Sitaru-Schweitzer Type Inequality for Fuzzy and Pseudo-Integrals

Bayaz Daraby, Mortaza Tahmourasi and Asghar Rahimi

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 1
Pages: 31-45

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.2003349.1344

Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 1 (2024), 31-45
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.2003349.1344

Sitaru-Schweitzer Type Inequality for Fuzzy and Pseudo-Integrals

Bayaz Daraby[1]*, Mortaza Tahmourasi[2] and Asghar Rahimi[3]

Abstract. In this paper, we have proved and stated the Sitaru-Schweitzer type inequality for fuzzy integrals and also we state this inequality for pseudo-integrals in two classes. The first one is for pseudo-integrals where pseudo-addition and pseudo-multiplication are constructed by a monotone continuous function $g : [0,\infty] \to [0,\infty]$. Another one is given by the semiring $([a,b], \max, \odot)$ where an increasing function generates pseudo-multiplication.

1. Introduction

Fuzzy measure and fuzzy integral (Sugeno integral), which were initially introduced by Sugeno in 1974 [24], are essential analytical methods of measuring uncertain information [12]. Several papers discussed the study of inequalities for the fuzzy integral, initiated by Román-Flores-Flores et al., was discussed in several papers. Recently, the fuzzy integral counterparts of several classical inequalities, including Chebyshev's, Markov's and Hardy's inequalities, were given by Flores-Franulič and Román-Flores (see [13], [14], [22]). Also, many researchers have investigated fuzzy integral inequalities and generalized some of those, such as H. Agahi et al. in [1, 2] and D. Zhang and E. Pap (see [25]).

The concept of pseudo-analysis is derived from classical analysis, which is one of the most widely used and interesting generalizations of classical analysis, which is based on the structure of semirings on the real interval $[a,b] \subseteq [-\infty,+\infty]$ with pseudo-addition and pseudo-multiplication operators (see [17–19]).

2020 *Mathematics Subject Classification.* 03E72, 26E50, 28E10.
Key words and phrases. Fuzzy integrals, Schweitzer type inequality, Fuzzy integral inequality, Pseudo-integrals.
Received: 27 May 2023, Accepted: 17 July 2023.
* Corresponding author.

One of the advantages of pseudo-analysis is its broader scope, which can include nonlinear and indeterminate problems from different branches, as well as the use of mathematical tools in various fields. Based on the semiring structure in pseudo-analysis, the concepts of pseudo-measure and pseudo-integral have been developedand accordingly, many classical integral inequalities relative to pseudo-sum have been extended. Daraby et al. have popularized some fuzzy integral inequalities for the Sugeno integrals and pseudo-integrals in [3–11, 15]. In the classical mathematical analysis, the Sitaru-Schweitzer type inequality is as follows:

Theorem 1.1 ([23])**.** *If $f : [a, b] \to [l, L]$, with $l > 0$ is an integrable function such that $\frac{1}{f}$ is also integrable, then*

$$\int_a^b f dx \cdot \int_a^b \frac{1}{f} dx \leq \frac{(l+L)^2}{4lL}(b-a)^2,$$

holds.

In this paper, we have also organized the article as the following: In Section 2, we have described the definitions, properties and results of fuzzy measure, fuzzy integrals and pseudo-integrals. In Section 3, we have stated and proved the Sitaru-Schweitzer type inequality for the fuzzy integrals. In Section 4, we have stated and proved the Sitaru-Schweitzer type inequality for pseudo-integrals. In Section 5, we deal with it through further discussionsand finally, this paper has finished with a short conclusion.

2. Preliminaries

In this section, we provide some definitions and concepts for the next sections.

2.1. **Sugeno integrals.** We denote by $\mathbb{R}$, the set of all real numbers. Let X be a non-empty set and Σ be a $\sigma-$algebra of subsets of X. Throughout this paper, all considered subsets are supposed to be in Σ.

Definition 2.1 ([21])**.** A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$,

(FM2) $A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B)$,

(FM3) $A_1 \subseteq A_2 \subseteq \dots \quad \Rightarrow \quad \lim \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$,

(FM4) $A_1 \supseteq A_2 \supseteq \dots$ and $\mu(A_1) < \infty \quad \Rightarrow \quad \lim \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$.

