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Generalized Ostrowski-Grüss Like Inequality on Time Scales

Faraz Mehmood[1*], Asif Raza Khan[2] and Muhammad Awais Shaikh[3]

Abstract. In this paper, we present a generalization of the Montgomery Identity to various time scale versions, including the discrete case, continuous case, and the case of quantum calculus. By obtaining this generalization of Montgomery Identity we establish results about the generalization of Ostrowski-Grüss like inequality to the several time scales, namely discrete case, continuous case and the case of quantum calculus. Additionally, we recapture several published results from different authors in various papers, thus unifying the corresponding discrete and continuous versions. Furthermore, we demonstrate the applicability of our derived consequence to the case of quantum calculus.

1. Introduction

Dragomir et. al. have derived the Ostrowski−Grüss like inequality with the help of Grüss & Ostrowski inequalities in 1997, (see [8]) which is stated as below:

Theorem 1.1. *Suppose function $g : I \to \mathbb{R}$ is differentiable in the open interval $I \subset \mathbb{R}$; $\kappa, \ell \in I$ and $\ell > \kappa$. If for real constants γ, Γ; $\gamma \leq g'(\theta) \leq \Gamma$ $\forall \theta \in [\kappa, \ell]$. Then*

$$\left| g(\theta) - \frac{1}{\ell - \kappa} \int_{\kappa}^{\ell} g(\tau) d\tau - \frac{g(\ell) - g(\kappa)}{\ell - \kappa} \left(\theta - \frac{\kappa + \ell}{2} \right) \right| \leq \frac{\ell - \kappa}{4} (\Gamma - \gamma),$$

$\forall \theta \in [\kappa, \ell]$.

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The above inequality has a relationship between Ostrowski [23] & Grüss inequality [22]. It may be used for the estimation of error bounds for some special means and some numerical quadrature formulae. For other closed consequences of similar inequalities (see [7, 14, 16, 24]).

The main objective of the paper is to drive a generalization of Ostrowski-Grüss like inequality to an arbitrary time scales.

2. Essentials of Time Scales

The idea of time scales calculus' theory was initiated and introduced by Hilger (1988) in his PhD thesis [11], supervised by Aulbach, to unify discrete & continuous analysis and expand the theories to cases "in between". Since then, mathematical research in this field has exceeded more than 1000 publications and numerous applications in various scientific fields, including operations research, economics, physics, engineering, statistics, finance and biology [6]. The time scale calculus theory finds applications in most branches of science where dynamic processes are explained by discrete-time or continuous-time models. For an introduction to single-variable time scale calculus and its implementations, we recommend the book by Bohner & Peterson [2].

In 2004, Bohner introduced the variations' calculus on time scale, employing the delta derivative and delta integral [3]. Since then, it has been further developed by several authors in various publications (see [1, 10, 12, 13, 17, 18]). Many classical results of the calculus of variations, such as necessary or sufficient conditions of optimality, have been generalized to arbitrary time scales. The following definitions and propositions are extracted from [2].

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers.

The most important examples of time scales are $\mathbb{R}, \mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^\ell | \ell \in \mathbb{N}_0\}$.

Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(\theta) := \inf\{\tau \in \mathbb{T} | \tau > \theta\}$ for all $\theta \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(\theta) := \sup\{\tau \in \mathbb{T} | \tau < \theta\}$ for all $\theta \in \mathbb{T}$. If $\sigma(\theta) > \theta$, then we say that θ is right-scattered, while if $\rho(\theta) < \theta$ then we say that θ is left-scattered.

Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(\theta) > \theta$, the θ is called right-dense, and if $\rho(\theta) < \theta$ then θ is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

Definition 2.3. Suppose $\theta \in \mathbb{T}$, then two mappings $u, v : \mathbb{T} \to [0, +\infty)$ satisfying

$$u(\theta) := \sigma(\theta) - \theta, \qquad v(\theta) := \theta - \rho(\theta)$$

are called the graininess functions.

We now introduce the set $\mathbb{T}^c$ which is derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum θ, then $\mathbb{T}^c := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^c := \mathbb{T}$. Furthermore for a function $g : \mathbb{T} \to \mathbb{R}$, we define the function $g^\sigma : \mathbb{T} \to \mathbb{R}$ by $g^\sigma(\theta) = g(\sigma(\theta))$ for all $\theta \in \mathbb{T}$.

