Numerical Solution of Differential Equations of Elastic Curves in 3-dimensional Anti-de Sitter Space

Samira Latifi, Nemat Abazari and Ghader Ghasemi

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 20
Number: 4
Pages: 171-189

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.556633.1135

Sahand Communications in Mathematical Analysis (SCMA) Vol. 20 No. 4 (2023), 171-189
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.556633.1135

Numerical Solution of Differential Equations of Elastic Curves in 3-dimensional Anti-de Sitter Space

Samira Latifi[1], Nemat Abazari[2] and Ghader Ghasemi[3]*

Abstract. In this paper, we aim to extend the Darboux frame field into 3-dimensional Anti-de Sitter space and obtain two cases for this extension by considering a parameterized curve on a hypersurface; then we carry out the Euler-Lagrange equations and derive differential equations for non-null elastic curves in AdS_3 (i.e. 3-dimensional Anti-de Sitter space). In this study, we investigate the elastic curves in AdS_3 and obtain equations through which elastic curves are found out. Therefore, we solve these equations numerically and finally plot and design some elastic curves.

1. Introduction

One of the most classical themes in the calculus of variations in a space is elastic curve. Another one is named elastica, which would be either a surface or a manifold, characterized as the curve fulfilling a variational condition proper for interpolation problems [11]. Therefore, it can be considered as a solution of the variational problem introduced by Daniel Bernoulli and Leonhard Euler in 1744 [7]. The mathematical idealization of this issue declines the integral of the squared curvature with determined boundary conditions [14].

The purpose of the current study is to prosper and build up a relating context for elastic curves in AdS_3 which is considered. Einstein used highly symmetrical solution for calculating the equations with absirptive cosmological constant. In reality, this highly symmetrical solution is not

---

2020 *Mathematics Subject Classification.* 74B99, 53D25.
Key words and phrases. Darboux frame, Elastic curve, Geodesic curvature, Anti-de Sitter space, Euler-Lagrange equations.
Received: 27 June 2022, Accepted: 24 September 2022.
* Corresponding author.

as attractive as it must be; however, it would be considered as a sort of regularization of long distance actions of gravity.

It is worth mentioning that the conformal boundary of asymptotically Anti-de Sitter space is impressively different from asymptotically flat space times. Therefore, this aspect of anti-de Sitter is normally vital in case of Anti-de Sitter emerges in mathematical physics [2].

If we view a one sheeted hyperboloid

$$X_1^2 + X_2^2 + \cdots + X_n^2 - X_{n+1}^2 = 1.$$

In Minkowski space, we provide a space with a Lorentzian metric of constant curvature which is called as dS_n (i.e. n-dimensional de Sitter space).

AdS_n (i.e. n-dimensional Anti-de Sitter space) is addressed as the quadric

$$X_1^2 + X_2^2 + \cdots + X_{n-1}^2 - U^2 - V^2 = -1, \tag{1.1}$$

which is deeply put in a flat $n+1$ dimensional space with the metric

$$ds^2 = dX_1^2 + \cdots + dX_{n-1}^2 - dU^2 - dV^2.$$

In general, the topology of adS_n is $R^{n-1} \otimes S^1$ and the topology of dS_n is $S^{n-1} \otimes R$ [2]. Now, if we put $n = 3$ in the formula (1.1), then we get AdS_3.

In the present research, we investigate the elastic curves in AdS_3 and propose equations which help us to study the issues in relation to these subjects. Particularly, our aim is to develop some differential equations of the non-null elastic curves in AdS_3 [14, 1].

Then, a definition of an elastica in AdS_3 is proposed as an extremal point of the functional

$$\int_0^l (k_g^2(s) + \sigma) ds$$

where

$$k_g^2(s) = k_{g_1}^2(s) + k_{g_2}^2(s)$$

in the space M of non-null curves

$$\begin{array}{ll} \gamma : [0, l] \to AdS_3, & \|\gamma'\| = 1, \\ \gamma(0) = P_0, & \gamma(l) = P_l, \\ \gamma'(0) = V_0, & \gamma'(l) = V_l, \\ \gamma''(0) = a_0, & \gamma''(l) = a_l, \end{array}$$

where $k_g = k_g(s)$ denotes the geodesic curvature of the curve as a function of the arc length parameter s and σ is constant.

It is suffice to say that a non-null curve is either timelike or spacelike. Each of these features is put under investigation separately in the present paper. However, before illustrating these cases, a definition of Darboux frame field into 3-dimensional AdS_3 space is necessary. In the area of differential geometry, one of the most important instruments for studying curves and surfaces is the frame field. The most famous and reputable frame fields are the Frenet-Serret frame along a space curve and the Darboux frame along a surface curve.

In Euclidean 3-space, the Darboux frame is developed by the speed of the curve and the common vector of the surface though the Frenet serret frame is built from the velocity and the speed of the curve, see references [3, 8, 9, 10, 13] for further reading.

The estimations of segments 2 and 3 of the article are taken from reference [4], For additional reading, refer to the reference section.

We dedicate Section 4.1 of the main text to spacelike elastica in case 1 ED-frame field and develop differential equations and afterwards tackle these equations numerically and plot them. In Section 4.2, we consider timelike elastica in case 1 from ED-frame field and develop differential equations and solve them just like previous section. In section 5.1, we consider spacelike elastica in case 2 from ED-frame field and develop differential equations with their solutions and then plot them. We allocate the last section 5.2 to timelike elastica in case from ED-frame field and after deriving differential equations with the help of Matlab, we solve and plot them. As a result, in all four modes, we provide equations that show geodesic curvature of γ must satisfy them.

2. The Construction of the Extended Darboux Frame Field

We can construct the extended Darboux frame field along the Frenet curve β as follows:

Case 1. If the set $\{N, T, \beta''\}$ is linearly independent, then using the Gram-Schmidt orthonormalization method gives the orthonormal set $\{N, T, E\}$ where

$$E = \frac{\beta'' - \langle \beta'', N \rangle N}{\|\beta'' - \langle \beta'', N \rangle N\|}.$$

Case 2. If the set $\{N, T, \beta''\}$ is linearly dependent, i.e. if β'' is in the direction of the normal vector N. Applying the Gram-Schmidt orthonormalization method to $\{N, T, \beta'''\}$ yields the orthonormal set $\{N, T, E\}$, where

$$E = \frac{\beta''' - \langle \beta''', N \rangle N - \langle \beta''', T \rangle T}{\|\beta''' - \langle \beta''', N \rangle N - \langle \beta''', T \rangle T\|}.$$

in each case, if we define $D = N \otimes T \otimes E$ we have four unit vector fields T, E, D and N which are mutually orthogonal at each point of β.

Thus, we have a new orthonormal frame field $\{T, E, D, N\}$ along the curve β instead of its Frenet frame field. It is obvious $E(s)$ and $D(s)$ are also tangent to the hypersurface M for all s.