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

For the non-negative real-valued function f on X, we will denote

$$F_\alpha = L_\alpha f = \{x \in X \mid f(x) \geq \alpha\} = \{f \geq \alpha\},$$

the α-level of f, for $\alpha > 0$. The support of f is $L_0 f = \overline{\{x \in X \mid f(x) > 0\}} = \operatorname{supp}(f)$. We know that:

$$\alpha \leq \beta \quad \Rightarrow \quad \{f \geq \beta\} \subseteq \{f \geq \alpha\}.$$

If μ is a fuzzy measure on (X, Σ), assume

$$\mathcal{F}^\mu(X) = \{f : X \to [0, \infty) | f \text{ is } \mu - \text{measurable}\}.$$

Definition 2.2 ([24])**.** Let μ be a fuzzy measure on (X, Σ). If $f \in \mathcal{F}^\mu(X)$ and $A \in \Sigma$, then the fuzzy integral of f on A, with respect to the fuzzy measure μ, is defined as

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)].$$

Where $\vee$ and $\wedge$ denote the operations *sup* and *inf* on $[0, \infty]$, respectively. In particular, if $A = X$ then

$$⨍_X f d\mu = ⨍ f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)].$$

The following properties of the fuzzy integral can be found in [24].

Proposition 2.3 ([24])**.** *Let (X, Σ, μ) be a fuzzy measure space and $A, B \in \sum$ and $f, g \in \mathcal{F}^\mu(X)$. We have*

(i) $⨍_A f d\mu \leq \mu(A)$.
(ii) $⨍_A k d\mu = k \wedge \mu(A)$, *for non-negative constant* k.
(iii) *If* $A \subseteq B$, *then* $⨍_A f d\mu \leq ⨍_B f d\mu$.
(iv) *If* $f \leq g$ *on* A, *then* $⨍_A f(x) d\mu \leq ⨍_A g(x) d\mu$.
(v) $⨍_{A \cup B} f d\mu \geq ⨍_A f d\mu \vee ⨍_B f d\mu$.
(vi) *If* $\mu(A) < \infty$, *then* $⨍_A f d\mu \geq \alpha \quad \Leftrightarrow \quad \mu(A \cap \{f \geq \alpha\}) \geq \alpha$.

2.2. **Pseudo integrals.** Let $[a, b]$ be a closed or semiclosed subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$. In [17], the operations $\oplus$ and $\odot$ are defined. Those operations are named pseudo-addition and pseudo-multiplication, respectively. The operation $\oplus$ is a commutative, non-decreasing function (with respect to $\preceq$), associative and with a zero (neutral) element indicated by $\mathbf{0}$. The operation $\odot$ is a commutative, positively non- decreasing function, associative and for each $x \in [a, b]$,

$\mathbf{1} \odot x = x$. Also, we assume $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$.

Case I: The pseudo-addition is an idempotent operation and the pseudo-multiplication is not.

Case II: The pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}\big(g(x) + g(y)\big)$ and $x \odot y = g^{-1}\big(g(x)g(y)\big)$.

Case III: Both operations are idempotent. For example $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$ on the interval $[a, b]$.

In the sequel, we consider the semiring $([a, b], \oplus, \odot)$ for two significant cases. The first case is when pseudo-operations are produced by a monotone and continuous function such as $g : [a, b] \to [0, \infty)$. Therefore, the pseudo-integral for a function $f : [0, 1] \to [a, b]$ scales down the $g-$integral

$$\int_{[0,1]}^{\oplus} f(x)dx = g^{-1}\left(\int_0^1 g(f(x))dx\right). \tag{2.1}$$

The second class is when $x \oplus y = \sup(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ be given as follows:

$$\int_{\mathbb{R}}^{\sup} f \odot dm = \sup_{x\in\mathbb{R}} \left(f(x) \odot \psi(x)\right),$$

where the function $\psi : \mathbb{R} \to [a, b]$ defines a sup-measure m by $m(A) = \sup_{x\in A} \psi(x)$.