Definition 2.4. Suppose $g : \mathbb{T} \to \mathbb{R}$ is a function on time scales. Then for $\theta \in \mathbb{T}^c$, we define $g^\Delta(\theta)$ to be the number, if one exists, such that for all $0 < \varepsilon$ there is a neighborhood U of θ such that for all $\tau \in U$

$$\left|g^\sigma(\theta) - g(\tau) - g^\Delta(\theta)(\sigma(\theta) - \tau)\right| \le \varepsilon|\sigma(\theta) - \tau|.$$

We say that g is $\Delta-$differentiable on $\mathbb{T}^c$ provided $g^\Delta(\theta)$ exists for all $\theta \in \mathbb{T}^c$.

Definition 2.5. A function $g : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) provided if it satisfies

- (i) g is continuous at each right-dense point or maximal element of $\mathbb{T}$.
- (ii) The left-sided limit $\lim\limits_{\tau \to \theta-} g(\tau) = g(\theta-)$ exists at each left-dense point θ of $\mathbb{T}$.

Remark 2.6. It follows from [2, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 2.7. A function $G : \mathbb{T} \to \mathbb{R}$ is called a $\Delta-$antiderivative of $g : \mathbb{T} \to \mathbb{R}$ provided $G^\Delta(\theta) = g(\theta)$ holds for all $\theta \in \mathbb{T}^c$. Then the $\Delta-$integral of g is defined by

$$\int_\kappa^\ell g(\theta)\Delta\theta = G(\ell) - G(\kappa).$$

Proposition 2.8. *Suppose g, h are rd-continuous, $\kappa, \ell, l \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

- (i) $\int_\kappa^\ell [\alpha g(\theta) + \beta h(\theta)]\Delta\theta = \alpha \int_\kappa^\ell g(\theta)\Delta\theta + \beta \int_\kappa^\ell h(\theta)\Delta\theta$,
- (ii) $\int_\kappa^\ell g(\theta)\Delta\theta = -\int_\ell^\kappa g(\theta)\Delta\theta,$
- (iii) $\int_\kappa^\ell g(\theta)\Delta\theta = \int_\kappa^l g(\theta)\Delta\theta + \int_l^\ell g(\theta)\Delta\theta$,
- (iv) $\int_\kappa^\ell g(\theta)h^\Delta(\theta)\Delta\theta = (gh)(\ell) - (gh)(\kappa) - \int_\kappa^\ell g^\Delta(\theta)h(\sigma(\theta))\Delta\theta$,
- (v) $\int_\kappa^\kappa g(\theta)\Delta\theta = 0.$

Definition 2.9. Suppose $f_c : \mathbb{T}^2 \to \mathbb{R}$, $c \in \mathbb{N}_0$ is defined by

$$f_0(\theta, \tau) = 1, \quad \forall \tau, \theta \in \mathbb{T}$$

and then recursively by

$$f_{c+1}(\theta,\tau)=\int_{\tau}^{\theta} f_c(s,\tau)\Delta s, \quad \forall \tau,\theta\in\mathbb{T}.$$

The present paper is motivated by the following results: Grüss inequality on time scales and Ostrowski inequality on time scales by Bohner & Matthews and also generalized Grüss inequality and generalized Ostrowski inequality on time scales by authors in papers [19, 20].

In 2007, Bohner & Matthews proved the Grüss inequality on time scales in [4, Theorem 3.1] which is stated as:

Theorem 2.10. *Suppose $\kappa,\ell,\tau\in\mathbb{T}$, $g,h\in C_{rd}$ and $g,h:[\kappa,\ell]\to\mathbb{R}$. Then for*

$$m_1\le g(\tau)\le M_1, \qquad m_2\le h(\tau)\le M_2,$$

we have

$$\left|\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)h^{\sigma}(\tau)\Delta\tau-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau\cdot\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} h^{\sigma}(\tau)\Delta\tau\right| \le\frac{1}{4}(M_1-m_1)(M_2-m_2).$$

In 2008, Bohner & Matthews also proved the Ostrowski inequality on time scales in [5, Theorem 3.5] which is stated as:

Theorem 2.11. *Suppose $\kappa,\ell,\tau,\theta\in\mathbb{T}$, $\kappa<\ell$ and $g:[\kappa,\ell]\to\mathbb{R}$ is differentiable. Then*

$$\left|g(\theta)-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau\right|\le\frac{M}{\ell-\kappa}(f_2(\theta,\kappa)+f_2(\theta,\ell)), \tag{2.1}$$

where

$$M=\sup_{\kappa<\theta<\ell}\left|g^{\Delta}(\theta)\right|.$$

This inequality is sharp in the sense that the right-hand side of (2.1) *cannot be replaced by a smaller one.*

In 2009, Liu et. al. derived an inequality of Ostrowski-Grüss like on time scales and obtained unified results for discrete & continuous versions. They also extended the result to the case of quantum calculus. In this paper, we will establish a generalized inequality of Ostrowski-Grüs like on time scales by Theorem 2.10. Additionally, we will recapture the results from different papers [5, 9, 15, 21]. Moreover, we will apply our established result to the case of quantum calculus.