Thus, the set $\{T(s), E(s), D(s)\}$ spans the tangent hyperplane of the hypersurface at the point $\beta(s)$ [4, 5, 6].

3. The Derivative Equations

Let us now express the derivatives of these vector fields in terms of themselves in each case. Since $\{T, E, D, N\}$ is orthonormal, we have

$$\begin{array}{ll} g(T,T) = {}_1, & g(E,E) = {}_2, \\ g(D,D) = {}_3, & g(N,N) = {}_4, \end{array}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}$, and

$$\begin{aligned} T' &= \epsilon_1 g(T',T)T + \epsilon_2 g(T',T)E + \epsilon_4 g(T',D)D + \epsilon_4 (T'N)N, \\ E' &= \epsilon_1 g(E',T)T + \epsilon_2 g(E',E)E + \epsilon_3 g(E',D)D + \epsilon_4 g(E',N)N, \\ D' &= \epsilon_1 g(D',T)T + \epsilon_2 g(D',E)E + \epsilon_3 g(D',D)D + \epsilon_4 g(D',N)N, \\ N' &= \epsilon_1 g(N',T)T + \epsilon_2 g(N',E)E + \epsilon_3 g(N',D)D + \epsilon_4 g(N',N)N. \end{aligned}$$

In Case 1, ED-frame field is of the first kind. Since we have

$$\begin{aligned} E &= \frac{\beta'' - \langle \beta'', N \rangle N}{\|\beta'' - \langle \beta'', N \rangle N\|} \\ &= \frac{T' - \langle T', N \rangle N}{\|T' - \langle T', N \rangle N\|}, \end{aligned}$$

we get

$$T' = \big\|T' - \big\langle T', N \big\rangle N\big\| E + \big\langle T', N \big\rangle N.$$

i.e. $\langle T', D \rangle = 0$.

In Case 2, ED-frame field is of the second kind. Thus $\{N, T, \beta''\}$ is linearly dependent and

$$E = \frac{\beta''' - \langle \beta''', N \rangle N - \langle \beta''', T \rangle T}{\|\beta''' - \langle \beta''', N \rangle N - \langle \beta''', T \rangle T\|}. \tag{3.1}$$

The linear dependency of $\{N, T, \beta''\}$ gives $\beta'' = \lambda N$, that is $\langle T', E \rangle = \langle T', D \rangle = 0$. Moreover, if we substitute $\beta''' = \lambda' N + \lambda N'$ into (3.1), we obtain $\langle N', D \rangle = 0$. we denote

$$\big\langle E', N \big\rangle = T_{g_1}, \qquad \big\langle D', N \big\rangle = T_{g_2}$$

and call T_{g_i} the geodesic torsion of order i. similarly, we put

$$\big\langle T', E \big\rangle = k_{g_1}, \qquad \big\langle E', D \big\rangle = k_{g_2}$$

and define k_{g_i} as the geodesic curvature of order i.

Lastly, if we use $\langle T', N\rangle = k_n$, we obtain the differential equations of ED-frame fields in matrix notation

$$\text{Case 1:} \begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & k_{g_1} & 0 & k_n \\ -k_{g_1} & 0 & k_{g_2} & T_{g_1} \\ 0 & -k_{g_2} & 0 & T_{g_2} \\ -k_n & -T_{g_1} & -T_{g_2} & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} \tag{3.2}$$

$$\text{Case 2:} \begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & k_n \\ 0 & 0 & k_{g_2} & T_{g_1} \\ 0 & -k_{g_2} & 0 & 0 \\ -k_n & -T_{g_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix} \tag{3.3}$$

see reference [4]. Now let us consider four cases as follow.

4. Case 1

4.1. **Spacelike Elastica.** In this section, we consider spacelike elastica in Case 1 from ED-frame field (3.2). We will begin this section by considering Darboux basis. Let us consider an arc length parameterized spacelike curve in AdS_3 described by the embedding $\gamma = \gamma(s)$ where $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_3)$.

At a point $\gamma(s)$ of γ, let $T = \gamma'$ denote the unit spacelike tangent vector to γ, let N, E be timelike vectors and D a spacelike vector on Ads_3. The derivatives T', E', D', N' in the Darboux basis T, E, D, N satisfy

$$\begin{aligned} T' &= -k_{g_1} E - \frac{1}{r} N, \\ E' &= -k_{g_1} T + k_{g_2} D, \\ D' &= k_{g_2} E, \\ N' &= -\frac{1}{r} T. \end{aligned} \tag{4.1}$$

that

$$\begin{aligned} T' &= +\langle T', T\rangle T - \langle T', E\rangle E + \langle T', D\rangle D - \langle T', N\rangle N, \\ E' &= +\langle E', T\rangle T - \langle E', E\rangle E + \langle E', D\rangle D - \langle E', N\rangle N, \\ D' &= +\langle D', T\rangle T - \langle D', E\rangle E + \langle D', D\rangle D - \langle D', N\rangle N, \\ N' &= +\langle N', T\rangle T - \langle N', E\rangle E + \langle N', D\rangle D - \langle N', N\rangle N. \end{aligned}$$

where

$$\begin{aligned} &\langle T', E\rangle = k_{g_1}, \quad \langle T', N\rangle = k_n, \\ &\langle E', D\rangle = k_{g_2}, \quad \langle E', N\rangle = T_{g_1}, \\ &\langle D', N\rangle = T_{g_2}, \\ &T_{g_1} = T_{g_2} = 0. \end{aligned}$$

From (4.1) we obtain the relation

$$\left\langle E', E' \right\rangle = k_{g_1}^2 + k_{g_2}^2 \tag{4.2}$$

By substituting the quantities (4.1) in relation (4.2) we have

$$\begin{aligned} \left\langle E', E' \right\rangle &= \langle -k_{g_1} T + k_{g_2} D, -k_{g_1} T + k_{g_2} D \rangle \\ &= k_{g_1}^2 \langle T, T \rangle + k_{g_2}^2 \langle D, D \rangle \\ &= k_{g_1}^2 + k_{g_2}^2. \end{aligned}$$

We want to minimize the functional

$$\int_0^1 \left(\left\langle E', E' \right\rangle + \sigma \right) ds$$

under the constraints

$$\langle E, E \rangle = -1, \qquad E = \gamma'', \qquad \langle \gamma, \gamma \rangle = -r^2, \qquad \left\langle \gamma', \gamma' \right\rangle = \langle T, T \rangle = 1.$$

Thus, it is possible to apply the Euler-lagrange equations to the functional

$$\begin{aligned} F = \left\langle E', E' \right\rangle + \lambda\left(\langle E, E \rangle + 1 \right) + \eta\left(\left\langle \gamma', \gamma' \right\rangle - 1 \right) \\ + \mu\left(\langle \gamma, \gamma \rangle + r^2 \right) + 2 \left\langle \Lambda, \gamma'' - E \right\rangle. \end{aligned}$$

where λ, μ, η are scalars and

$$\Lambda = \langle \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 \rangle$$

is a vector in E_2^4 (for calculus of variations see [11]).

