\right)^{\rho}\left(dw\right)^{\rho}\\
&=\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\frac{1}{\Gamma(1+\rho)}\int_0^{1/2}\left|G(\kappa_1+w\Im\left(\kappa_2,\kappa_1\right))\right|\\
&\times\left[\left|F^{(\rho)}\left(\kappa_1\right)\right|\left((1-w)^{2\rho}-\left(\frac{1}{2}\right)^{2\rho}\right)\left(dw\right)^{\rho}\right.
\end{aligned}
$$

$$\times \left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|\left(\left(\frac{1}{2}\right)^{2\rho}-w^{2\rho}\right)\Bigg](dw)^{\rho}.$$

By substituting $\upsilon=\kappa_1+w\zeta\left(\kappa_2,\kappa_1\right)$ for $w\in[0,1]$ and using $|G|_\infty=\sup_{s\in[\kappa_1,\ \kappa_1+\Im(\kappa_2,\kappa_1)]}|G(\upsilon)|$

$$\begin{aligned}(2.15)\qquad J_1 \le\ & \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\frac{\left|F^{(\rho)}\left(\kappa_1\right)\right|}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{\rho}}|G|_\infty\frac{1}{\Gamma(\rho+1)}\\ &\times\int_{\kappa_1}^{\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)}\left(\left(1-\frac{\upsilon-\kappa_1}{\Im\left(\kappa_2,\kappa_1\right)}\right)^{2\rho}-\left(\frac{1}{2}\right)^{2\rho}\right)(d\upsilon)^{\rho}\\ &+\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\frac{\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{\rho}}|G|_\infty\frac{1}{\Gamma(\rho+1)}\\ &\times\int_{\kappa_1}^{\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)}\left(\left(\frac{1}{2}\right)^{2\rho}-\left(\frac{\upsilon-\kappa_1}{\Im\left(\kappa_2,\kappa_1\right)}\right)^{2\rho}\right)(d\upsilon)^{\rho}.\end{aligned}$$

Similarly, by changing integration order in the second term and assuming that G is symmetric w.r.t. $\kappa_1+\frac{1}{2}\Im\left(\kappa_2,\kappa_1\right)$, we have

$$\begin{aligned}J_2 =\ & \frac{1}{\Gamma(1+\rho)}\int_{1/2}^{1}\left|G(\kappa_1+(1-w)\,\Im\left(\kappa_2,\kappa_1\right))\right|\frac{1}{\Gamma(1+\rho)}\\ &\times\int_{1/2}^{w}\left[(1-s)^{\rho}\left|F^{(\rho)}\left(\kappa_1\right)\right|+s^{\rho}\left|F^{(\rho)}\left(\kappa_2\right)\right|\right](ds)^{\rho}(dw)^{\rho}\\ =\ & \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\frac{1}{\Gamma(1+\rho)}\int_{1/2}^{1}\left|G(\kappa_1+(1-w)\,\Im\left(\kappa_2,\kappa_1\right))\right|\\ &\times\left[\left|F^{(\rho)}\left(\kappa_1\right)\right|\left(\left(\frac{1}{2}\right)^{2\rho}-(1-w)^{2\rho}\right)\right.\\ &\left.+\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|\left(w^{2\rho}-\left(\frac{1}{2}\right)^{2\rho}\right)\right](dw)^{\rho}.\end{aligned}$$

By substituting $\upsilon=\kappa_1+(1-w)\Im\left(\kappa_2,\kappa_1\right)$ and using the assumption that $|G|_\infty=\sup_{s\in[\kappa_1,\kappa_1+\Im(\kappa_2,\kappa_1)]}|G(x)|$

$$\begin{aligned}(2.16)\qquad J_2 \le\ & \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\frac{\left|F^{(\rho)}\left(\kappa_1\right)\right|}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{\rho}}|G|_\infty\frac{1}{\Gamma(1+\rho)}\\ &\times\int_{\kappa_1}^{\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)}\left(\left(\frac{1}{2}\right)^{2\rho}-\left(\frac{\upsilon-\kappa_1}{\Im\left(\kappa_2,\kappa_1\right)}\right)^{2\rho}\right)(d\upsilon)^{\rho}\\ &+\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\frac{\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|}{\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{\rho}}|G|_\infty\frac{1}{\Gamma(1+\rho)}\end{aligned}$$

$$\times \int_{\kappa_1}^{\kappa_1+\frac{1}{2}\Im(\kappa_2,\kappa_1)} \left(\left(1 - \frac{\upsilon - \kappa_1}{\Im(\kappa_2,\kappa_1)}\right)^{2\rho} - \left(\frac{1}{2}\right)^{2\rho} \right) (d\upsilon)^{\rho}.$$