3. Generalized Ostrowski-Grüss Like Inequality on Time Scales

To prove the our main Theorem 3.9, we require the following generalized Montgomery Identity.

Lemma 3.1 (Generalized Montgomery Identity)**.** *Suppose* $\kappa, \ell, \tau, \theta \in \mathbb{T}$, $\kappa < \ell$ *and* $g : [\kappa, \ell] \to \mathbb{R}$ *is differentiable and* $\lambda \in [0, 1]$*. Then*

$$(1-\lambda)g(\theta)+\lambda\frac{g(\kappa)+g(\ell)}{2} = \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau+\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau, \tag{3.1}$$

where

$$p(\theta,\tau) = \begin{cases} \tau - \left(\kappa + \lambda\frac{\ell-\kappa}{2}\right), & \kappa \le \tau < \theta, \\ \tau - \left(\ell - \lambda\frac{\ell-\kappa}{2}\right), & \theta \le \tau \le \ell. \end{cases}$$

Proof. Apply Proposition 2.8 (iv), we have

$$\begin{aligned} &\int_{\kappa}^{\theta} \left(\tau - \left(\kappa + \lambda\frac{\ell-\kappa}{2}\right)\right) g^{\Delta}(\tau)\Delta\tau \\ &\quad = \left(\theta - \left(\kappa + \lambda\frac{\ell-\kappa}{2}\right)\right) g(\theta) + \lambda\frac{\ell-\kappa}{2}g(\kappa) - \int_{\kappa}^{\theta} g^{\sigma}(\tau)\Delta\tau \end{aligned}$$

and similarly

$$\begin{aligned} &\int_{\theta}^{\ell} \left(\tau - \left(\ell - \lambda\frac{\ell-\kappa}{2}\right)\right) g^{\Delta}(\tau)\Delta\tau \\ &\quad = -\left(\theta - \left(\ell - \lambda\frac{\ell-\kappa}{2}\right)\right) g(\theta) + \lambda\frac{\ell-\kappa}{2}g(\ell) - \int_{\theta}^{\ell} g^{\sigma}(\tau)\Delta\tau. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau + \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau \\ &\quad = \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau + \frac{1}{\ell-\kappa}\Bigg[(\ell-\kappa)(1-\lambda)g(\theta) \\ &\qquad + \lambda\frac{\ell-\kappa}{2}(g(\kappa)+g(\ell)) - \int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau\Bigg] \\ &\quad = (1-\lambda)g(\theta) + \lambda\frac{g(\kappa)+g(\ell)}{2}, \end{aligned}$$

i.e., (3.1) holds. □

Remark 3.2. If put $\lambda = 0$ in Lemma 3.1, then we recapture the Montgomery Identity on time scales which are cited in [5, Lemma 3.1] and

[15, Lemma 1].

Discrete Case:

Corollary 3.3. *We suppose* $\mathbb{T} = \mathbb{Z}$. *Let* $\kappa = 0$, $\ell = n$, $\tau = b$, $\theta = a$ *and* $g(c) = y_c$. *Then*

$$(1-\lambda)y_a + \lambda\left(\frac{y_0 + y_n}{2}\right) = \frac{1}{n}\sum_{b=1}^{n} y_b + \frac{1}{n}\sum_{b=0}^{n-1} p(a,b)\Delta y_b,$$

where $p(a,0) = 0$ *and*

$$p(a,b) = \begin{cases} b - \frac{\lambda n}{2}, & 0 \le b \le a-1, \\ b - n\left(1 - \frac{\lambda}{2}\right), & a \le b \le n-1. \end{cases}$$

as we just require $1 \le a \le n$, $0 \le b \le n-1$.

Remark 3.4. If put $\lambda = 0$ in Corollary 3.3, then we recapture [5, Corollary 3.2] and [9, Theorem 2.1].