Now, we can carry out the Euler-Lagrange equations

$$\begin{cases} \dfrac{\partial F}{\partial \gamma} - \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial \gamma'} \right) = 0, \\ \dfrac{\partial F}{\partial \gamma'} - \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial \gamma''} \right) = 0, \\ \dfrac{\partial F}{\partial E} - \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial E'} \right) = 0. \end{cases}$$

So, we obtain from the Euler-Lagrange equations

$$\begin{array}{lll} \dfrac{\partial F}{\partial \gamma} = 2\mu\gamma & \dfrac{\partial F}{\partial E} = 2\lambda E - 2\Lambda & \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial \gamma'} \right) = 2\eta\gamma'' \\ \dfrac{\partial F}{\partial \gamma'} = 2\eta\gamma' & \dfrac{\partial F}{\partial E'} = 2E' & \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial \gamma''} \right) = 2\Lambda' \\ \dfrac{\partial F}{\partial \gamma''} = 2\Lambda & & \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial E'} \right) = 2E'' \end{array}$$

$$\begin{aligned} &\mu\gamma - \eta\gamma'' = 0 \\ &\mu \cdot (-\gamma_1, -\gamma_2, \gamma_3, \gamma_4) - \eta\left(-\gamma_1'', -\gamma_2'', \gamma_3'', \gamma_4'' \right) = 0, \end{aligned} \tag{4.3}$$

(4.4) $\eta\gamma - \Lambda' = 0$

$$\eta \cdot \left(-\gamma_1', -\gamma_2', \gamma_3', \gamma_4'\right) - \left(-\Lambda_1', -\Lambda_2', \Lambda_3', \Lambda_4'\right) = 0,$$

(4.5) $\lambda E - E'' = \Lambda$

$$\lambda \cdot (-e_1, -e_2, e_3, e_4) - \left(-e_1'', -e_2'', e_3'', e_4''\right) = (-\Lambda_1, -\Lambda_2, -\Lambda_3, \Lambda_4).$$

if we take first derivative of (4.5) with respect to s, we have

$$\lambda' E + \lambda E' - E''' = \Lambda'$$

and combine with (4.4), and then derive the achieved equation (4.4), we have

$$\eta'\gamma' + \eta\gamma'' = \lambda'' E + 2\lambda' E' + \lambda E'' - E'''$$

and combine with (4.3), $\gamma = -rN$, $\gamma' = T$, we obtain

$$\lambda'' E + 2\lambda' E' + \lambda E'' - E''' = \eta' T - r\mu N,$$

and

$$\begin{aligned} (4.6) \qquad & \lambda'' (-e_1, -e_2, e_3, e_4) + 2\lambda' \left(-e_1', -e_2', e_3', e_4'\right) \\ & + \lambda \left(-e_1'', -e_2'', e_3'', e_4''\right) - \left(-e_1'''', -e_2'''', e_3'''', e_4''''\right) \\ & = \eta' (-t_1, -t_2, t_3, t_4) - r\mu (-n_1, -n_2, n_3, n_4). \end{aligned}$$

By using the equations (4.1) and considering the relation $k_g^2 = k_{g_1}^2 + k_{g_2}^2$, we have some derivatives of E as following

$$\begin{aligned} E' &= -k_{g_1} T + k_{g_2} D, \\ E'' &= -k_{g_1}' T + k_g^2 E + k_{g_2}' D + \frac{1}{r} k_{g_1} N, \\ E''' &= \left(-k_{g_1}'' - k_{g_1} k_g^2 - \frac{1}{r^2} k_{g_1}\right) T + \left(k_{g_1} k_{g_1}' + 2k_g k_g' + k_{g_2} k_{g_2}'\right) E \\ &\quad + \left(k_g^2 k_{g_2} + k_{g_2}''\right) D + \left(\frac{2}{r} k_{g_1}'\right) N. \end{aligned}$$

and

$$\begin{aligned} E'''' = -&\left(k_{g_1}''' + k_{g_1}' k_g^2 + 2k_{g_1} k_g k_g' + \frac{1}{r^2} k_{g_1}' + k_{g_1}^2 k_{g_1}' + 2k_{g_1} k_g k_g' \right. \\ &\left. + k_{g_1} k_{g_2} k_{g_2}' - \frac{2}{r^2} k_{g_1}'\right) T \\ &+ \left(k_{g_1} k_{g_1}'' + k_{g_1}^2 k_g^2 + \frac{1}{r^2} k_{g_1}^2 + \left(k_{g_1}'\right)^2 + k_{g_1} k_{g_1}'' + 2\left(k_g'\right)^2 \right. \\ &\left. + 2k_g k_g'' + k_{g_2}'^2 + k_{g_2} k_{g_2}'' + k_{g_2}^2 k_g^2 + k_{g_2}'' k_{g_2}\right) E \\ &+ \left(k_{g_2} k_{g_1} k_{g_1}' + 4k_{g_2} k_g k_g' + k_{g_2}^2 k_{g_2}' + k_g^2 k_{g_2}' + k_{g_2}'''\right) D \end{aligned}$$

$$+\left(\frac{1}{r}k''_{g_1}+\frac{1}{r}k_{g_1}k_g^2+\frac{1}{r^3}k_{g_1}+\frac{2}{r}k''_{g_1}\right)N.$$

from (4.6), we have

$$\lambda''E+2\lambda'E'+\lambda E''-E''''-\eta'T+\mu rN=0,$$

then

$$\begin{aligned}&\lambda''E+2\lambda'\left(-k_{g_1}T+k_{g_2}D\right)+\lambda\left(-k'_{g_1}T+k_g^2E+k'_{g_2}D+\frac{1}{r}k_{g_1}N\right)\\&\quad+\Big(k'''_{g_1}+k'_{g_1}k_g^2+2k_{g_1}k_gk'_g+\frac{1}{r^2}k'_{g_1}+k_{g_1}^2k'_{g_1}+2k_{g_1}k_gk'_g\\&\quad+k_{g_1}k_{g_2}k'_{g_2}-\frac{2}{r^2}k'_{g_1}\Big)T-\Big(k_{g_1}k''_{g_1}+k_{g_1}^2k_g^2+\frac{1}{r^2}k_{g_1}^2+\left(k'_{g_1}\right)^2\\&\quad+k_{g_1}k''_{g_1}+2\left(k'_g\right)^2+2k_gk''_g+\left(k'_{g_2}\right)^2+k_{g_2}k''_{g_2}+k_{g_2}^2k_g^2+k''_{g_2}k_{g_2}\Big)E\\&\quad-\left(k_{g_2}k_{g_1}k'_{g_1}+4k_{g_2}k_gk'_g+k_{g_2}^2k'_{g_2}+k_g^2k'_{g_2}+k''''_{g_2}\right)D\\&\quad-\left(\frac{1}{r}k''_{g_1}+\frac{1}{r}k_{g_1}k_g^2+\frac{1}{r^3}k_{g_1}+\frac{2}{r}k''_{g_1}\right)N-\eta'T+\mu rN\\&=0.\end{aligned}$$