Using (2.15) and (2.16) in (2.14), we get our required result. □

Corollary 2.4. *If we take* $\Im(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, *then from* (2.12), *we have*

$$\left| F\left(\frac{\kappa_1+\kappa_2}{2}\right) {}_{\kappa_1}J_{\kappa_2}^{(\rho)} G(\upsilon) - {}_{\kappa_1}J_{\kappa_2}^{(\rho)} (FG)(\upsilon) \right| \le |G|_\infty \frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)} (\kappa_2-\kappa_1)^{2\rho} \left[1 - 2\left(\frac{1}{2}\right)^{3\rho}\right] \times \left(\left|F^{(\rho)}(\kappa_1)\right| + \left|F^{(\rho)}(\kappa_2)\right| \right). \tag{2.17}$$

Corollary 2.5. *For* $\rho = 1$ *and* $|G|_\infty = 1$ *and* $\Im(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, *we obtain Theorem 2.2. of citeKirmaci:*

$$\left| F\left(\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)\right) - \frac{1}{\Im(\kappa_2,\kappa_1)} \int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} F(\upsilon) \right| \le \frac{1}{8} \left(\Im(\kappa_2,\kappa_1)\right) \left[\left|F'(\kappa_1)\right| + \left|F'(\kappa_1+\Im(\kappa_2,\kappa_1))\right| \right]. \tag{2.18}$$

Theorem 2.6. *For an open invex subset* V *of* $\mathbb{R}$, *there is a function* $\Im$ *such that* $\Im : V \times V \to \mathbb{R}^\rho$. *Let* F *be a differentiable function such that* $F : V \to \mathbb{R}^\rho$ *where* $F^{(\rho)} \in L[\kappa_1, \kappa_1 + \Im(\kappa_2,\kappa_1)]$ *and* $\Im(\kappa_2,\kappa_1) > 0$. *Let* $G : [\kappa_1, \kappa_1 + \Im(\kappa_2,\kappa_1)] \to [0,\infty)$ *be a function symmetric to* $\kappa_1 + \frac{1}{2}\Im(\kappa_2,\kappa_1)$ *and it is also an integrable function. If* $\left|F^{(\rho)}\right|$ *is a preinvex function on* V *then for every* $\kappa_1, \kappa_2 \in V$, *we have the following result:*

$$\left| \left[\frac{F(\kappa_1) + F(\kappa_1+\Im(\kappa_2,\kappa_1))}{2}\right] {}_{\kappa_1}J_{(\kappa_1+\Im(\kappa_2,\kappa_1))}^{(\rho)} G(\upsilon) - {}_{\kappa_1}J_{(\kappa_1+\Im(\kappa_2,\kappa_1))}^{(\rho)} (FG)(\upsilon) \right| \le \frac{|G|_\infty}{2\Gamma(1+2\rho)} \left(\Im(\kappa_2,\kappa_1)\right)^{\rho} \left(\left|F^{(\rho)}(\kappa_1)\right| + \left|F^{(\rho)}(\kappa_1+\Im(\kappa_2,\kappa_1))\right| \right) \times \left[(\kappa_1+\Im(\kappa_2,\kappa_1))^{\rho} - (\kappa_1)^{\rho} \right]. \tag{2.19}$$

Proof. Applying modulus on both sides of (2.6),

$$\left| \left[\frac{F(\kappa_1) + F(\kappa_1+\Im(\kappa_2,\kappa_1))}{2}\right] {}_{\kappa_1}J_{(\kappa_1+\Im(\kappa_2,\kappa_1))}^{(\rho)} G(\upsilon) - {}_{\kappa_1}J_{(\kappa_1+\Im(\kappa_2,\kappa_1))}^{(\rho)} (FG)(\upsilon) \right| \tag{2.20}$$

$$= \left| \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \frac{1}{\Gamma(1+\rho)} \left[\int_0^1 \left(\int_0^s G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho \right.\right.\right.$$
$$\left.\left.\left. - \int_s^1 G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho \right) F^{(\rho)}(\kappa_1 + s\Im(\kappa_2,\kappa_1))\,(ds)^\rho \right] \right|.$$

