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Results via Partial-b Metric and Solution of a Pair of Elliptic Boundary Value Problem

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Abstract. We give a method to establish a fixed point via partial b-metric for multivalued mappings. Since the geometry of multivalued fixed points perform a significant role in numerous real-world problems and is fascinating and innovative, we introduce the notions of fixed circle and fixed disc to frame hypotheses to establish fixed circle/ disc theorems in a space that permits non-zero self-distance with a coefficient more significant than one. Stimulated by the reality that the fixed point theorem is the frequently used technique for solving boundary value problems, we solve a pair of elliptic boundary value problems. Our developments cannot be concluded from the current outcomes in related metric spaces. Examples are worked out to substantiate the validity of the hypothesis of our results.

1. Introduction

Recently fixed point theory has gained remarkable significance in pure and applied mathematics, engineering, computer science, and physical sciences. In 1922, Stefan Banach [3] established a fixed point in a complete metric space of contractive mapping and this conclusion is famous as a Banach contraction principle. Later, this principle has been generalized by many researchers in one way or another in various directions (see, for instance, [1, 9, 11, 21, 23, 40, 42, 45] and references therein). A few of them are Nadler [24], who generalized it considering multivalued contraction; Matthews [22], who introduced with the partial metric; Bhaktin [4] (Czerwick [8]), who introduced with the b-metric, Shukla

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[32], who introduced with the partial b-metric as an enhancement of both b-metric as well as partial metric (see also [35–41]) and Özgür and Taş [25, 26], who initiated the study of the geometry of fixed points. In a metric space (Q, d), if $C(s_0, \mathfrak{r}) = \{s \in Q : d(s, s_0) = \mathfrak{r}\}$ is a circle centred at s_0 and radius $\mathfrak{r}$ and if $\mathcal{T}s = s$ for every $s \in C(s_0, \mathfrak{r})$ then the circle $C(s_0, \mathfrak{r})$ is called the fixed circle [25] of a single-valued self mapping $\mathcal{T}$. However, if $D(s_0, \mathfrak{r}) = \{s \in Q : d(s, s_0) \leq \mathfrak{r}\}$ is a disc centred at s_0 and radius $\mathfrak{r}$ and if $\mathcal{T}s = s$ for every $s \in D(s_0, \mathfrak{r})$, then the disc $D(s_0, \mathfrak{r})$ is the fixed disc [28] of $\mathcal{T}$.

The property of non-zero self-distance of any point with a coefficient greater than one in a partial b-metric makes it significant, different, and more generalized than usual metric space. Consequently, we determine the fixed point in a complete partial b-metric space, which can not be concluded from the outcomes in related metric spaces. Also, we demonstrate in this work that our discontinuous multivalued mappings not only fix one element of the space under consideration but also fix a set of fixed points under appropriate conditions which may include some geometrical shape. Further, encouraged by the reality that the theory of fixed point is frequently used for solving boundary value problems, we solve a pair of elliptic boundary value problems.

2. Preliminaries

Definition 2.1 ([32]). A partial b-metric on a non-empty set Q is a mapping $p : Q \times Q \to \mathbb{R}^+$ satisfying:

(i) $s = r$ iff $p(s, s) = p(s, r) = p(r, r)$;
(ii) $p(s, s) \leq p(s, r)$;
(iii) $p(s, r) = p(r, s)$;
(iv) $\exists$ a real number $\alpha \geq 1$ satisfying $p(s, w) \leq \alpha(p(s, r) + p(r, w)) - p(r, r), s, r, w \in Q$.

Here, (Q, p) is a partial b-metric space for constant $\alpha \geq 1$.

Remark 2.2 ([32]). If p is a partial b-metric $p(s, r) = 0 \Rightarrow s = r$, $s, r \in Q$, however reverse implication need not be valid.

Remark 2.3 ([32]). It is fascinating to notice that each partial metric is also a partial b-metric for $\alpha = 1$ and each b-metric is also a partial b-metric wherein the distance between any point is zero and the same coefficient but the reverse may not be valid.

Example 2.4 ([32]). Let $Q = \mathbb{R}^+$ and $p : Q \times Q \to \mathbb{R}^+$ be defined as

$$p(s, r) = [\max\{s, r\}]^k + |s - r|^k, \quad k > 1 \text{ and } s, r \in Q.$$

Then, p is a partial b-metric for $\alpha = 2^k > 1$, however, it is none of a partial metric or a b-metric.