Then

$$\begin{aligned}&\Big(-2\lambda'k_{g_1}-\lambda k'_{g_1}-\eta'+k'''_{g_1}+k'_{g_1}k_g^2+4k_{g_1}k_gk'_g+\frac{1}{r^2}k'_{g_1}\\&\quad+k_{g_1}^2k'_{g_1}+k_{g_1}k_{g_2}k'_{g_2}-\frac{2}{r^2}k'_{g_1}\Big)T\\&\quad+\Big(\lambda''+\lambda k_g^2-k_{g_1}k''_{g_1}-k_{g_1}^2k_g^2-\frac{1}{r^2}k_{g_1}^2-(k'_{g_1})^2-k_{g_1}k''_{g_1}-2(k'_g)^2\\&\quad-2k_gk''_g-(k'_{g_2})^2-2k_{g_2}k''_{g_2}-k_{g_2}^2k_g^2\Big)E+\big(2\lambda'k_{g_2}+\lambda k'_{g_2}\\&\quad-k_{g_2}k_{g_1}k'_{g_1}-4k_{g_2}k_gk'_g-k_{g_2}^2k'_{g_2}-k_g^2k'_{g_2}-k'''_{g_2}\big)D\\&\quad+\left(\frac{1}{r}\lambda k_{g_1}+\mu r-\frac{1}{r}k''_{g_1}-\frac{1}{r}k_{g_1}k_g^2-\frac{1}{r^3}k_{g_1}-\frac{2}{r}k''_{g_1}\right)N\\&=0.\end{aligned}$$

Considering the linear independence of the vectors T, E, D, N we have the following theorem:

Theorem 4.1. *γ is a spacelike elastica in AdS_3 if and only if the geodesic curvature of γ satisfies in the four below equations:*

$$\begin{aligned}&-2\lambda'k_{g_1}-\lambda k'_{g_1}-\eta'-k'''_{g_1}+k'_{g_1}k_g^2+4k_{g_1}k_gk'_g\\&\quad+\frac{1}{r^2}k'_{g_1}+k_{g_1}^2k'_{g_1}+k_{g_1}k_{g_2}k'_{g_2}-\frac{2}{r^2}k'_{g_1}=0,\end{aligned}$$

$$\lambda'' + \lambda k_g^2 - 2k_{g_1}k''_{g_1} - k_{g_1}^2 k_g^2 - \frac{1}{r^2}k_{g_1}^2 - (k'_{g_1})^2$$
$$- 2k_g'^2 - 2k_g k_g'' - k_{g_2}'^2 - 2k_{g_2}k''_{g_2} - k_{g_2}^2 k_g^2 = 0,$$

$$2\lambda' k_{g_2} + \lambda k'_{g_2} - k_{g_2}k_{g_1}k'_{g_1} - 4k_{g_2}k_g k'_g - k_{g_2}^2 k'_{g_2}$$
$$- k_g^2 k'_{g_2} - k'''_{g_2} = 0,$$

$$\frac{1}{r}\lambda k_{g_1} + \mu r - \frac{1}{r}k''_{g_1} - \frac{1}{r}k_{g_1}k_g^2 - \frac{1}{r^3}k_{g_1} - \frac{2}{r}k''_{g_1} = 0.$$

we solve this equations numerically and plot them at Figure 1.

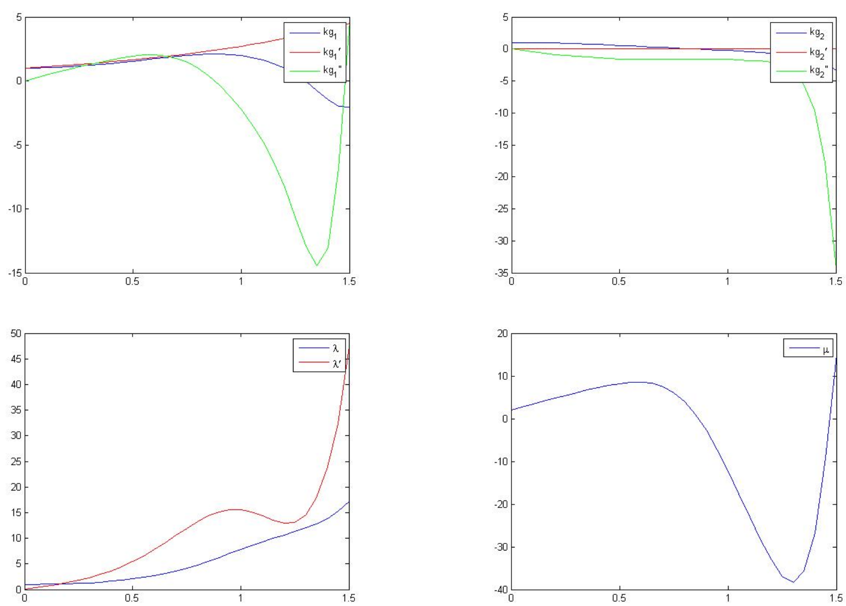

FIGURE 1. Spacelike1

4.2. **Timelike Elastica.** We start with preliminaries on the geometry of timelike elastica in Case 1 from ED-frame field in AdS_3. Let γ be an arc length parameterized timelike curve in AdS_3. At a point $\gamma(s)$ of γ, let $T = \gamma'$ denote the unit timelike tangent vector to γ.

Let N, E be spacelike vectors and D a timelike vector on AdS_3. The derivatives T', E', D', N' in Extension of the Darboux basis T, E, D, N

satisfy

$$\begin{aligned}
T' &= k_{g_1}E + \frac{1}{r}N,\\
E' &= k_{g_1}T - k_{g_2}D,\\
D' &= -k_{g_2}E,\\
N' &= \frac{1}{r}T.
\end{aligned}$$

We have

$$\left\langle E', E' \right\rangle = -\left(k_{g_1}^2 + k_{g_2}^2\right)$$

we want to minimize the functional

$$\int_0^l \left(-\left\langle E', E'\right\rangle + \sigma\right) ds$$

under the constraints

$$\langle E, E\rangle = 1, \qquad E = \gamma'', \qquad \langle \gamma, \gamma\rangle = -r^2, \qquad \left\langle \gamma', \gamma'\right\rangle = -1.$$

Thus, it is possible to apply the Euler-Lagrange equations to the functional

$$\begin{aligned}
F = -\left\langle E', E'\right\rangle + \lambda\left(\langle E, E\rangle - 1\right) + \eta\left(\left\langle \gamma', \gamma'\right\rangle + 1\right)\\
+ \mu\left(\langle \gamma, \gamma\rangle + r^2\right) + 2\left\langle \Lambda, \gamma'' - E\right\rangle.
\end{aligned}$$