Continuous Case:

Corollary 3.5. *We suppose* $\mathbb{T} = \mathbb{R}$. *Then*

$$(1-\lambda)g(\theta)+\lambda\left(\frac{g(\kappa)+g(\ell)}{2}\right) = \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g(\tau)d\tau + \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g'(\tau)d\tau.$$

Remark 3.6. If put $\lambda = 0$ in Corollary 3.5, then we recapture Montgomery identity in the continuous case which may be seen in [21, p. 565] and [9, Theorem 2.1].

Quantum Calculus Case:

Corollary 3.7. *We suppose* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $\kappa = q^m$, $\ell = q^n$ *and* $\tau = q^c$ *with* $m < n$. *Then*

$$\begin{aligned}(1-\lambda)g(\theta) &+ \lambda\left(\frac{g(q^n)+g(q^m)}{2}\right) \\ &= \frac{1}{q^n - q^m}\sum_{c=m}^{n-1} g\left(q^{c+1}\right) + \frac{1}{q^n - q^m}\sum_{c=m}^{n-1}\left[g\left(q^{c+1}\right) - g\left(q^c\right)\right] p(\theta, q^c),\end{aligned}$$

where

$$p\left(\theta, q^c\right) = \begin{cases} q^c - \left(q^m + \lambda\frac{q^n - q^m}{2}\right), & q^m \le q^c < \theta, \\ q^c - \left(q^n - \lambda\frac{q^n - q^m}{2}\right), & \theta \le q^c \le q^n. \end{cases}$$

Remark 3.8. If put $\lambda = 0$ in Corollary 3.7, then we recapture [5, Corollary 3.4].

Theorem 3.9. *Suppose $\kappa, \ell, \tau, \theta \in \mathbb{T}$, $\kappa < \ell$ and $g : [\kappa, \ell] \to \mathbb{R}$ is differentiable and $\lambda \in [0,1]$. If g^{Δ} is rd-continuous and*

$$\gamma \leq g^{\Delta}(\theta) \leq \Gamma, \quad \forall \theta \in [\kappa, \ell].$$

Then we have

$$\begin{aligned}(3.2)\qquad &\Bigg|(1-\lambda)g(\theta) + \lambda\frac{g(\kappa)+g(\ell)}{2} - \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau \\ &\quad - \frac{g(\ell)-g(\kappa)}{(\ell-\kappa)^2}\left(f_2(\theta,\kappa) - f_2(\theta,\ell)\right)\Bigg| \leq \frac{1}{4}(\ell-\kappa)(\Gamma-\gamma),\end{aligned}$$

for all $\theta \in [\kappa, \ell]$.

Proof. By Lemma 3.1, we obtain

$$\begin{aligned}(3.3)\qquad (1-\lambda)g(\theta) + \lambda\frac{g(\kappa)+g(\ell)}{2} - \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\sigma}(\tau)\Delta\tau \\ = \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau.\end{aligned}$$

It is clear that $\forall \theta \in [\kappa, \ell]$ and $\tau \in [\kappa, \ell]$ we have

$$\theta - \ell \leq p(\theta,\tau) \leq \theta - \kappa,$$

using Theorem 2.10 to the mappings $p(\theta,\cdot)$ & $g^{\Delta}(\cdot)$, we obtain

(3.4)

$$\begin{aligned}&\left|\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau - \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} p(\theta,\tau)\Delta\tau \cdot \frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\Delta}(\tau)\Delta\tau\right| \\ &\quad \leq \frac{1}{4}[(\theta-\kappa)-(\theta-\ell)](\Gamma-\gamma) \\ &\quad \leq \frac{1}{4}(\ell-\kappa)(\Gamma-\gamma),\end{aligned}$$

by usual calculation we obtain

$$\begin{aligned}\int_{\kappa}^{\ell} p(\theta,\tau)\Delta\tau &= \int_{\kappa}^{\theta} \tau - \left(\kappa + \lambda\frac{\ell-\kappa}{2}\right)\Delta\tau + \int_{\theta}^{\ell} \tau - \left(\ell - \lambda\frac{\ell-\kappa}{2}\right)\Delta\tau \\ &= \int_{\kappa}^{\theta} \tau - \left(\kappa + \lambda\frac{\ell-\kappa}{2}\right)\Delta\tau - \int_{\ell}^{\theta} \tau - \left(\ell - \lambda\frac{\ell-\kappa}{2}\right)\Delta\tau \\ &= f_2(\theta,\kappa) - f_2(\theta,\ell)\end{aligned}$$

and

$$\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell} g^{\Delta}(\tau)\Delta\tau = \frac{g(\ell)-g(\kappa)}{\ell-\kappa}.$$

By combining (3.3), (3.4) and the above two equalities, we obtain (3.2). □

Remark 3.10. If put $\lambda = 0$ in Theorem 3.9, then we recapture [15, Theorem 4].