Theorem 2.4 ([16]). *Let m be a* sup*-measure on $([0, \infty], \mathbb{B}[0, \infty])$, where $\mathbb{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = \text{ess}\ \sup_{\mu}(\psi(x)|x \in A)$ and $\psi : [0, \infty] \to [0, \infty]$ is a continuous function. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0, \infty], \mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power $\lambda, \lambda \in (0, \infty)$) such that $\lim_{\lambda\to\infty} m_\lambda = m$.*

Theorem 2.5 ([16]). *Let $([0, \infty], \sup, \odot)$ be a semiring , when $\odot$ is a generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0, \infty)$. Let m be the same as in Theorem 2.4, Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$,*

$$\begin{aligned} \int^{\sup} f \odot dm &= \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda \\ &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int g^\lambda(f(x))dx\right). \end{aligned} \tag{2.2}$$

Theorem 2.6 ([20])**.** *Let $\varphi : [a,b] \to [a,b]$ be a convex and non-increasing function. If a generator $g : [a,b] \to [a,b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then for any measurable function $f : [a,b] \to [a,b]$, we have*

$$\varphi\left(\int_{[0,1]}^{\oplus} f(x)dx\right) \leq \int_{[0,1]}^{\oplus} \varphi(f(x))dx. \tag{2.3}$$

Note that, if φ and g ane non-decreasing functions, then the reverse of (2.3) *holds.*

Theorem 2.7. *[20] Let $\varphi : [a,b] \to [a,b]$ be a convex and non-increasing function and the pseudo-multiplication $\odot$ is represented by a convex and increasing generator g. Let m be the same as in Theorem 2.5. Then for any continuous function $f : [0,1] \to [a,b]$ we have*

$$\varphi\left(\int_{[0,1]}^{\sup} f \odot dm\right) \leq \int_{[0,1]}^{\sup} \varphi(f)dm. \tag{2.4}$$

3. Sitaru-Schweitzer Type Inequality for Fuzzy Integrals

In this section, we investigate the Sitaru-Schweitzer's inequality for fuzzy integrals.

Lemma 3.1. *Let $L > 0$ and $l \leq L$, then $\dfrac{(l+L)^2}{4lL} \geq 1$.*

Proof. If in the contrary, we suppose $\dfrac{(l+L)^2}{4lL} < 1$. Therefore

$$\begin{aligned}
&(l+L)^2 < 4lL\\
&l^2 + 2mM + L^2 - 4lL < 0\\
&l^2 - 2mM + L^2 < 0\\
&(l-L)^2 < 0.
\end{aligned}$$

Which is a contradiction. Hence, the proof is complete. □

Theorem 3.2. *Let $f : [a,b] \to [l,L]$ and $\dfrac{1}{f}$ be a fuzzy measurable functions. If $l > 0$ and μ is a regular fuzzy measure, then the inequality*

$$⨍_a^b f d\mu \cdot ⨍_a^b \frac{1}{f} d\mu \leq \frac{(l+L)^2}{4lL}(b-a)^2 \tag{3.1}$$

holds.

Proof. From part (i) of Proposition 2.3, we have

$$\fint_a^b f(x)d\mu \leq \mu[a,b] = b-a, \qquad \fint_a^b \frac{1}{f(x)}d\mu \leq \mu[a,b] = b-a.$$

Therefore

$$\fint_a^b f d\mu \cdot \fint_a^b \frac{1}{f}d\mu \leq (b-a)\cdot(b-a) = (b-a)^2. \tag{3.2}$$

Applying Lemma 3.1 and Inequality (3.2), we have

$$\fint_a^b f d\mu \cdot \fint_a^b \frac{1}{f}d\mu \leq \frac{(l+L)^2}{4lL}(b-a)^2.$$

Proof is now complete. □

In the following by some examples, we illustrate the validity of Theorem 3.2.

Example 3.3. We define $f : [0,1] \to [1,2]$, by $f(x) = 1 + x$ and assume that μ is a Lebesgue measure. Then by some simple calculations, we have

$$\begin{aligned}
\fint_0^1 f(x)d\mu &= \fint_0^1 (1+x)d\mu \\
&= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([0,1] \cap \{x :\ x+1 \geq \alpha\}\right)\right] \\
&= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([0,1] \cap \{x :\ x \geq \alpha - 1\}\right)\right] \\
&= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([0,1] \cap [\alpha-1, 1]\right)\right] \\
&= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([\alpha-1, 1]\right)\right] \\
&= \sup_{\alpha\in[0,1]} \left[\alpha \wedge (1-(\alpha-1)\right] \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\fint_0^1 \frac{1}{f(x)}d\mu &= \fint_0^1 \frac{1}{1+x}d\mu \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left([0,1] \cap \left\{x :\ \frac{1}{1+x} \geq \alpha\right\}\right)\right) \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left([0,1] \cap \left\{x :\ x \leq \frac{1-\alpha}{\alpha}\right\}\right)\right) \\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left([0,1] \cap \left[0, \frac{1-\alpha}{\alpha}\right]\right)\right)
\end{aligned}$$