Definition 2.5 ([32])**.** Let $\{s_n\}$ be a sequence in a partial b-metric space (Q, p) for any constant $\alpha \geq 1$. Then

(i) $\{s_n\}$ converges to s if $\lim_{n\to\infty} p(s_n, s) = p(s, s)$.

(ii) $\{s_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} p(s_n, s_m)$ exists and is finite.

(iii) (Q, p) is a complete partial b-metric if, for every Cauchy sequence $\{s_n\}$ in Q, there exists $s \in Q$ satisfying $\lim_{n,m\to\infty} p(s_n, s_m) = \lim_{n\to\infty} p(s_n, s) = p(s, s)$.

It is fascinating to notice that via partial b-metric, the limit of a convergent sequence need not be unique.

If map $T : Q \to Q$ of a non-empty set Q is a multivalued map, (Q, ρ) is a partial metric space [22] and $CB(Q)$ denotes the set of all -empty, bounded, and closed subsets of (Q, ρ), then an element $s \in Q$ is called a fixed point of T if $s \in Ts$ [24].

Definition 2.6 ([2])**.** Let $P,\ R,\ S \in CB(Q)$ and $H(P, R) = \max\{\delta_\rho(P, R), \delta_\rho(R, P)\}$, where, $\delta_\rho(P, R) = \sup\{\rho(p, R) : p \in P\}$ and $\delta_\rho(R, P) = \sup\{\rho(r, P) : r \in R\}$. We have

(i) $H(P, R) = 0$ implies that $P = R$;

(i) $H(P, P) \leq H(P, R)$;

(iii) $H(P, R) = H(R, P)$;

(iv) $H(P, R) \leq H(P, S) + H(S, R) - \inf_{s\in S} \rho(s, s)$.

$H : CB(Q) \times CB(Q) \to [0, \infty)$ is known as partial Pompeiu-Hausdorff metric induced by ρ.

Lemma 2.7 ([2])**.** *Let P and R be non-empty bounded and closed subsets of a partial metric space (Q, ρ) and $\xi > 1$. Then for any $p \in P$ there exists $r = r(p) \in R$ such that $\rho(p, r) \leq \xi H(P, R)$.*

Lemma 2.8 ([2])**.** *Let P and R be non-empty, closed, and bounded subsets of a partial metric space (Q, ρ) and $\alpha \in P$, then for $\epsilon > 0, \exists\ \beta \in \mathcal{R}$ satisfying $\rho(\alpha, \beta) \leq H(P, \mathcal{R}) + \epsilon$.*

3. Main Results

First, we establish the fixed point of a discontinuous multivalued mapping making use of iterations via the Hausdorff metric.

Theorem 3.1. *Suppose $\mathcal{T} : Q \to CB(Q)$ be multivalued mapping in a complete partial b-metric (Q, p) for $\alpha > 1$, satisfying the conditions*

$$H(\mathcal{T}s, \mathcal{T}r) \leq \alpha_1 p(s, \mathcal{T}s) + \alpha_2 p(r, \mathcal{T}r) + \alpha_3 p(s, \mathcal{T}r) + \alpha_4 p(r, \mathcal{T}s) + \alpha_5 p(s, r) + \alpha_6 \frac{p(s, \mathcal{T}s)(1 + p(s, \mathcal{T}s))}{1 + p(s, r)}; \tag{3.1}$$

$s, r \in Q$ and $\alpha_i \geq 0$, $1 \leq i \leq 6$, with $\alpha_1 + \alpha_2 + 2\alpha\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 < 1$ and $\alpha_3 \geq \alpha_4$. Then, $\mathcal{T}$ has a fixed point.

Proof. Choose any $s_0 \in Q$. Suppose $s_1 \in \mathcal{T}s_0$. By Lemma 2.8, we may select $s_2 \in \mathcal{T}s_1$, satisfying

$$p(s_1, s_2) \leq H(\mathcal{T}s_0, \mathcal{T}s_1) + (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6). \tag{3.2}$$