Corollary 3.11. *Under all the assumptions of Theorem* 3.9 *with* $\lambda = 1$. *Then*

$$\left|\frac{g(\theta)}{2} + \frac{g(\kappa)+g(\ell)}{4} - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau - \frac{g(\ell)-g(\kappa)}{2(\ell-\kappa)}\left(\theta - \frac{\kappa+\ell}{2}\right)\right| \le \frac{1}{4}(\ell-\kappa)(\Gamma-\gamma),$$

for all $\theta \in [\kappa, \ell]$.

Corollary 3.12. *Under all the assumptions of Theorem* 3.9 *with* $\lambda = 1$. *Then*

$$\left|\frac{g(\kappa)+g(\ell)}{2} - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right| \le \frac{1}{4}(\ell-\kappa)(\Gamma-\gamma),$$

for all $\theta \in [\kappa, \ell]$.

If g^Δ is bounded in the interval $[\kappa, \ell]$ then we have the corollary below:

Corollary 3.13. *Under all the assumptions of Theorem* 3.9*, if* $|g^\Delta(\theta)| \le M$ *for all* $\theta \in [\kappa, \ell]$ *and some positive constant* M. *Then*

$$\left|(1-\lambda)g(\theta) + \lambda\frac{g(\kappa)+g(\ell)}{2} - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau - \frac{g(\ell)-g(\kappa)}{(\ell-\kappa)^2}\left(f_2(\theta,\kappa) - f_2(\theta,\ell)\right)\right| \le \frac{1}{2}(\ell-\kappa)M, \tag{3.5}$$

$\forall\theta \in [\kappa, \ell]$.

Remark 3.14. If put $\lambda = 0$ in Corollary 3.13, then we recapture [15, Corollary 4].

Furthermore, choosing $\theta = (\kappa+\ell)/2$ and $\theta = \ell$, respectively, in (3.2), we have the corollary below.

Corollary 3.15. *Under all the assumptions of Theorem* 3.9. *Then*

$$\left|(1-\lambda)g\left(\frac{\kappa+\ell}{2}\right) + \lambda\frac{g(\kappa)+g(\ell)}{2} - \frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right| \le \frac{1}{4}(\ell-\kappa)(\Gamma-\gamma) \qquad \textit{if } \frac{\kappa+\ell}{2} \in \mathbb{T} \tag{3.6}$$

and

(3.7)
$$\left|(1-\lambda)g(\ell)+\lambda\frac{g(\kappa)+g(\ell)}{2}-\frac{g(\ell)-g(\kappa)}{(\ell-\kappa)^2}f_2(\ell,\kappa)-\frac{1}{\ell-\kappa}\int_\kappa^\ell g^\sigma(\tau)\Delta\tau\right|$$
$$\leq\frac{1}{4}(\ell-\kappa)(\Gamma-\gamma).$$

Remark 3.16. If put $\lambda=0$ in (3.6) and (3.7), then we recapture inequalities (13) and (14) of [15] respectively.

If we implement the Ostrowski-Grüss like inequalety to various time scales, we would obtain some well-known and new results.

Discrete Case:

Corollary 3.17. *We suppose $\mathbb{T}=\mathbb{Z}$. Let $\kappa=0$, $\ell=n$, $\tau=b$, $\theta=a$ and $g(c)=y_c$. With these, since $f_c(\theta,\tau)=\binom{\theta-\tau}{c}$, $\forall\theta,\tau\in\mathbb{Z}$. Therefore,*

$$\begin{aligned}
f_2(\theta,0)&=\binom{\theta-\lambda\frac{\ell}{2}}{2}\\
&=\frac{(\theta-\lambda\frac{\ell}{2})(\theta-\lambda\frac{\ell}{2}-1)}{2}\\
f_2(\theta,n)&=\binom{\theta-n+\lambda\frac{n}{2}}{2}\\
&=\frac{(\theta-n+\lambda\frac{n}{2})(\theta-n+\lambda\frac{n}{2}-1)}{2}
\end{aligned}$$

Thus, we have

$$\left|(1-\lambda)y_a+\lambda\left(\frac{y_0+y_n}{2}\right)-\frac{1}{n}\sum_{b=1}^{n}y_b-\frac{(y_n-y_0)}{n}(1-\lambda)\left(a-\frac{1+n}{2}\right)\right|$$
$$\leq\frac{1}{4}n(\Gamma-\gamma),$$

for all $a=1,\ldots,n$, where $\gamma\leq\Delta y_a\leq\Gamma$.