$$= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left(\left[0, \frac{1-\alpha}{\alpha}\right]\right)\right)$$
$$= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \frac{1-\alpha}{\alpha}\right)$$
$$= 0.6180,$$

also

$$\frac{(l+L)^2}{4lL}(b-a)^2 = \frac{(1+2)^2}{4(1)(2)}(1)^2 = \frac{9}{8}.$$

Finally, $0.6180 \leq 1.125$.

Example 3.4. Let $f : [0,1] \to \left[\frac{1}{2}, \frac{3}{2}\right]$ and $f(x) = \frac{3}{2} - x$ and μ be a Lebesgue measure. Then with simple calculation we have

$$\fint_0^1 f(x)d\mu = \fint_0^1 \left(\frac{3}{2} - x\right) d\mu$$
$$= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([0,1] \cap \left\{x : \ \frac{3}{2} - x \geq \alpha\right\}\right)\right]$$
$$= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([0,1] \cap \left\{x : \ x \leq \frac{3}{2} - \alpha\right\}\right)\right]$$
$$= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left([0,1] \cap \left[0, \frac{3}{2} - \alpha\right]\right)\right]$$
$$= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \mu\left(\left[0, \frac{3}{2} - \alpha\right]\right)\right]$$
$$= \sup_{\alpha\in[0,1]} \left[\alpha \wedge \left(\frac{3}{2} - \alpha\right)\right]$$
$$= \frac{3}{4},$$

and

$$\fint_0^1 \frac{1}{f(x)}d\mu = \fint_0^1 \frac{1}{\frac{3}{2} - x}d\mu$$
$$= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left([0,1] \cap \left\{x : \ \frac{1}{\frac{3}{2} - x} \geq \alpha\right\}\right)\right)$$
$$= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu\left([0,1] \cap \left\{x : \ x \geq \frac{\frac{3}{2}\alpha - 1}{\alpha}\right\}\right)\right)$$

$$
\begin{aligned}
&= \sup_{\alpha\in[0,1]}\left(\alpha\wedge\mu\left([0,1]\cap\left[\frac{\frac{3}{2}\alpha-1}{\alpha},1\right]\right)\right)\\
&= \sup_{\alpha\in[0,1]}\left(\alpha\wedge\mu\left(\left[\frac{\frac{3}{2}\alpha-1}{\alpha},1\right]\right)\right)\\
&= \sup_{\alpha\in[0,1]}\left(\alpha\wedge\left(1-\frac{\frac{3}{2}\alpha-1}{\alpha}\right)\right)\\
&= 0.7807,
\end{aligned}
$$

also

$$\frac{(l+L)^2}{4lL}(b-a)^2=\frac{4}{3}\times 1=1.333,$$

finally,

$$0.5855\leq\frac{4}{3}=1.333.$$

4. Sitaru-Schweitzer Type Inequality for Pseudo-Integrals

In this section, we prove the Sitaru-Schweitzer's inequality for pseudo integrals.

Theorem 4.1. *Let $f:[a,b]\to[l_0,L_0]$ be a measurable function, $l_0>0$ and $g:[l_0,L_0]\to[l,L]$ be a continuous function. Then*

$$\int_{[a,b]}^{\oplus} f(x)dx\odot\int_{[a,b]}^{\oplus}(\mathbf{1}\oslash f(x))\,dx\leq g^{-1}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right). \tag{4.1}$$