Using equation (3.1) in equation (3.2), we obtain

$$\begin{aligned}
p(s_1, s_2) &\leq \alpha_1 p(s_0, \mathcal{T}s_0) + \alpha_2 p(s_1, \mathcal{T}s_1) + \alpha_3 p(s_0, \mathcal{T}s_1) + \alpha_4 p(s_1, \mathcal{T}s_0) \\
&\quad + \alpha_5 p(s_0, s_1) + \alpha_6 \frac{p(s_0, \mathcal{T}s_0)(1 + p(s_0, \mathcal{T}s_0))}{1 + p(s_0, s_1)} \\
&\quad + (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6) \\
&\leq \alpha_1 p(s_0, s_1) + \alpha_2 p(s_1, s_2) + \alpha_3 p(s_0, s_2) + \alpha_4 p(s_1, s_1) \\
&\quad + \alpha_5 p(s_0, s_1) + \alpha_6 \frac{p(s_0, s_1)(1 + p(s_0, s_1))}{1 + p(s_0, s_1)} \\
&\quad + (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6) \\
&= (\alpha_1 + \alpha_5 + \alpha_6) p(s_0, s_1) + \alpha_2 p(s_1, s_2) + \alpha_3 p(s_0, s_2) \\
&\quad + \alpha_4 p(s_1, s_1) + (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6) \\
&\leq (\alpha_1 + \alpha_5 + \alpha_6) p(s_0, s_1) + \alpha_2 p(s_1, s_2) \\
&\quad + \alpha_3(\alpha(p(s_0, s_1) + p(s_1, s_2)) - p(s_1, s_1))) + \alpha_4 p(s_1, s_1) \\
&\quad + (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6) \\
&= (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6) p(s_0, s_1) + (\alpha_2 + \alpha\alpha_3) p(s_1, s_2) \\
&\quad - (\alpha_3 - \alpha_4) p(s_1, s_1) + (\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6),
\end{aligned}$$

that is,

$$\begin{aligned}
p(s_1, s_2) &\leq \left(\frac{\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6}{1 - (\alpha_2 + \alpha\alpha_3)}\right) p(s_0, s_1) - \left(\frac{\alpha_3 - \alpha_4}{1 - (\alpha_2 + \alpha\alpha_3)}\right) p(s_1, s_1) \\
&\quad + \frac{(\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6)}{1 - (\alpha_2 + \alpha\alpha_3)} \\
&\leq \left(\frac{\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6}{1 - (\alpha_2 + \alpha\alpha_3)}\right) p(s_0, s_1) + \left(\frac{\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6}{1 - (\alpha_2 + \alpha\alpha_3)}\right).
\end{aligned}$$

Using Lemma 2.8, there exists $s_3 \in \mathcal{T}s_2$ such that

$$p(s_2, s_3) \leq H(\mathcal{T}s_1, \mathcal{T}s_2) + \frac{(\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6)^2}{1 - (\alpha_2 + \alpha\alpha_3)}. \tag{3.3}$$

Using equation (3.1) in equation (3.3), we get

$$p(s_2, s_3) \leq \alpha_1 p(s_1, \mathcal{T}s_1) + \alpha_2 p(s_2, \mathcal{T}s_2) + \alpha_3 p(s_1, \mathcal{T}s_2) + \alpha_4 p(s_2, \mathcal{T}s_1)$$

$$\begin{aligned}
&+ \alpha_5 p(s_1,s_2) + \alpha_6 \frac{p(s_1,\mathcal{T}s_1)(1+p(s_1,\mathcal{T}s_1))}{1+p(s_1,s_2)}\\
&+ \frac{(\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)^2}{1-(\alpha_2+\alpha\alpha_3)}\\
\leq\ & \alpha_1 p(s_1,s_2) + \alpha_2 p(s_2,s_3) + \alpha_3 p(s_1,s_3) + \alpha_4 p(s_2,s_2)\\
&+ \alpha_5 p(s_1,s_2) + \alpha_6 \frac{p(s_1,s_2)(1+p(s_1,s_2))}{1+p(s_1,s_2)}\\
&+ \frac{(\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)^2}{1-(\alpha_2+\alpha\alpha_3)}\\
\leq\ & \alpha_1 p(s_1,s_2) + \alpha_2 p(s_2,s_3) + \alpha_3(\alpha(p(s_1,s_2)+p(s_2,s_3))\\
&- p(s_2,s_2)) + \alpha_4 p(s_2,s_2) + \alpha_5 p(s_1,s_2)\\
&+ \alpha_6 \frac{p(s_1,s_2)(1+p(s_1,s_2))}{1+p(s_1,s_2)} + \frac{(\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)^2}{1-(\alpha_2+\alpha\alpha_3)}\\
=\ & (\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)p(s_1,s_2) + (\alpha_2+\alpha\alpha_3)p(s_2,s_3)\\
&- (\alpha_3-\alpha_4)p(s_2,s_2) + \frac{(\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)^2}{1-(\alpha_2+\alpha\alpha_3)}\\
\leq\ & (\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)p(s_1,s_2) + (\alpha_2+\alpha\alpha_3)p(s_2,s_3)\\
&+ \frac{(\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6)^2}{1-(\alpha_2+\alpha\alpha_3)}.
\end{aligned}$$