Remark 3.18. If put $\lambda=0$ in Corollary 3.17, we recapture the [15, Corollary 2].

Continuous Case:

Corollary 3.19. *We suppose $\mathbb{T}=\mathbb{R}$. Then our delta integral is the common Riemann integral. Hence, $f_c(\theta,\tau)=\frac{(\theta-\tau)^2}{2}$, $\forall\theta,\tau\in\mathbb{R}$.*

This guides us to describe the inequality below:

$$\Bigg|(1-\lambda)g(\theta)+\lambda\frac{g(\kappa)+g(\ell)}{2}-\frac{1}{\ell-\kappa}\int_{\kappa}^{\ell}g(\tau)d\tau$$
$$-\frac{g(\ell)-g(\kappa)}{(\ell-\kappa)}(1-\lambda)\left(\theta-\frac{\kappa+\ell}{2}\right)\Bigg|$$
$$\leq\frac{1}{4}(\ell-\kappa)(\Gamma-\gamma),$$

$\forall\theta\in[\kappa,\ell]$, *where* $\gamma\leq g'(\theta)\leq\Gamma$.

Remark 3.20. If put $\lambda=0$ in Corollary 3.19, we recapture the [15, Corollary 1], which is the Ostrowski-Grüss like integral inequality.

Quantum Calculus Case:

Corollary 3.21. *We suppose* $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $\kappa=q^m$ *and* $\ell=q^n$ *with* $m<n$. *In this situation, one has*

$$f_c(\theta,\tau)=\prod_{v=0}^{c-1}\frac{\theta-q^v\tau}{\sum_{u=0}^{v}q^u},\qquad\forall\theta,\tau\in\mathbb{T}.$$

Therefore,

$$f_2(\theta,q^m)=\frac{\left(\theta-q^m-\lambda\frac{q^n-q^m}{2}\right)\left(\theta-q^{m+1}-\lambda\frac{q^n-q^{m+1}}{2}\right)}{1+q},$$
$$f_2(\theta,q^n)=\frac{\left(\theta-q^n+\lambda\frac{q^n-q^m}{2}\right)\left(\theta-q^{n+1}+\lambda\frac{q^{n+1}-q^m}{2}\right)}{1+q}.$$

Then

$$\Bigg|\left(1-\frac{\lambda}{2}\right)\theta-\left(1-\frac{\lambda}{2}\right)(q^{m+1}+q^{n+1})-\lambda\theta\frac{(q^m+q^n)}{(1+q)(q^n-q^m)}$$
$$+\lambda^2\frac{(q^m+q^n)}{4(1+q)}\Bigg|$$
$$\leq\frac{1}{4}(\Gamma-\gamma),$$

where $\forall\ \theta\in[\kappa,\ell]$.

Remark 3.22. If put $\lambda=0$ in Corollary 3.21, we recapture the [15, Corollary 3].

4. Conclusion

We proved the generalized Montgomery Identity for various time scale versions, including the discrete case, continuous case, and the case of quantum calculus. Through this generalization, we obtain results regarding the generalization of the Ostrowski-Grüss-like inequality to several time scales. As a result, our findings unify corresponding discrete and continuous versions of previously proven results by different researchers in various papers [5, 9, 15, 21]. Furthermore, we apply our derived result to the case of quantum calculus.

Limitations and Advantages/Future of the Work.

(i) This work only for the bounded functions.
(ii) Function must be differentiable.
(iii) Function must be rd-continuous.
(iv) We will enhance our work for weighted version.
(v) We will also enhance our work for three steps and five steps kernels.
(vi) We can do same work for L_p space.

Declarations

Availability of data and material

Not applicable.

Conflict-of-interest

The authors declare that they have no conflict-of-interest.

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Authors' Contributions
Faraz Mehmood made the main contribution in conceiving the presented research. Asif R. Khan and Muhammad Awais Shaikh worked jointly on each section while Faraz Mehmood drafted the manuscript. All authors read and approved the final manuscript.

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