To get the differential equations which govern the extremals

$$\begin{cases}
\dfrac{\partial F}{\partial \gamma} - \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial \gamma'}\right) = 0,\\
\dfrac{\partial F}{\partial \gamma'} - \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial \gamma''}\right) = 0,\\
\dfrac{\partial F}{\partial E} - \dfrac{d}{ds}\left(\dfrac{\partial F}{\partial E'}\right) = 0.
\end{cases}$$

To substitution, we have:

$$\begin{cases}
\mu\gamma - \eta\gamma'' = 0,\\
\eta\gamma' - \Lambda' = 0,\\
\lambda E - \Lambda + E'' = 0.
\end{cases}$$

Combining these equations and considering that $\gamma = rN$, we yields

$$\lambda'' E + 2\lambda' E' + \lambda E'' + E'''' = r\eta' N' + \mu r N,$$

Substituting E, E', E'', E'''', N, N', we have

$$\begin{aligned}
(4.7)\quad \lambda''\left(-e_1, -e_2, e_3, e_4\right) + 2\lambda'\left(-e_1', -e_2', e_3', e_4'\right) + \lambda\left(-e_1'', -e_2'', e_3'', e_4''\right)\\
+ \left(-e_1'''', -e_2'''', e_3'''', e_4''''\right)\\
= r\eta'\left(-n_1', -n_2', n_3', n_4'\right) + \mu r\left(-n_1, -n_2, n_3, n_4\right).
\end{aligned}$$

The derivatives of E expressed in the Darboux basis are given by

$$\begin{aligned}
E' &= k_{g_1}T - k_{g_2}D,\\
E'' &= k'_{g_1}T + k_g^2 E + \frac{1}{r}k_{g_1}N - k'_{g_2}D,\\
E''' &= \left(k''_{g_1} + k_g^2 k_{g_1} + \frac{1}{r^2}k_{g_1}\right)T + \left(k_{g_1}k'_{g_1} + 2k_g k'_g + k_{g_2}k'_{g_2}\right)E\\
&\quad + \left(\frac{2}{r}k'_{g_1}\right)N - \left(k_{g_2}k_g^2 + k''_{g_2}\right)D,\\
E'''' &= \left(k'''_{g_1} + 2k_g k'_g k_{g_1} + k_g^2 k'_{g_1} + \frac{1}{r^2}k'_{g_1} + k_{g_1}^2 k'_{g_1} + 2k_g k'_g k_{g_1}\right.\\
&\quad \left. + k_{g_1}k_{g_2}k'_{g_2} + \frac{2}{r^2}k'_{g_1}\right)T + \left(k_{g_1}k''_{g_1} + k_g^2 k_{g_1}^2 + \frac{1}{r^2}k_{g_1}^2 + k'^2_{g_1}\right.\\
&\quad \left. + k_{g_1}k''_{g_1} + 2k'^2_g + 2k_g k''_g + (k'_{g_2})^2 + k_{g_2}k''_{g_2} + k_{g_2}^2 k_g^2 + k_{g_2}k''_{g_2}\right)E\\
&\quad + \left(\frac{1}{r}k''_{g_1} + \frac{1}{r}k_g^2 k_{g_1} + \frac{1}{r^3}k_{g_1} + \frac{2}{r}k''_{g_1}\right)N\\
&\quad - \left(k_{g_1}k'_{g_1}k_{g_2} + 2k_g k'_g k_{g_2} + k_{g_2}^2 k'_{g_2} + k'_{g_2}k_g^2 + k'''_{g_2} + 2k_{g_2}k_g k'_g\right)D.
\end{aligned}$$

From (4.7), we have

$$\begin{aligned}
&\left(2\lambda' k_{g_1} + \lambda k'_{g_1} - \eta' + k''''_{g_1} + 4k_g k'_g k_{g_1} + k_g^2 k'_{g_1} + \frac{3}{r^2}k_{g_1}\right.\\
&\qquad \left. + k_{g_1}^2 k'_{g_1} + k_{g_1}k_{g_2}k'_{g_2}\right)T + \left(\lambda'' + \lambda k_g^2 + 2k_{g_1}k''_{g_1} + k_g^2 k_{g_1}^2\right.\\
&\qquad \left. + \frac{1}{r^2}k_{g_1}^2 + k'^2_{g_1} + 2k_g k''_g + k'^2_{g_2} + 2k_{g_2}k''_{g_2} + k_g^2 k_{g_2}^2\right)E\\
&\qquad - \left(\frac{\lambda}{r}k_{g_1} - \mu r + \frac{1}{r}k''_{g_1} + \frac{1}{r}k_{g_1}k_g^2 + \frac{1}{r^3}k_{g_1} + \frac{2}{r}k''_{g_1}\right)N\\
&\qquad - \left(2\lambda' k_{g_2} + \lambda k'_{g_2} + k_{g_1}k'_{g_1}k_{g_2} + 4k_g k_{g_2}k'_g + k_{g_2}^2 k'_{g_2}\right.\\
&\qquad \left. + k'_{g_2}k_g^2 + k'''_{g_2}\right)D\\
&\quad = 0.
\end{aligned}$$

From these we have the following theorem:

Theorem 4.2. *γ is a timelike elastica in AdS_3 if and only if the geodesic curvature of γ satisfies in the four below equations:*

$$2\lambda' k_{g_1} + \lambda k'_{g_1} - \eta' + k'''_{g_1} + 4k_g k'_g k_{g_1} + \frac{3}{r^2}k'_{g_1}$$

$$+ k_g^2 k'_{g_1} + k_{g_1}^2 k'_{g_1} + k_{g_1} k_{g_2} k'_{g_2} = 0,$$

$$\lambda'' + \lambda k_g^2 + 2k_{g_1} k''_{g_1} + k_g^2 k_{g_1}^2 + \frac{1}{r^2} k_{g_1}^2 + k'^2_{g_1} + 2k'^2_g + 2k_g k''_g + k'^2_{g_2} + 2k_{g_2} k''_{g_2} + k_g^2 k_{g_2}^2 = 0,$$

$$2\lambda' k_{g_2} + \lambda k'_{g_2} + k_{g_1} k'_{g_1} k_{g_2} + 4k_g k_{g_2} k'_g + k'''_{g_2} + k_{g_2}^2 k'_{g_2} + k'_{g_2} k_g^2 = 0,$$

$$\frac{1}{r}\lambda k_{g_1} - \mu r + \frac{1}{r} k''_{g_1} + \frac{1}{r} k_{g_1} k_g^2 + \frac{1}{r^3} k_{g_1} + \frac{2}{r} k''_{g_1} = 0.$$

We plot them in Figure 2.