From preinvexity of $\left|F^{(\rho)}\right|$ on V and Lemma 2.2, we have

(2.21)
$$\left| \left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2,\kappa_1))}{2} \right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) \right.$$
$$\left. - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}(FG)(\upsilon) \right|$$
$$\leq \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \left[\int_0^1 \left| \frac{1}{\Gamma(1+\rho)} \int_0^s G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho \right.\right.$$
$$- \frac{1}{\Gamma(1+\rho)} \int_s^1 G(\kappa_1 + w\Im(\kappa_2,\kappa_1))\,(dw)^\rho |$$
$$\left. \times (1-s)^\rho \left| F^{(\rho)}(\kappa_1) \right| + s^\rho \left| F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1)) \right| (ds)^\rho \right].$$

After simplification, (2.21) becomes,

$$\left| \left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2,\kappa_1))}{2} \right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) \right.$$
$$\left. - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}(FG)(\upsilon) \right|$$
$$\leq \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \int_0^1 \left(\frac{1}{\Gamma(1+\rho)} \int_0^s |G(\kappa_1 + w\Im(\kappa_2,\kappa_1))|\,(dw)^\rho \right)$$
$$\times \left((1-s)^\rho \left| F^{(\rho)}(\kappa_1) \right| + s^\rho \left| F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1)) \right| \right) (ds)^\rho$$
$$- \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \int_0^1 \left(\frac{1}{\Gamma(1+\rho)} \int_s^1 |G(\kappa_1 + w\Im(\kappa_2,\kappa_1))|\,(dw)^\rho \right)$$
$$\times \left((1-s)^\rho \left| F^{(\rho)}(\kappa_1) \right| + s^\rho \left| F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1)) \right| \right) (ds)^\rho.$$

By the change of the order of the integration, we have

$$\left| \left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2,\kappa_1))}{2} \right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) \right.$$
$$\left. - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}(FG)(\upsilon) \right|$$
$$\leq \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \int_0^1 |G(\kappa_1 + w\Im(\kappa_2,\kappa_1))| \frac{1}{\Gamma(1+\rho)}$$

$$\times \int_w^1 \left((1-s)^\rho \left|F^{(\rho)}(\kappa_1)\right| + s^\rho \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right|\right)(ds)^\rho (dw)^\rho$$
$$+ \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \int_0^1 |G(\kappa_1 + w\Im(\kappa_2,\kappa_1))| \frac{1}{\Gamma(1+\rho)}$$
$$\times \int_0^w \left((1-s)^\rho \left|F^{(\rho)}(\kappa_1)\right| + s^\rho \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right|\right)(ds)^\rho (dw)^\rho .$$
$$= \frac{(\Im(\kappa_2,\kappa_1))^{2\rho}}{2\Gamma(1+\rho)} \int_0^1 |G(\kappa_1 + w\Im(\kappa_2,\kappa_1))| \frac{1}{\Gamma(1+\rho)}$$
$$\times \int_0^1 \left((1-s)^\rho \left|F^{(\rho)}(\kappa_1)\right| + s^\rho \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right|\right)(ds)^\rho (dw)^\rho$$

After some straightforward calculation, we obtain

$$\left|\left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2,\kappa_1))}{2}\right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) \right.$$
$$\left. - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} (FG)(\upsilon)\right|$$
$$\le \frac{\Gamma(1+\rho)}{2\Gamma(1+2\rho)} (\Im(\kappa_2,\kappa_1))^{2\rho} \left(\left|F^{(\rho)}(\kappa_1)\right| + \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right|\right)$$
$$\times \frac{1}{\Gamma(1+\rho)} \int_0^1 |G(\kappa_1 + w\Im(\kappa_2,\kappa_1))| (dw)^\rho .$$