Proof. Based on pseudo-integrals definition, we have

$$
\begin{aligned}
&\int_{[a,b]}^{\oplus} f(x)dx\odot\int_{[a,b]}^{\oplus}(\mathbf{1}\oslash f(x))\,dx\\
&\quad= g^{-1}\left(g\int_{[a,b]}^{\oplus} f(x)dx\cdot g\int_{[a,b]}^{\oplus}\mathbf{1}\oslash f(x)dx\right)\\
&\quad= g^{-1}\left[g\left(g^{-1}\int_a^b g(f(x))dx\right)\cdot g\left(g^{-1}\int_a^b g\left(\mathbf{1}\oslash f(x)\right)dx\right)\right]\\
&\quad= g^{-1}\left[g\left(g^{-1}\int_a^b g(f(x))dx\right)\cdot g\left(g^{-1}\int_a^b g\left(g^{-1}\left(\frac{g(\mathbf{1})}{g(f(x))}\right)\right)dx\right)\right]\\
&\quad= g^{-1}\left[g(\mathbf{1})\cdot\int_a^b g(f(x))dx\cdot\int_a^b\frac{1}{g(f(x))}dx\right].
\end{aligned}
$$

Now, by using the classical form, we have

$$\int_{[a,b]}^{\oplus} f(x)dx \odot \int_{[a,b]}^{\oplus} (\mathbf{1} \oslash f(x))\, dx \leq g^{-1}\left[g(\mathbf{1}) \cdot \left(\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right)\right]$$
$$= \mathbf{1} \odot g^{-1}\left[\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right],$$

in continue, from $\mathbf{1} \odot x = x$, we have

$$\int_{[a,b]}^{\oplus} f(x)dx \odot \int_{[a,b]}^{\oplus} (\mathbf{1} \oslash f(x))\, dx = g^{-1}\left[\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right].$$

Proof is now complete. □

Example 4.2. Define the functions $f : [0,1] \to [1, \sqrt{2}]$ as $f(x) = \sqrt{x+1}$ and $g : [1, \sqrt{2}] \to [1, 2]$ as $g(x) = x^2$. With simple calculation, we have

$$\begin{aligned}\int_{[0,1]}^{\oplus} f(x)dx &= g^{-1} \int_0^1 g(f(x))dx \\ &= g^{-1} \int_0^1 (x+1)dx \\ &= g^{-1}\left(\frac{3}{2}\right) = \sqrt{\frac{3}{2}},\end{aligned}$$

and

$$\begin{aligned}\int_{[0,1]}^{\oplus} (\mathbf{1} \oslash f(x))\, dx &= g^{-1} \int_0^1 g\left(\mathbf{1} \oslash g(x)\right) dx \\ &= g^{-1} \int_0^1 g g^{-1}\left(\frac{g(\mathbf{1})}{g(f(x))}\right) dx \\ &= g^{-1} \int_0^1 \left(\frac{1}{x+1}\right) dx \\ &= \sqrt{0.69315}.\end{aligned}$$

And for the right side of inequality, we have

$$g^{-1}\left(\frac{(l+L)^2}{4lL} \cdot (1-0)^2\right) = g^{-1}\left(\frac{5.82}{5.46}\right) = \sqrt{\frac{5.82}{5.46}}.$$

By replacing in inequality, we obtain

$$\sqrt{\frac{3}{2}} \odot 0.8325 \leq \sqrt{\frac{5.82}{5.46}}.$$

Consequently,

$$1.0196 \leq 1.0324.$$

In the sequel, we generalize the Sitaru-Schweitzer type inequality by the semiring $([a,b], \max, \odot)$, where $\odot$ is generated.

Theorem 4.3. *Let $f : [a,b] \to [l_0, L_0]$ be a measurable function, $l_0 > 0$ and $g : [l_0, L_0] \to [l, L]$ be a continuous function. Let m be the same as in Theorem 2.4. Then*

$$\int_{[a,b]}^{\sup} f(x) \odot dm \odot \int_{[a,b]}^{\sup} (\mathbf{1} \oslash f(x)) \odot dm \leq g^{-\lambda}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right). \tag{4.2}$$