Which implies

$$p(s_2,s_3) \leq \left(\frac{\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6}{1-(\alpha_2+\alpha\alpha_3)}\right)p(s_1,s_2) + \left(\frac{\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6}{1-(\alpha_2+\alpha\alpha_3)}\right)^2.$$

Continuing this procedure and using the principle of mathematical induction, we acquire a sequence $\{s_n\}$, where $s_n \in \mathcal{T}s_{n-1}$ and $s_{n+1} \in \mathcal{T}s_n$, such that

$$\begin{aligned}
p(s_n,s_{n+1}) \leq\ & \left(\frac{\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6}{1-(\alpha_2+\alpha\alpha_3)}\right)p(s_{n-1},s_n)\\
&+ \left(\frac{\alpha_1+\alpha\alpha_3+\alpha_5+\alpha_6}{1-(\alpha_2+\alpha\alpha_3)}\right)^n, \quad n \in N.
\end{aligned}$$

Therefore

$$\begin{aligned}
p(s_n,s_{n+1}) &\leq kp(s_{n-1},s_n) + k^n\\
&\leq k\left(kp(s_{n-2},s_{n-1}) + k^{n-1}\right) + k^n\\
&\leq k^2 p(s_{n-2},s_{n-1}) + 2k^n\\
&\ \vdots
\end{aligned}$$

$$p(s_n, s_{n+1}) \leq k^n p(s_0, s_1) + nk^n,$$

where

$$k = \left(\frac{\alpha_1 + \alpha\alpha_3 + \alpha_5 + \alpha_6}{1 - (\alpha_2 + \alpha\alpha_3)} \right).$$

Now,

$$\begin{aligned}
p(s_n, s_{n+p}) &\leq \alpha p(s_n, s_{n+1}) + \alpha^2 p(s_{n+1}, s_{n+2}) + \cdots + \alpha^p p(s_{n+p-1}, s_{n+p}) \\
&\quad - (p(s_{n+1}, s_{n+1}) + p(s_{n+2}, s_{n+2}) + \cdots + p(s_{n+p-1}, s_{n+p-1})) \\
&\leq \alpha[k^n p(s_0, s_1) + nk^n] + \alpha^2[k^{n+1} p(s_0, s_1) + (n+1)k^{n+1}] + \cdots \\
&\quad + \alpha^p[k^{n+p-1} p(s_0, s_1) + (n+p-1)k^{n+p-1}] \\
&\leq [\alpha k^n + \alpha^2 k^{n+1} + \cdots + \alpha^p k^{n+p-1}] p(s_0, s_1) + [\alpha n k^n \\
&\quad + \alpha^2 (n+1) k^{n+1} + \cdots + \alpha^p (n+p-1) k^{n+p-1}] \\
&\leq \left[\sum \alpha k^n \right] p(s_0, s_1) + \left[\sum \alpha n k^n \right].
\end{aligned}$$

Since, $0 < k < 1$, $\sum \alpha k^n$ and $\sum \alpha n k^n$ have the same radius of convergence and $\{s_n\}$ is a Cauchy sequence. Completeness of Q implies that there exist $\nu \in Q$ satisfying