By substituting $v = \kappa_1 + w\zeta(\kappa_2,\kappa_1)$ and using the assumption that $|G|_\infty = \sup_{s\in[\kappa_1,\kappa_2]} |G(x)|$

$$\left|\left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2,\kappa_1))}{2}\right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} G(\upsilon) \right.$$
$$\left. - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))} (FG)(\upsilon)\right|$$
$$\le \frac{\Gamma(1+\rho)}{2\Gamma(1+2\rho)} (\Im(\kappa_2,\kappa_1))^\rho |G|_\infty$$
$$\times \left(\left|F^{(\rho)}(\kappa_1)\right| + \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right|\right) \frac{1}{\Gamma(1+\rho)} \int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} (d\upsilon)^\rho$$
$$= \frac{|G|_\infty \Gamma(1+\rho)}{2\Gamma(1+2\rho)} \frac{(\Im(\kappa_2,\kappa_1))^\rho}{\Gamma(1+\rho)} \left(\left|F^{(\rho)}(\kappa_1)\right| + \left|F^{(\rho)}(\kappa_1 + \Im(\kappa_2,\kappa_1))\right|\right)$$
$$\times \left[(\kappa_1 + \Im(\kappa_2,\kappa_1))^\rho - (\kappa_1)^\rho\right],$$

which is as required. □

Corollary 2.7. *If we take $\Im(\kappa_2,\kappa_1) = \kappa_2 - \kappa_1$, then from (2.19), we have*

$$\left|\left[\frac{F(\kappa_1) + F(\kappa_2)}{2}\right] {}_{\kappa_1}J^{(\rho)}_{\kappa_2} G(\upsilon) - {}_{\kappa_1}J^{(\rho)}_{\kappa_2} (FG)(\upsilon)\right| \tag{2.22}$$

$$\leq \frac{1}{2\Gamma(1+2\rho)} (\kappa_2 - \kappa_1)^{2\rho} |G|_\infty \left(\left| F^{(\rho)}(\kappa_1) \right| + \left| F^{(\rho)}(\kappa_2) \right| \right).$$

Corollary 2.8. *For $\rho = 1$ and $|G|_\infty = 1$, we get the similar inequality [See [3, Theorem 2.2]]*

$$\left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2, \kappa_1))}{2} \right] - \frac{1}{\Im(\kappa_2, \kappa_1)} \int_{\kappa_1}^{\kappa_1 + \Im(\kappa_2,\kappa_1)} F(\upsilon) \tag{2.23}$$

$$\leq \frac{1}{4} (\Im(\kappa_2, \kappa_1)) \left(\left| F'(\kappa_1) \right| + \left| F'(\kappa_1 + \Im(\kappa_2, \kappa_1)) \right| \right).$$

3. Examples

Example 3.1. Let $F(\upsilon) = -|\upsilon|$ is a preinvex function with respect to $\Im$, where

$$\Im(\kappa_1, \kappa_2) = \begin{cases} \kappa_1 - \kappa_2, & \kappa_1, \kappa_2 \leq 0 \text{ or } \kappa_1, \kappa_2 \geq 0, \\ \kappa_2 - \kappa_1, & \text{otherwise,} \end{cases}$$

If we take $G(\upsilon) = 1$, $\kappa_1 = 1$, $\kappa_2 = 3$ then $\Im(\kappa_2, \kappa_1) = -2$, then the left-handed side of (2.12) is:

$$\left| F\left(\kappa_1 + \frac{1}{2} \Im(\kappa_2, \kappa_1) \right) {}_{\kappa_1}J^{(\rho)}_{(\kappa_1 + \Im(\kappa_2,\kappa_1))} G(\upsilon) - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1 + \Im(\kappa_2,\kappa_1))} (FG)(\upsilon) \right|$$
$$= 1$$

The right-handed side of (2.12) is:

$$\frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)} (\Im(\kappa_2, \kappa_1))^{2\rho} |G|_\infty \left[1 - 2\left(\frac{1}{2} \right)^{3\rho} \right]$$
$$\times \left(\left| F^{(\rho)}(\kappa_1) \right| + \left| F^{(\rho)}(\kappa_1 + \Im(\kappa_2, \kappa_1)) \right| \right)$$
$$= 2$$

Thus, the assumption for (2.3) is satisfied.