Proof. Based on definition, we have

$$\begin{aligned}
&\int_{[a,b]}^{\sup} f(x) \odot dm \odot \int_{[a,b]}^{\sup} (\mathbf{1} \oslash f(x)) \odot dm \\
&= \left(\lim_{\lambda\to\infty} \int_{[a,b]}^{\oplus_\lambda} f \odot dm_\lambda\right) \odot \left(\lim_{\lambda\to\infty} \int_{[a,b]}^{\oplus_\lambda} (1 \oslash f) \odot dm_\lambda\right) \\
&= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_a^b g^\lambda(f(x))dx \odot \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_a^b g^\lambda(1 \oslash f)dx \\
&= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_a^b g^\lambda(f(x))dx \odot \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_a^b g^\lambda \left(g^{-\lambda}(\frac{g^\lambda(1)}{g^\lambda(f)}\right) dx \\
&= g^{-\lambda}\left(g^\lambda \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_a^b g^\lambda(f(x))dx \cdot g^\lambda \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \int_a^b \frac{g^\lambda(1)}{g^\lambda(f(x))}dx\right) \\
&= g^{-\lambda}\left(\lim_{\lambda\to\infty} g^\lambda \left(g^\lambda\right)^{-1} \int_a^b g^\lambda(f(x))dx \cdot \lim_{\lambda\to\infty} g^\lambda \cdot \left(g^\lambda\right)^{-1} \int_a^b \frac{g^\lambda(1)}{g^\lambda(f(x))}dx\right) \\
&= g^{-\lambda}\left(\lim_{\lambda\to\infty} \int_a^b g^\lambda f(x)dx \cdot \lim_{\lambda\to\infty} \int_a^b \frac{g^\lambda(1)}{g^\lambda(f(x))}dx\right) \\
&= \lim_{\lambda\to\infty} \left(g^{-\lambda}\right) \left[\int_a^b g^\lambda(f(x))dx \cdot \int_a^b \frac{g^\lambda(1)}{g^\lambda(f(x))}dx\right) \\
&= \lim_{\lambda\to\infty} \left(g^{-\lambda}\right) \left[g^\lambda(1) \cdot \int_a^b g^\lambda(f(x))dx \cdot \int_a^b \frac{1}{g^\lambda(f(x))}dx\right] \\
&\leq \lim_{\lambda\to\infty} \left(g^{-\lambda}\right) \left[g^\lambda(1) \cdot \left(\frac{(l+L)^2}{4lL}\right) \cdot (b-a)^2\right] \\
&= \left(g^\lambda\right)^{-1} \left[g^\lambda(1) \cdot \frac{(l+L)^2}{4lL} \cdot (b-a)^2\right] \\
&= 1 \odot g^{-\lambda} \left[\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right] \\
&= g^{-\lambda} \left[\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right], \quad (\text{Because } \mathbf{1} \odot x = x).
\end{aligned}$$

Therefore, we have

$$\int_{[a,b]}^{\sup} f(x) \odot dm \odot \int_{[a,b]}^{\sup} (\mathbf{1} \oslash f(x)) \odot dm \leq g^{-\lambda}\left[\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right].$$

Proof is now complete. □

Example 4.4. Let $g^{\lambda}(x) = e^{\lambda x}$ and $\psi(x)$ be the same as Theorem 2.4. Then

$$x \oplus y = \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} + e^{\lambda y}\right) = \max(x, y)$$

and

$$x \odot y = \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(e^{\lambda x} \cdot e^{\lambda y}\right) = x + y.$$

Therefore, we get

$$\begin{aligned}\sup&\left(\sup(f(x) \odot \psi(x)) \odot \sup\left(\frac{1}{f(x)} \odot \psi(x)\right)\right) dx\\ &\leq g^{-\lambda}\left(\frac{(l+L)^2}{4lL} \cdot (b-a)^2\right).\end{aligned}$$

Note that the third important case $\oplus = \max$ and $\odot = \min$ has been studied in Theorem 3.2 and the pseudo-integral in such a case yields the Sugeno integrals.

5. Further Discussions

In this section, we provide a strengthened version of Sitaru-Schweitzer type integral inequalities for pseudo-integrals.

Theorem 5.1. *Let $f : [a,b] \to [l_0, L_0]$ be a measurable function and let a generator $g : [l_0, L_0] \to [l, L)$ of the pseudo addition $\oplus$ and the pseudo-multiplication $\odot$ be a monotone function. If $\varphi : [l_0, L_0] \to [l, L]$ is a continuous and strictly increasing function such that φ commutes with $\odot$, then the inequality*

$$\begin{aligned}\varphi^{-1}&\left(\left(\varphi \int_{[a,b]}^{\oplus} f d\mu\right) \odot \left(\varphi \int_{[a,b]}^{\oplus} (1 \oslash f)\, d\mu\right)\right)\\ &\leq \varphi^{-1}\left(g^{-1}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right)\right)\end{aligned}$$

holds.