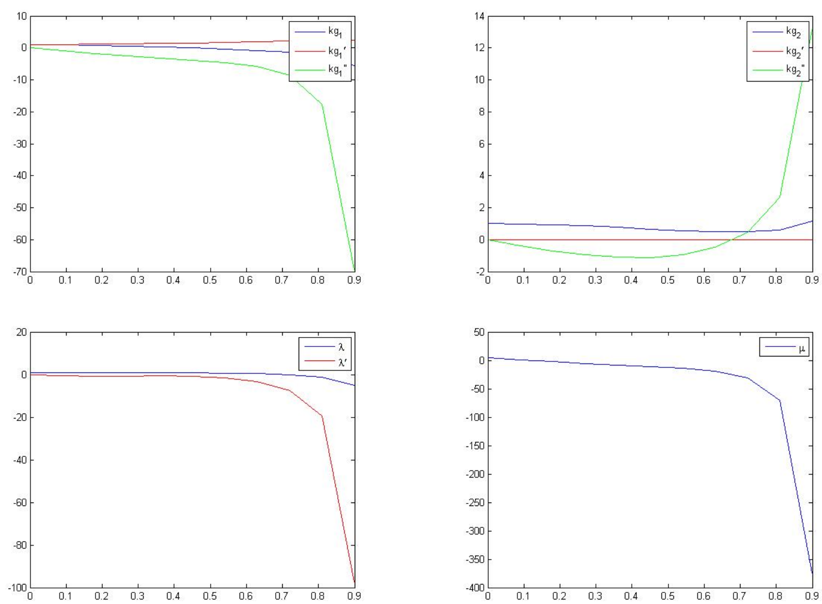

FIGURE 2. Timelike1

5. CASE 2

5.1. **Spacelike Elastica.** In this section, we aim at obtaining spacelike elastica in Case 2 from ED-frame field (3.3). Assuming the same conditions of the previous case about γ, let $T = \gamma'$ denote the unit spacelike tangent vector to γ, let N, D be timelike vectors and E a spacelike vector on AdS_3. The derivatives T', E', D', N' in the Darboux basis T, E, D, N satisfy

$$\begin{aligned} T' &= -\frac{1}{r} N, \\ E' &= -k_{g_2} D, \\ D' &= -k_{g_2} E, \\ N' &= -\frac{1}{r} T. \end{aligned}$$

where

$$k_{g_2} = k_g.$$

We obtain that

$$\left\langle E', E' \right\rangle = -k_g^2.$$

We want to minimize the functional

$$\int \left(- \left\langle E', E' \right\rangle + \sigma \right) ds$$

under the constraints

$$\langle E, E \rangle = 1, \quad E = \gamma''', \quad \langle \gamma, \gamma \rangle = -r^2, \quad \left\langle \gamma', \gamma' \right\rangle = 1, \quad \left\langle \gamma'', \gamma'' \right\rangle = -1.$$

Then we have

$$\begin{aligned} F = &- \left\langle E', E' \right\rangle + \sigma + \lambda \left(\langle E, E \rangle - 1 \right) + \eta \left(\left\langle \gamma'', \gamma'' \right\rangle + 1 \right) \\ &+ \mu \left(\left\langle \gamma', \gamma' \right\rangle - 1 \right) + \delta \left(\langle \gamma, \gamma \rangle + r^2 \right) + 2 \left\langle \Lambda, \gamma'' - E \right\rangle . \end{aligned}$$

Now, we can carry out the Euler-Lagrange equations

$$\begin{cases} \dfrac{\partial F}{\partial \gamma} - \dfrac{d}{ds} \left(\dfrac{\partial F}{\partial \gamma'} \right) = 0, \\ \dfrac{\partial F}{\partial \gamma'} - \dfrac{d}{ds} \left(\dfrac{\partial F}{\partial \gamma''} \right) = 0, \\ \dfrac{\partial F}{\partial \gamma''} - \dfrac{d}{ds} \left(\dfrac{\partial F}{\partial \gamma'''} \right) = 0, \\ \dfrac{\partial F}{\partial E} - \dfrac{d}{ds} \left(\dfrac{\partial F}{\partial E'} \right) = 0. \end{cases}$$

So, we obtain from the Euler-Lagrange equations

$$\frac{\partial F}{\partial \gamma} = 2\delta\gamma \qquad \frac{\partial F}{\partial E} = 2\lambda E - 2\Lambda \qquad \frac{d}{ds}\left(\frac{\partial F}{\partial \gamma'}\right) = 2\mu\gamma''$$
$$\frac{\partial F}{\partial \gamma'} = 2\mu\gamma' \qquad \frac{\partial F}{\partial E'} = -2E' \qquad \frac{d}{ds}\left(\frac{\partial F}{\partial \gamma''}\right) = 2\eta\gamma'''$$
$$\frac{\partial F}{\partial \gamma''} = 2\eta\gamma'' \qquad \frac{d}{ds}\left(\frac{\partial F}{\partial \gamma'''}\right) = 2\Lambda'$$
$$\frac{\partial F}{\partial \gamma'''} = 2\Lambda \qquad \frac{d}{ds}\left(\frac{\partial F}{\partial E'}\right) = -2E''$$

with substitution have

$$\delta\gamma - \mu\gamma'' = 0, \tag{5.1}$$

$$\mu\gamma' - \eta\gamma''' = 0, \tag{5.2}$$

$$\eta\gamma'' - \Lambda' = 0, \tag{5.3}$$

$$\lambda E - \Lambda + E'' = 0. \tag{5.4}$$

If we take first derive of (5.4) with respect to s, we have

$$\lambda' E + \lambda E' + E''' = \Lambda'$$

and combine with (5.3), and then derive the achieved equation (5.3), we have

$$\lambda'' E + 2\lambda' E' + \lambda E'' + E'''' = \eta'\gamma'' + \eta\gamma'''$$

and combine with (5.2) , and then derive the achieved equation (5.2), we have

$$\begin{aligned}\lambda''' E + \lambda'' E' + 2\lambda'' E' + 2\lambda' E'' + \lambda' E'' + \lambda E''' + E'''''\\ = \eta''\gamma'' + \eta'\gamma''' + \mu'\gamma' + \mu\gamma''.\end{aligned}$$

and with assuming $\gamma' = T$, $\gamma''' = E$, $\gamma'' = -rN''$ and using (5.1), we obtain

$$\begin{aligned}&\left(\eta'' + \mu\right)\left(-rN''\right) + \mu' T + \eta' E\\ &\quad = \lambda''' E + 3\lambda'' E' + 3\lambda' E'' + \lambda E''' + E'''''.\end{aligned} \tag{5.5}$$

the derivatives of E are

$$\begin{aligned}E' &= -k_g D,\\ E'' &= -k_g' D - k_g D' = -k_g' D + k_g^2 E,\\ E''' &= 3k_g k_g' E - \left(k_g'' + k_g^3\right) D,\\ E'''' &= \left(3k_g'^2 + 4k_g k_g'' + k_g^4\right) E - \left(6k_g^2 k_g' + k_g'''\right) D,\\ E''''' &= \left(6k_g' k_g'' + 4k_g' k_g'' + 4k_g k_g''' + 4k_g^3 k_g'\right) E\\ &\quad + \left(3k_g'^2 + 4k_g k_g'' + k_g^4\right)(-k_g D) - (12k_g k_g'^2\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
&+ 6k_g^2 k_g'''')D - \left(6k_g^2 k_g' + k_g'''\right)(-k_g E)\\
&= \left(10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g'''\right) E - \left(15k_g k_g'^2\right.\\
&\left.+10k_g^2 k_g'' + k_g^5 + k_g''''\right) D.
\end{aligned}
$$