$$\begin{aligned}
\lim_{n \to \infty} p(s_n, \nu) &= \lim_{m,n \to \infty} p(s_n, s_m) \\
&= p(\nu, \nu).
\end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned}
p(\nu, \mathcal{T}\nu) &\leq \alpha(p(\nu, s_{n+1}) + p(s_{n+1}, \mathcal{T}\nu)) - p(s_{n+1}, s_{n+1}) \\
&= \alpha\left(p(\nu, s_{n+1}) + p(\mathcal{T}s_n, \mathcal{T}\nu)\right) - p(s_{n+1}, s_{n+1}) \\
&\leq \alpha p(\nu, s_{n+1}) + \alpha H(\mathcal{T}s_n, \mathcal{T}\nu) - p(s_{n+1}, s_{n+1}) \\
&\leq \alpha p(\nu, s_{n+1}) + \alpha \Bigg(\alpha_1 p(s_n, \mathcal{T}s_n) + \alpha_2 p(\nu, \mathcal{T}\nu) + \alpha_3 p(s_n, \mathcal{T}\nu) \\
&\quad + \alpha_4 p(\nu, \mathcal{T}s_n) + \alpha_5 p(s_n, \nu) + \alpha_6 \frac{p(s_n, \mathcal{T}s_n)(1 + p(s_n, \mathcal{T}s_n))}{1 + p(s_n, \nu)} \Bigg) \\
&\quad - p(s_{n+1}, s_{n+1}) \\
&\leq \alpha p(\nu, s_{n+1}) + \alpha \Bigg(\alpha_1 p(s_n, s_{n+1}) + \alpha_2 p(\nu, \mathcal{T}\nu) + \alpha_3 p(s_n, \mathcal{T}\nu) \\
&\quad + \alpha_4 p(\nu, s_{n+1}) + \alpha_5 p(s_n, \nu) \alpha_6 \frac{p(s_n, s_{n+1})(1 + p(s_n, s_{n+1}))}{1 + p(s_n, \nu)} \Bigg) \\
&\quad - p(s_{n+1}, s_{n+1}).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$(1 - \alpha(\alpha_2 + \alpha_3)) p(\nu, \mathcal{T}\nu) \leq 0.$$

This implies $p(\nu, \mathcal{T}\nu) = 0$, i.e., $\nu \in \mathcal{T}\nu$. $\square$

Example 3.2. Let (Q, p) be the complete partial b-metric space, where $Q = [0, 3]$ and $p(s, r) = [\max\{s, r\}]^2 + |s - r|^2$. Suppose the mapping $T : Q \to CB(Q)$ is defined as

$$\mathcal{T}s = \begin{cases} [0, 1], & 0 \leq s \leq 2 \\ \{2\}, & 2 \leq s \leq 3. \end{cases}$$

Taking $\alpha_1 = 0$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{20}$, $\alpha_4 = 0$, $\alpha_5 = \frac{3}{10}$, $\alpha_6 = \frac{7}{20}$, $s = 2^2 > 1$, $\alpha_1 + \alpha_2 + 2\alpha\alpha_3 + \alpha_4 + \alpha_5 = \frac{19}{20} < 1$ and $\alpha_3 \geq \alpha_4$. Now, we have the following cases:

Case I: when $s, r \in [0, 2)$,

$$H(\mathcal{T}s, \mathcal{T}r) = 0 \leq 0 + 8\alpha_2 + 8\alpha_3. + 0 + 8\alpha_5 + 8\alpha_6 \leq \frac{48}{5}.$$

Case II: when $s \in [0, 2)$ and $r \in [2, 3]$,

$$H(\mathcal{T}s, \mathcal{T}r) = 2 \leq 0 + 8\alpha_2 + 18\alpha_3 + 0 + 18\alpha_5 + \frac{6}{19}\alpha_6 \leq \frac{1598}{190}.$$

Case III: when $s \in [2, 3]$ and $r \in [0, 2)$,

$$H(\mathcal{T}s, \mathcal{T}r) = 2 \leq 0 + 2\alpha_2 + 8\alpha_3 + 0 + 18\alpha_5 + \frac{72}{19}\alpha_6 \leq \frac{1449}{190}.$$

Case IV: when $s, r \in [2, 3]$,

$$H(\mathcal{T}s, \mathcal{T}r) = 0 \leq 0 + 2\alpha_2 + 2\alpha_3 + 0 + 18\alpha_5 + \frac{6}{19}\alpha_6 \leq \frac{1161}{190}.$$

Thus, all the hypotheses of the Theorem 3.1 are validated and $\mathbf{F} = [0, 1] \cup \{2\}$ are the fixed points of a discontinuous multivalued mapping $\mathcal{T}$. Further, one may verify that $p(s, s), s \in \mathbf{F}$ is not equal to zero.

Next, we determine the common fixed point of discontinuous multivalued mappings using of the iterations via the Hausdorff metric. Notably, the containment of range space of the underlying pair of mappings is not exploited, which is frequently used to establish a common fixed point in numerous settings.