Proof. From the Inequality (2.3), we obtain

(5.1)

$$\left(\varphi \int_{[a,b]}^{\oplus} f d\mu\right) \odot \varphi\left(\int_{[a,b]}^{\oplus} (1 \oslash f)\, d\mu\right) \leq \int_{[a,b]}^{\oplus} \varphi(f) d\mu \odot \int_{[a,b]}^{\oplus} \varphi(1 \oslash f) d\mu.$$

Applying Theorem 4.1 and the Inequality (5.1), we have

$$\left(\varphi \int_{[a,b]}^{\oplus} f d\mu\right) \odot \varphi\left(\int_{[a,b]}^{\oplus} (1 \oslash f)\, d\mu\right) \leq g^{-1}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right). \tag{5.2}$$

It follows that

$$\varphi^{-1}\left(\left(\varphi \int_{[a,b]}^{\oplus} f d\mu\right) \odot \varphi\left(\int_{[a,b]}^{\oplus} (1 \oslash f)\, d\mu\right)\right)$$
$$\leq \varphi^{-1}\left(g^{-1}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right)\right).$$

Therefore, the theorem is proved. □

Corollary 5.2. *Assuming $\varphi(x) = x^s$ that $\infty > s \geq 0$ and by considering the condition Theorem 5.1, the inequalty*

$$\left(\left(\int_{[a,b]}^{\oplus} f d\mu\right)^s \odot \left(\int_{[a,b]}^{\oplus} (1 \oslash f)\, d\mu\right)^s\right)^{\frac{1}{s}} \leq \left(g^{-1}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right)\right)^{\frac{1}{s}}.$$

holds for all $\infty > s \geq 0$ where $(.)^s$ commutes with $\odot$.

Now we consider the second class, when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}\big(g(x)g(y)\big)$.

Theorem 5.3. *Let $f : [a, b] \to [l_0, L_0]$ be a measurable function and $\odot$ be represented by a monotone multiplicative generator g and m be the same as in Theorem 2.4. If $\varphi : [l_0, L_0] \to [l, L]$ is a continuous and strictly increasing function such that φ commutes with $\odot$, then the following inequality*

$$\varphi^{-1}\left(\left(\varphi \int_{[a,b]}^{\sup} f \odot dm\right) \odot \left(\varphi \int_{[a,b]}^{\sup} (1 \oslash f) \odot dm\right)\right)$$
$$\leq \varphi^{-1}\left(g^{-\lambda}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right)\right)$$

holds.

Proof. From the Inequality (2.4), we obtain

$$\left(\varphi \int_{[a,b]}^{\sup} f \odot dm\right) \odot \varphi\left(\int_{[a,b]}^{\sup} (1 \oslash f) \odot dm\right) \tag{5.3}$$
$$\leq \int_{[a,b]}^{\oplus} \varphi(f) \odot dm \odot \int_{[a,b]}^{\sup} \varphi(1 \oslash f) \odot dm.$$

Applying Theorem (4.3) and Inequality (5,3), we have

$$\left(\varphi \int_{[a,b]}^{\sup} f \odot dm\right) \odot \left(\varphi \int_{[a,b]}^{\sup} (1 \oslash f) \odot dm\right) \leq g^{-\lambda}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right). \tag{5.4}$$

It follows that

$$\varphi^{-1}\left(\left(\varphi \int_{[a,b]}^{\sup} f \odot dm\right) \odot \left(\varphi \int_{[a,b]}^{\sup} (1 \oslash f) \odot dm\right)\right) \leq \varphi^{-1}\left(g^{-\lambda}\left(\frac{(l+L)^2}{4lL}(b-a)^2\right)\right).$$

Therefore, the theorem is proved. □

6. Conclusion

The classical Sitaru-Schewitzer type integral inequality is one of the most important inequalities and it is deeply connected with the study of singular integral theory. In this paper, we indicated Sitaru-Schewitzer type inequality for the fuzzy integral and generalized this inequality for the pseudo-integrals. In the sequel, several illustrated examples are given. Moreover, a strengthened version of Sitaru-Schewitzer inequality for pseudo-integrals is proved.

Acknowledgment. The author would like to gratefully thank the anonymous referees for their careful reading of the paper and the suggestions and corrections.