From (5.5) and (5.6), we have:

$$
\begin{aligned}
&- \eta' E - \mu' T + \left(10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g'''\right) E\\
&\quad - \left(15k_g k_g'^2 + 10k_g^2 k_g'' + k_g^5 + k_g''''\right) D + 1/r\left(\eta'' + \mu\right) N\\
&\quad + \lambda''' E - 3\lambda'' k_g D + 3\lambda'\left(-k_g' D + k_g^2 E\right) + \lambda\left(3k_g k_g' E\right.\\
&\quad \left.- (k_g'' + k_g^3) D\right)\\
&= 0,
\end{aligned}
$$

then

$$
\begin{aligned}
&- \mu' T + \left(\lambda''' + 3\lambda' k_g^2 + 3\lambda k_g k_g'\right.\\
&\quad \left.-\eta' + 10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g'''\right) E - \left(3\lambda'' k_g + 3\lambda' k_g' + \lambda k_g''\right.\\
&\quad \left.+\lambda k_g^3 + 15k_g k_g'^2 + 10k_g^2 k_g'' + k_g^5 + k_g''''\right) D + 1/r\left(\eta'' + \mu\right) N\\
&= 0,
\end{aligned}
$$

then

$$
\begin{cases}
\mu' = 0 \to \mu = \text{constant},\\
1/r\,(\eta'' + \mu) = 0 \xrightarrow{r \neq 0} \eta'' = -\mu,\\
\lambda''' + 3\lambda' k_g^2 + 3\lambda k_g k_g' - \eta' + 10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g''' = 0,\\
3\lambda'' k_g + 3\lambda' k_g' + \lambda k_g'' + \lambda k_g^3 + 15k_g k_g'^2 + 10k_g^2 k_g'' + k_g^5 + k_g'''' = 0.
\end{cases}
$$

Theorem 5.1. *γ is a spacelike elastica in AdS_3 if and only if the geodesic curvature of γ satisfies in the two below equations:*

$$
\begin{aligned}
&\lambda''' + 3\lambda' k_g^2 + 3\lambda k_g k_g' - \eta' + 10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g''' = 0,\\
&3\lambda'' k_g + 3\lambda' k_g' + \lambda k_g'' + \lambda k_g^3 + 15k_g k_g'^2 + 10k_g^2 k_g'' + k_g^5 + k_g'''' = 0.
\end{aligned}
$$

we plot them in Figure 3.

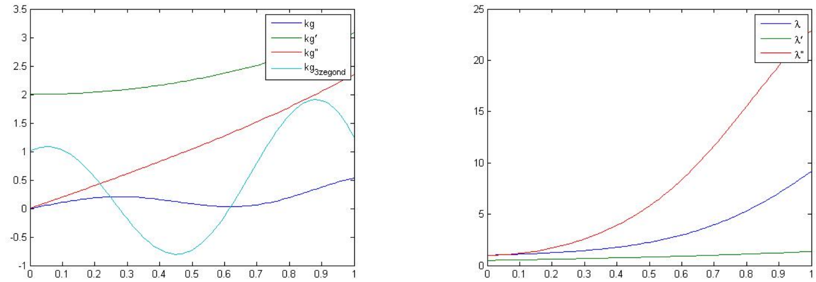

FIGURE 3. Spacelike2

5.2. **Timelike Elastica.** This section is devoted to timelike elastica in case 2 from ED-frame field. Assuming the same conditions of the previous case about γ, let $T = \gamma'$ denote the unit timelike tangent vector to γ, let N, D be spacelike vectors and E a timelike vector on AdS_3.

The derivatives T', E', D', N' in the Darboux basis T, E, D, N satisfy

$$
\begin{aligned}
T' &= +\frac{1}{r}N, \\
E' &= k_{g_2} D, \\
D' &= k_{g_2} E, \\
N' &= \frac{1}{r}T,
\end{aligned}
$$

where

$$k_{g_2} = k_g.$$

We have

$$\langle E', E' \rangle = k_g^2,$$

we want to minimize the functional

$$\int \left(\langle E', E' \rangle + \sigma \right) ds,$$

under the constraints

$$
\begin{aligned}
&\langle E, E \rangle = -1, E = \gamma''', \\
&\langle \gamma, \gamma \rangle = -r^2, \langle \gamma', \gamma' \rangle = -1, \langle \gamma'', \gamma'' \rangle = 1.
\end{aligned}
$$

We have

$$
\begin{aligned}
F = {} & \langle E', E' \rangle + \sigma + \lambda \left(\langle E, E \rangle + 1 \right) + \eta \left(\langle \gamma'', \gamma'' \rangle - 1 \right) \\
& + \mu \left(\langle \gamma', \gamma' \rangle + 1 \right) + \delta \left(\langle \gamma, \gamma \rangle + r^2 \right) + 2 \langle \Lambda, \gamma''' - E \rangle .
\end{aligned}
$$

similar to the previous mode, we get the differential equations which govern the extremals

$$\delta\gamma - \mu\gamma'' = 0, \tag{5.7}$$

$$\mu\gamma' - \eta\gamma''' = 0, \tag{5.8}$$

$$\eta\gamma'' - \Lambda' = 0, \tag{5.9}$$

$$\lambda E - \Lambda - E'' = 0. \tag{5.10}$$

If we take first derivative of (5.10) with respect to s, we have

$$\lambda' E + \lambda E' - E''' = \Lambda'$$

and combine with (5.9), and then derivative the achieved equation (5.9), we have

$$\eta'\gamma'' + \eta\gamma''' = \lambda'' E + 2\lambda' E' + \lambda E'' - E''''$$

and combine with (5.8), $\gamma'' = rN''$, $\gamma' = T$, $\gamma''' = E$, we obtain

$$\begin{aligned} &\lambda''' E + 3\lambda'' E' + 3\lambda' E'' + \lambda E''' - E''''' \\ &= \left(\eta'' + \mu\right) rN'' + \mu' T + \eta' E. \end{aligned} \tag{5.11}$$