Theorem 3.3. *Suppose $\mathcal{T}, \mathcal{S} : Q \to CB(Q)$ be multivalued mappings of a complete partial b-metric (Q, p) for $\alpha > 1$, satisfying the conditions:*
(3.5)
$$H(\mathcal{T}s, \mathcal{S}r) \leq \alpha_1 p(s, \mathcal{T}s) + \alpha_2 p(r, \mathcal{S}r) + \alpha_3 p(s, \mathcal{S}r) + \alpha_4 p(r, \mathcal{T}s) + \alpha_5 p(s, r),$$

$s, r \in Q$ and $\alpha_i \geq 0$, $1 \leq i \leq 5$, with $(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4)(\alpha^2 + \alpha) + 2\alpha\alpha_5 < 2$, $\sum_{i=1}^{5} \alpha_i = 1$ and $\alpha_3 \geq \alpha_4$, Then, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

Proof. Choose $s_0 \in Q$ and let $s_1 \in \mathcal{T}s_0$ and By Lemma 2.8, we select $s_2 \in \mathcal{S}s_1$ such that

$$\begin{aligned}
p(s_1,s_2) &\leq H(\mathcal{T}s_0,\mathcal{S}s_1)+(\alpha_1+\alpha_5+\alpha\alpha_3)\\
&\leq \alpha_1 p(s_0,\mathcal{T}s_0)+\alpha_2 p(s_1,\mathcal{S}s_1)+\alpha_3 p(s_0,\mathcal{S}s_1)+\alpha_4 p(s_1,\mathcal{T}s_0)\\
&\quad +\alpha_5 p(s_0,s_1)+(\alpha_1+\alpha_5+\alpha\alpha_3)\\
&\leq \alpha_1 p(s_0,s_1)+\alpha_2 p(s_1,s_2)+\alpha_3 p(s_0,s_2)+\alpha_4 p(s_1,s_1)\\
&\quad +\alpha_5 p(s_0,s_1)+(\alpha_1+\alpha_5+\alpha\alpha_3)\\
&\leq \alpha_1 p(s_0,s_1)+\alpha_2 p(s_1,s_2)+\alpha_3\alpha\left(p(s_0,s_1)+p(s_1,s_2)\right)\\
&\quad -\alpha_3 p(s_1,s_1)+\alpha_4 p(s_1,s_1)+\alpha_5 p(s_0,s_1)+(\alpha_1+\alpha_5+\alpha\alpha_3).
\end{aligned}$$

On solving, we get

$$\begin{aligned}
(1-(\alpha_2+\alpha\alpha_3))p(s_1,s_2) &\leq (\alpha_1+\alpha\alpha_3+\alpha_5)p(s_0,s_1)-\alpha_3 p(s_1,s_1)\\
&\quad +\alpha_4 p(s_1,s_1)+(\alpha_1+\alpha_5+\alpha\alpha_3).
\end{aligned} \tag{3.6}$$

By symmetry, we have

$$\begin{aligned}
p(s_1,s_2) &= p(s_2,s_1)\\
&= p(\mathcal{S}s_1,Ts_0)\\
&\leq H(\mathcal{S}s_1,\mathcal{T}s_0)+(\alpha_2+\alpha\alpha_4+\alpha_5)\\
&\leq \alpha_1 p(s_1,\mathcal{T}s_1)+\alpha_2 p(s_0,\mathcal{S}s_0)+\alpha_3 p(s_1,\mathcal{S}s_0)+\alpha_4 p(s_0,\mathcal{T}s_1)\\
&\quad +\alpha_5 p(s_1,s_0)+(\alpha_2+\alpha\alpha_4+\alpha_5)\\
&\leq \alpha_1 p(s_1,s_2)+\alpha_2 p(s_0,s_1)+\alpha_3 p(s_1,s_1)+\alpha_4 p(s_0,s_2)\\
&\quad +\alpha_5 p(s_1,s_0)+(\alpha_2+\alpha\alpha_4+\alpha_5)\\
&\leq \alpha_1 p(s_1,s_2)+\alpha_2 p(s_0,s_1)+\alpha_3 p(s_1,s_1)+\alpha_4\alpha(p(s_0,s_1)\\
&\quad +p(s_1,s_2))-\alpha_4 p(s_1,s_1)+\alpha_5 p(s_1,s_0)(\alpha_2+\alpha\alpha_4+\alpha_5).
\end{aligned}$$