References

1. H. Agahi, R. Mesiar and Y. Ouyang, *General Minkowski type inequality for Sugeno integral*, Fuzzy Sets Syst., 161 (2010), pp. 708-715.
2. H. Agahi, R. Mesiar and Y. Ouyang, *New general extensions of Chebyshev type inequality for Sugeno integrals*, International Jornal of Approxinate Reasoning, 51 (2002), pp. 135-140.
3. B. Daraby, *A review on some fuzzy integral inequalities*, Sahand Commun. Math. Anal., 18 (3) (2021), pp. 153-185.
4. B. Daraby, *Generalizations of related Fritz Carlson type inequalities for fuzzy integrals*, Sahand Commun. Math. Anal., 18, (4) (2021), pp. 131-153.
5. B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets Syst., 194 (2012), pp. 90-96.

6. B. Daraby and L. Arabi, *Related Fritz Carlson type inequalities for Sugeno integrals*, Soft Comput., 17 (10) (2013), pp. 1745-1750.
7. B. Daraby and F. Ghadimi, *General Minkowski type and related inequalities for seminormed fuzzy integrals*, Sahand Commun. Math. Anal., 1 (1) (2014), pp. 9-20.
8. B. Daraby, H. G. Asll and I. Sadeqi, *General related inequalities to Carlson-type inequality for the Sugeno integral*, Appl. Math. Comput., 305 (2017), pp. 323-329.
9. B. Daraby, A. Rahimi and M. Tahmouresi, *Generalization of a two-dimensional Hardy type inequality for pseudo-integrals*, Advances in Mathematical Sciencs and Applications, 32 (1) (2023), pp. 225-236.
10. B. Daraby, A. Shafiloo and A. Rahimi, *Generalizations of the Feng Qi type inequality for pseudo-integral*, Gazi University Journal of Science, 28 (4) (2015), pp. 695-702.
11. B. Daraby, R. Mesiar, F. Rostampour and A. Rahimi, *Related Thunsdorff type and Frank-Pick type inequalities for Sugeno integral*, Appl. Math. Comput., 414 (2022), 126683.
12. S. Ezghari, A. Zahi and K. Zenkouar, *A new nearest neighbor classification method based on fuzzy set theory and aggregation operators*, Expert Systems with Applications, 80 (2017), pp. 58-74.
13. A. Flores-Franulič and H. Román-Flores, *A Chebyshev type inequality for fuzzy integrals*, Appl. Math. Comput., 190 (2007), pp. 1178-1184.
14. A. Flores-Franulič, H. Román-Flores and Y. Chalco-Cano, *Markov type inequalities for fuzzy integrals*, Appl. Math. Comput., 207 (2009), pp. 242-247.
15. M. R. Karimzadeh, B. Daraby and A. Rahimi, *Diaz-Metcalf type inequality for Sugeno and pseudo-integrals*, Iran. J. Fuzzy Syst., 20 (2023), pp. 31-41.
16. R. Mesiar and E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets Syst., 102 (3) (1999), pp. 385-392.
17. E. Pap, *g-Calculus*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 23 (1) (1993), pp. 145-156.
18. E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, Mathematics and Its Applications 337, Dordrecht/ Boston/ London, 1995.
19. E. Pap, *Pseudo-additive measures and their applications*, In Handbook of measure theory (pp. 1403-1468). North-Holland, 2002.
20. E. Pap and M. Štrboja, *Generalization of the Jensen inequality for pseudo-integral*, Information Sciences, 180 (2010), pp. 543-548.
21. D. Ralescu, G. Adams, *The fuzzy integral*, Journal of Mathematical Analysis and Applications, 75(2) (1980), 562-570.

22. H. Román-Flores, A. Flores-Franulič and Y. Chalco-Cano, *A Hardy-type inequality for fuzzy integrals*, Appl. Math. Comput., 204 (1) (2008), pp. 178-183.
23. D. Sitaru, *Math Phenomenon*, Pitești: Editura Paralela, 45 (168) (2016).
24. Z. Wang and G.J. Klir, *Fuzzy Measure Theory*, Springer Science & Business Media, (1992).
25. D. Zhang and E. Pap, *Jensen's inequalities for pseudo-integrals*, Iran. J. Fuzzy Syst., 18 (3) (2021), pp. 99-109.

---

[1] Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.

Email address: bdaraby@maragheh.ac.ir

[2] Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.

Email address: mortazatahmoras@gmail.com

[3] Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.

Email address: rahimi@maragheh.ac.ir