Example 3.2. Let $F(\upsilon) = -|\upsilon|$, If we take $G(\upsilon) = 1$, $\kappa_1 = -3$, $\kappa_2 = 1$, and $\Im(\kappa_2, \kappa_1) = 4$, then the left-hand side of (2.19) is:

$$\left| \left[\frac{F(\kappa_1) + F(\kappa_1 + \Im(\kappa_2, \kappa_1))}{2} \right] {}_{\kappa_1}J^{(\rho)}_{(\kappa_1 + \Im(\kappa_2,\kappa_1)} G(\upsilon) \right.$$
$$\left. - {}_{\kappa_1}J^{(\rho)}_{(\kappa_1 + \Im(\kappa_2,\kappa_1)} (FG)(\upsilon) \right|$$
$$= -3.$$

The right-hand side of(2.19) is:

$$\frac{1}{2\Gamma(1+2\rho)} (\Im(\kappa_2, \kappa_1))^{\rho} |G|_\infty$$

$$
\begin{aligned}
&\times\left(\left|F^{(\rho)}\left(\kappa_1\right)\right|+\left|F^{(\rho)}\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)\right|\right)\\
&=-2.
\end{aligned}
$$

Thus, the assumption for (2.6) is satisfied.

4. Applications

4.1. **Random Variables.** There is generalized probability distribution function $G:\left[\kappa_1,\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right]\to\mathbb{R}^\rho$ for a random variable X where $s\in\left[\kappa_1,\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right]$ and it has lower and upper bounds, then

$$
\begin{aligned}
E^\rho\left(\upsilon\right)&=\frac{1}{\Gamma\left(\rho+1\right)}\int_{\kappa_1}^{\kappa_1+\Im\left(\kappa_2,\kappa_1\right)}\upsilon^\rho G\left(\upsilon\right)\left(d\upsilon\right)^\rho,\\
E_\tau^\rho\left(\upsilon\right)&=\frac{1}{\Gamma\left(\rho+1\right)}\int_{\kappa_1}^{\kappa_1+\Im\left(\kappa_2,\kappa_1\right)}\upsilon^{\rho\tau} G\left(\upsilon\right)\left(d\upsilon\right)^\rho,
\end{aligned}
$$

where $E^\rho\left(\upsilon\right)$ is the generalized expectation and $E_\tau^\rho\left(\upsilon\right)$ is the τ moment.

Proposition 4.1. *If we take* $F\left(\upsilon\right)=\upsilon^{\tau\rho,}$ *for* $\tau\geq 2$, *the function* $\left|F^\rho\left(\upsilon\right)\right|=\frac{\Gamma(1+\tau\rho)}{\Gamma(1+(\tau-1)\rho)}\upsilon^{(\tau-1)\rho}$ *and we have from* (2.12),

$$
\begin{aligned}
&\left|\left(\kappa_1+\frac{1}{2}\Im\left(\kappa_2,\kappa_1\right)\right)^{\tau\rho}\ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1))}G(\upsilon)-E_\tau^\rho\left(\upsilon\right)\right|\\
&\quad\leq\frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)}\frac{\Gamma(1+\tau\rho)}{\Gamma(1+(\tau-1)\rho)}\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{2\rho}\left|G\right|_\infty\\
&\qquad\times\left[1-2\left(\frac{1}{2}\right)^{3\rho}\right]\left[\kappa_1+\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)^{(\tau-1)\rho}\right].
\end{aligned}
$$

Proposition 4.2. *If we take* $F\left(\upsilon\right)=\upsilon^{\tau\rho,}$ *for* $\tau\geq 2$, *the function* $\left|F^\rho\left(\upsilon\right)\right|=\frac{\Gamma(1+\tau\rho)}{\Gamma(1+(\tau-1)\rho)}\upsilon^{(\tau-1)\rho}$ *and we have from* (2.19),

$$
\begin{aligned}
&\left|\left[\frac{\kappa_1^{\tau\rho}+\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)^{\tau\rho}}{2}\right]\ {}_{\kappa_1}J^{(\rho)}_{(\kappa_1+\Im(\kappa_2,\kappa_1)}G(\upsilon)-\quad E_\tau^\rho\left(\upsilon\right)\right|\\
&\quad\leq\frac{1}{2\Gamma(1+2\rho)}\left(\Im\left(\kappa_2,\kappa_1\right)\right)^{\rho}\frac{\Gamma(1+\tau\rho)}{\Gamma(1+(\tau-1)\rho)}\left|G\right|_\infty\\
&\qquad\times\left[\kappa_1^{(\tau-1)\rho}+\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)^{(\tau-1)\rho}\right]\left[\left(\kappa_1+\Im\left(\kappa_2,\kappa_1\right)\right)^\rho-\left(\kappa_1\right)^\rho\right].
\end{aligned}
$$