On solving, we get

$$\begin{aligned}
(1-(\alpha_1+\alpha\alpha_4))p(s_1,s_2) &\leq (\alpha_2+\alpha\alpha_4+\alpha_5)p(s_0,s_1)+\alpha_3 p(s_1,s_1)\\
&\quad -\alpha_4 p(s_1,s_1)+\alpha_2+\alpha\alpha_4+\alpha_5.
\end{aligned} \tag{3.7}$$

By adding and solving equations (3.6) and (3.7), we get

$$\begin{aligned}
p(s_1,s_2) &\leq \left(\frac{\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4+2\alpha_5}{2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}\right)p(s_0,s_1)\\
&\quad +\left(\frac{\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4+2\alpha_5}{2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}\right).
\end{aligned} \tag{3.8}$$

Now consider $s_3 \in \mathcal{T}s_2$, we have

$$\begin{aligned}
p(s_2,s_3) &\leq H(\mathcal{T}s_1,\mathcal{S}s_2) + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)} \\
&\leq \alpha_1 p(s_1,\mathcal{T}s_1) + \alpha_2 p(s_2,\mathcal{S}s_2) + \alpha_3 p(s_1,\mathcal{S}s_2) + \alpha_4 p(s_2,\mathcal{T}s_1) \\
&\quad + \alpha_5 p(s_1,s_2) + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)} \\
&= \alpha_1 p(s_1,s_2) + \alpha_2 p(s_2,s_3) + \alpha_3 p(s_1,s_3) + \alpha_4 p(s_2,s_2) \\
&\quad + \alpha_5 p(s_1,s_2) + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)} \\
&\leq \alpha_1 p(s_1,s_2) + \alpha_2 p(s_2,s_3) + \alpha_3\alpha(p(s_1,s_2)+p(s_2,s_3)) \\
&\quad - \alpha_3 p(s_2,s_2) + \alpha_4 p(s_2,s_2) \\
&\quad + \alpha_5 p(s_1,s_2) + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}.
\end{aligned}$$

On solving, we get

(3.9)

$$\begin{aligned}
(1-(\alpha_2+\alpha\alpha_3))p(s_2,s_3) &\leq (\alpha_1+\alpha\alpha_3+\alpha_5)p(s_1,s_2) - \alpha_3 p(s_2,s_2) \\
&\quad + \alpha_4 p(s_2,s_2) + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}.
\end{aligned}$$

By symmetry, we have

$$\begin{aligned}
p(s_2,s_3) &= p(s_3,s_2) \\
&= p(Ss_2,Ts_1) \\
&\leq H(\mathcal{S}s_2,\mathcal{T}s_1) + (\alpha_1+\alpha_5+\alpha\alpha_3)^2 \\
&\leq \alpha_1 p(s_2,\mathcal{S}s_2) + \alpha_2 p(s_1,\mathcal{T}s_1) + \alpha_3 p(s_2,\mathcal{T}s_1) + \alpha_4 p(s_1,\mathcal{S}s_2) \\
&\quad + \alpha_5 p(s_2,s_1) + \frac{(\alpha_2+\alpha_5+\alpha\alpha_4)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)} \\
&\leq \alpha_1 p(s_2,s_3) + \alpha_2 p(s_1,s_2) + \alpha_3 p(s_2,s_2) + \alpha_4 p(s_1,s_3) \\
&\quad + \alpha_5 p(s_2,s_1) + \frac{(\alpha_2+\alpha_5+\alpha\alpha_4)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)} \\
&\leq \alpha_1 p(s_2,s_3) + \alpha_2 p(s_1,s_2) + \alpha_3 p(s_2,s_2) + \alpha_4\alpha(p(s_1,s_2) \\
&\quad + p(s_2,s_3)) - \alpha_4 p(s_2,s_2) + \alpha_5 p(s_2,s_1) \\
&\quad + \frac{(\alpha_2+\alpha_5+\alpha\alpha_4)^2}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}.
\end{aligned}$$

On solving, we get

(3.10)

$$(1-(\alpha_1+\alpha\alpha_4))p(s_2,s_3) \leq (\alpha_2+\alpha\alpha_4+\alpha_5)p(s_1,s_2)+\alpha_3 p(s_2,s_2) - \alpha_4 p(s_2,s_2)+\frac{(\alpha_2+\alpha\alpha_4+\alpha_5)^2}{2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}.$$