The derivatives of E are:

$$\begin{aligned} E' &= k_g D, \\ E'' &= k_g' D + k_g^2 E, \\ E''' &= \left(k_g'' + k_g^3\right) D + \left(3k_g k_g'\right) E, \\ E'''' &= \left(4k_g'' k_g + 3k_g'^2 + k_g^4\right) E + \left(6k_g' k_g^2 + k_g'''\right) D, \\ E''''' &= \left(10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g'''\right) E + \big(15k_g k_g'^2 \\ &\quad +10k_g^2 k_g'' + k_g^5 + k_g''''\big) D. \end{aligned} \tag{5.12}$$

From (5.11) and (5.12), we have:

$$\begin{aligned} &- \eta' E - \mu' T - \left(10k_g' k_g'' + 10k_g^3 k_g' + 5k_g k_g'''\right) E - \big(15k_g k_g'^2 \\ &\quad +10k_g^2 k_g'' + k_g^5 + k_g''''\big) D - 1/r \left(\eta'' + \mu\right) N \\ &\quad + \lambda''' E + 3\lambda'' k_g D + 3\lambda' \left(k_g' D + k_g^2 E\right) + \lambda \big(3k_g k_g' E \\ &\quad + \left(k_g'' + k_g^3\right) D\big) \\ &= 0, \end{aligned}$$

then

$$\begin{aligned} &- \mu' T + \big(\lambda''' + 3\lambda' k_g^2 + 3\lambda k_g k_g' - \eta' - 10k_g' k_g'' - 10k_g^3 k_g' \\ &\quad -5k_g k_g'''\big) E + \big(3\lambda'' k_g + 3\lambda' k_g' + \lambda k_g'' + \lambda k_g^3 - 15k_g k_g'^2 \\ &\quad -10k_g^2 k_g'' - k_g^5 - k_g''''\big) D - 1/r \left(\eta'' + \mu\right) N \\ &= 0. \end{aligned}$$

and

$$\begin{cases} \mu' = 0 \rightarrow \mu = \text{constant}, \\ 1/r\,(\eta'' + \mu) = 0 \xrightarrow{r \neq 0} \eta'' = -\mu, \\ \lambda''' + 3\lambda' k_g^2 + 3\lambda k_g k_g' - \eta' - 10 k_g' k_g'' - 10 k_g^3 k_g' - 5 k_g k_g''' = 0, \\ 3\lambda'' k_g + 3\lambda' k_g' + \lambda k_g'' + \lambda k_g^3 - 15 k_g k_g'^2 - 10 k_g^2 k_g'' - k_g^5 - k_g'''' = 0. \end{cases}$$

Theorem 5.2. *γ is a timelike elastica in AdS_3 if and only if the geodesic curvature of γ satisfies in the two below equations:*

$$\lambda''' + 3\lambda' k_g^2 + 3\lambda k_g k_g' - \eta' - 10 k_g' k_g'' - 10 k_g^3 k_g' - 5 k_g k_g''' = 0,$$

$$3\lambda'' k_g + 3\lambda' k_g' + \lambda k_g'' + \lambda k_g^3 - 15 k_g k_g'^2 - 10 k_g^2 k_g'' - k_g^5 - k_g'''' = 0.$$

we plot them in Figure 4.

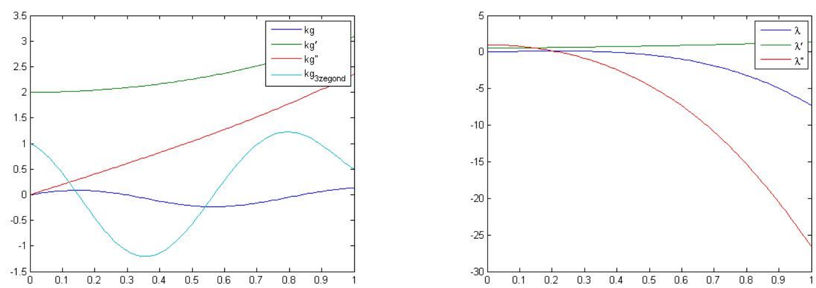

FIGURE 4. Timelike2

Theorems 4.1 and 5.1 shows spacelike elastica curves in AdS_3 space and theorems 4.2 and 5.2 shows differential equations that geodesic curvature of timelike elastica satisfies in them.

References

1. I. Boukaroura and S. Djabi and S. Khelladi, *Boundary Value Problems in Thermo Viscoplasticity*, Sahand Commun. Math. Anal., 18 (2) (2021), pp. 19-30.
2. I. Bengtsson, *Anti-de Sitter space*, lecture notes, 1998.
3. M. P. Do Carmo, *Differential Geometry of curves and surface*, Englewood Cliffs. NJ, USA: Prentice Hall, 1976.
4. M. Duldul and B.U. Duldul and N. Kuruoglu and E. Ozdamar, *Extension of the Darboux frame into Euclidean 4-space and its invariants*, Turkish J. Math., 41 (2017), pp. 1628-1639.
5. F. Pashaie and A. Mohammadpoiri, *L_k-biharmonic spacelike hypersurfaces in Minkowski 4-space $\mathbb{E}_1^4$*, Sahand Commun. Math. Anal., 5 (1) (2017), pp. 21-30.
6. F. Pashaie, *On bi-conservative hypersurfaces in the Lorentz-Minkowski 4-space* $\mathbb{E}_1^4$, Sahand Commun. Math. Anal., 20 (3) (2023), pp. 1-18.
7. A.E.H. Love, *A Treatise on The Mathematical Theory of Elasticity*, Cambridge University Press, Cambridge, England, 1927.

8. B. O'Neill, *Elementary Differential Geometry*, Burlington. MA. USA: Academic Press, 1966.
9. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 3, 3rd Editiion, Houston. TX. USA: Publish or perish, 1999.
10. D.J. Struik, *Lectures on Classical Differential Geometry Reading*, MA. USA: Addison-Wesley, 1950.
11. G.O. Tukel and A. Yucesan, *Elastic Curves in A Two-Dimensional Lightlike Cone*, Int. Electron. J. Geom., 8 (2) (2015), pp. 1-8.
12. R. Weinstock, *Calculus of Variations*, McGraw-Hill Book Company. USA, 1952.
13. T.J. Willmore, *An Introduction to Differential Geometry*, Oxford. UK: Clarendon Press, 1959.
14. A. Yucesan, *Elastica on 2-Dimensional Anti-De Sitter Space*, Int. J. Geom. Methods Mod. Phys., 8 (1) (2011), pp. 107-113.

---

[1] Department of Mathematics and Applications, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran.
Email address: samira.latifi5220@gmail.com

[2] Department of Mathematics and Applications, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran.
Email address: abazari@uma.ac.ir

[3] Department of Mathematics and Applications, Faculty of Science, University of Mohaghegh Ardabili, Ardabil, Iran.
Email address: ghasemi@uma.ac.ir