4.2. **Numerical Integration.** If $X_n:a=\kappa_1<a_1<\ldots<a_n=\kappa_2$ be a partition of the interval $\left[\kappa_1,\kappa_2\right]$, $\varepsilon_i\in\left[a_i,a_{i+1}\right]$, where $i=0,1,2,\ldots,n-1$. Then the midpoint quadrature rule is denoted $R_M\left(F,G,\varepsilon\right)$ and defined as:

$$
\frac{1}{\Gamma\left(\rho+1\right)}\int_{\kappa_1}^{\kappa_1+\Im\left(\kappa_2,\kappa_1\right)}F\left(s\right)G\left(s\right)\left(ds\right)^\rho=M\left(F,G,\varepsilon\right)+R_M\left(F,G,\varepsilon\right)
$$

and the trapezoidal quadrature rule is denoted by $R_T(F,G,\varepsilon)$ and defined as:

$$\frac{1}{\Gamma(\rho+1)}\int_{\kappa_1}^{\kappa_1+\Im(\kappa_2,\kappa_1)} F(s)\,G(s)\,(ds)^\rho = T(F,G,\varepsilon)+R_T(F,G,\varepsilon)$$

where

$$M(F,G,\varepsilon):=\frac{1}{\Gamma(\rho+1)}\sum_{i=0}^{n-1} F\left(\frac{\kappa_{1i}+\kappa_{1i+1}}{2}\right)\int_{\kappa_{1i}}^{\kappa_{1i+1}} G(s)\,(ds)^\rho,$$

$$T(F,G,\varepsilon):=\frac{1}{\Gamma(\rho+1)}\sum_{i=0}^{n-1}\frac{F(\kappa_{1i})+F(\kappa_{1i+1})}{2}\int_{\kappa_{1i}}^{\kappa_{1i+1}} G(s)\,(ds)^\rho,$$

Now, we have the following results.

Proposition 4.3. *Suppose all the assumptions of Theorem 2.3 are satisfied. Then the following weighted right-hand side of Hermite-Hadamard error estimate satisfies the inequality:*

$$\begin{aligned}|R_M(F,G,\varepsilon)| &\le \sum_{i=0}^{n-1}\frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)}(a_{i+1}-a_i)^{2\rho}|G|_\infty \\ &\quad\times\left[1-2\left(\frac{1}{2}\right)^{3\rho}\right]\left(\left|F^{(\rho)}(a_i)\right|+\left|F^{(\rho)}(a_{i+1})\right|\right).\end{aligned}$$

Proposition 4.4. *Suppose all the assumptions of Theorem 2.6 are satisfied. Then the following weighted left-hand side of Hermite-Hadamard error estimate satisfies the inequality:*

$$\begin{aligned}&|R_T(F,G,\varepsilon)| \\ &\quad\le \sum_{i=0}^{n-1}\frac{1}{2\Gamma(1+2\rho)}(a_{i+1}-a_i)^{2\rho}|G|_\infty\left(\left|F^{(\rho)}(a_i)\right|+\left|F^{(\rho)}(a_{i+1})\right|\right).\end{aligned}$$

4.3. **Special means.** We give the following special means for positive numbers κ_1, κ_2 where $\kappa_1<\kappa_2$.

(i) Arithmetic mean is denoted by $A(\kappa_1,\kappa_2)$ and it is defined as:

$$A(\kappa_1,\kappa_2):=\frac{\kappa_1+\kappa_2}{2}.$$

(ii) Harmonic mean is denoted by $H(\kappa_1,\kappa_2)$ and it is defined as:

$$H(\kappa_1,\kappa_2):=\frac{2}{\frac{1}{\kappa_1}+\frac{1}{\kappa_2}}.$$

(iii) Logarithmic mean is denoted by $L(\kappa_1,\kappa_2)$ and it is defined as:

$$L(\kappa_1,\kappa_2):=\frac{\kappa_2-\kappa_1}{\ln\kappa_2-\ln\kappa_1},\quad \kappa_1\neq\kappa_2.$$

(iv) Logarithmic mean is denoted by $L_n(\kappa_1, \kappa_2)$ and it is defined as:

$$L_n(\kappa_1, \kappa_2) := \left(\frac{\kappa_2 - \kappa_1}{(n+1)(\kappa_2 - \kappa_1)}\right)^{\frac{1}{n}}.$$

Proposition 4.5. *For $\kappa_1,\ \kappa_2 \in \mathbb{R}_+$ where $\kappa_1 < \kappa_2$ and $m \in \mathbb{N}$, $m > 1$, then the following inequality holds:*

$$A(\kappa_1^m,\ \kappa_2^m) - 2\frac{A(-\kappa_1^{m+1},\ \kappa_2^{m+1})}{(\kappa_2 - \kappa_1)(m+2)} \leq \frac{(m+1)(\kappa_2 - \kappa_1)}{2} A(-\kappa_1^m,\ \kappa_2^m).$$

Proof. For $F(\upsilon) = \upsilon^{m+1}$ and $|F'(\upsilon)| = (m+1)\upsilon^m$ is a convex function on $\mathbb{R}_+$ where $\upsilon \in \mathbb{R}_+$ and $m \in \mathbb{N}$, $m \geq 2$, $\Im(\kappa_2, \kappa_1) = \kappa_2 - \kappa_1$. From (2.23), we get the required inequality. □

5. Conclusion

We have established some new left and right-hand sides of Fejér type Hermite-Hadamard inequalities for preinvex functions on fractal sets. Our results have also been given special consideration.

References

1. G. Anastassiou, A. Kashuri and R. Liko, *Local fractional integrals involving generalized strongly m-convex mappings*, Arab J. Math., 8 (2019), pp. 95-107.
2. T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal., 60 (2005), pp. 1473-1484.
3. A. Barani, A.G. ghazanfari and S.S Dragomir, *Hermite Hadamard inequality for functions whose derivatives absolutes values are preinvex* . J. Inequal. Appl., 2012 (2012), pp. 247.
4. S.I. Butt, M. Nadeem and G. Farid, *On Caputo fractional derivatives via exponential s-convex functions*, Turkish Journal of Science, 5(2)020), pp. 140-146.
5. T. Du, T. Wang, M.A. Adil and Y. Zhang, *Certain integral inequalities considering generalized m-convexity on fractal sets and their applications*, World Scientific Publishing Co., 27(7) (2019), pp. 17.
6. A. Ekinci and M.E. Ozdemir, *Some new integral inequalities via Riemann-Liouville integral operators*, Appl. Comput. Math., 18(3) (2019).
7. S. Erden, M.Z. Sarikaya and N. Çelik, *Some generalized inequalities involving local fractional integrals and their applications for random variables and numerical integration*, J. Appl. Math. Stat. Inform., 12(2) (2016) pp. 49-65.

8. L. Fejér, *Uber die Fourierreihen*, II. Math. Naturwiss Anz. Ungar. Akad. Wiss., 24 (1906) pp. 369-390.
9. H. Kalsoom, M.A. Ali, M. Abbas, H. Budak and G. Murtaza, *Generalized quantum Montgomery identity and Ostrowski type inequalities for preinvex functions*, TWMS J. Pure Appl. Math., 13(1) (2022) pp. 72-90.
10. Ş. Kızıl and M.A. Ardıç, *Inequalities for strongly convex functions via Atangana-Baleanu Integral Operators*, Turkish Journal of Science, 6(2) (2021) pp. 96-109.
11. C. Luo, H. Wang and T. Du, *Fejér-Hermite-Hadamard type inequalities involving generalized h-convexity on fractal sets and their applications*, Chaos Solitons Fractals, 131 (2020) pp. 13.
12. H. Mo, X. Sui and D. Yu, *Generalized convex functions on fractal sets and two related inequalities*, Abstr. Appl. Anal., 2014 (2020) p. 7.
13. W. Sun, *Some local fractional integral inequalities for generalized preinvex functions and applications to numerical quadrature*, World Scientific Publishing Co., 27(5) (2019) pp. 14.
14. W. Sun, *Generalized preinvex functions and related Hermite-Hadamard type integral inequalities on fractal space*, J. Zhejiang Univ., 46(5) (2019) pp. 543-549.
15. W. Sun and L. Qiong, *New inequalities of Hermite Hadamard type for generalized convex functions on fractal sets and its applications*, J. Zhejiang Univ., 44(1) (2017) pp. 47-52.
16. T. Weir and B. Mond, *Preinvex functions in multiple objective optimization*, J. Math. Anal. Appl., 136(1) (1998) pp. 29-38.
17. X.J. Yang, *Advanced local fractional calculus and its applications*, WorldScience, New York, 2012.
18. U.S. Kırmacı, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., 147(1) (2004) pp. 137-146.

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