On adding and solving equations (3.9) and (3.10), we get

$$p(s_2,s_3) \leq \left(\frac{\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4+2\alpha_5}{2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}\right)p(s_1,s_2) + \left(\frac{\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4+2\alpha_5}{2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)}\right)^2. \tag{3.11}$$

Continuing like this and using the principle of mathematical induction, we get a sequence $\{s_n\}$, where $s_{2n+1} \in \mathcal{T}s_{2n}$, $s_{2n+2} \in \mathcal{S}s_{2n+1}$, satisfying.

$$\begin{aligned}
p(s_{2n+1},s_{2n+2}) &\leq H(\mathcal{T}s_{2n},\mathcal{S}s_{2n+1})+\frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^{2n+1}}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)^{2n}}\\
&\leq \alpha_1 p(s_{2n},\mathcal{T}s_{2n})+\alpha_2 p(s_{2n+1},\mathcal{S}s_{2n+1})+\alpha_3 p(s_{2n},\mathcal{S}s_{2n+1})\\
&\quad + \alpha_4 p(s_{2n+1},\mathcal{T}s_{2n})+\alpha_5 p(s_{2n},s_{2n+1})\\
&\quad + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^{2n+1}}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)^{2n}}\\
&\leq \alpha_1 p(s_{2n},s_{2n+1})+\alpha_2 p(s_{2n+1},s_{2n+2})+\alpha_3 p(s_{2n},s_{2n+2})\\
&\quad + \alpha_4 p(s_{2n+1},s_{2n+1})+\alpha_5 p(s_{2n},s_{2n+1})\\
&\quad + \frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^{2n+1}}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)^{2n}}\\
&\leq \alpha_1 p(s_{2n},s_{2n+1})+\alpha_2 p(s_{2n+1},s_{2n+2})+\alpha_3 s(p(s_{2n},s_{2n+1})\\
&\quad + p(s_{2n+1},s_{2n+2}))-\alpha_3 p(s_{2n+1},s_{2n+1})+\alpha_4 p(s_{2n+1},s_{2n+1})\\
&\quad + \alpha_5 p(s_{2n},s_{2n+1})+\frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^{2n+1}}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)^{2n}}.
\end{aligned}$$

On solving, we get

$$\begin{aligned}
&(1-(\alpha_2+\alpha\alpha_3))p(s_{2n+1},s_{2n+2})\\
&\quad\leq (\alpha_1+\alpha\alpha_3+\alpha_5)p(s_{2n},s_{2n+1})-\alpha_3 p(s_{2n+1},s_{2n+1})\\
&\quad\quad + \alpha_4 p(s_{2n+1},s_{2n+1})+\frac{(\alpha_1+\alpha_5+\alpha\alpha_3)^{2n+1}}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4)^{2n}}.
\end{aligned}\tag{3.12}$$

By Symmetry, we have

$$\begin{aligned}
p(s_{2n+1},s_{2n+2}) &= p(s_{2n+2},s_{2n+1})\\
&\leq H(\mathcal{S}s_{2n+1},\mathcal{T}s_{2n})+\frac{(\alpha_2+\alpha\alpha_4+\alpha_5)^{2n+1}}{(2-(\alpha_1+\alpha_2+\alpha\alpha_3+\alpha\alpha_4))^{2n}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_1 p(s_{2n+1}, \mathcal{S}s_{2n+1}) + \alpha_2 p(s_{2n}, \mathcal{T}s_{2n}) + \alpha_3 p(s_{2n+1}, \mathcal{T}s_{2n}) \\
&\quad + \alpha_4 p(s_{2n}, \mathcal{S}s_{2n+1}) + \alpha_5 p(s_{2n+1}, s_{2n}) \\
&\quad + \frac{(\alpha_2 + \alpha\alpha_4 + \alpha_5)^{2n+1}}{(2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4))^{2n}} \\
&\leq \alpha_1 p(s_{2n+1}, s_{2n+2}) + \alpha_2 p(s_{2n}, s_{2n+1}) + \alpha_3 p(s_{2n+1}, s_{2n+1}) \\
&\quad + \alpha_4 p(s_{2n}, s_{2n+2}) + \alpha_5 p(s_{2n+1}, s_{2n}) \\
&\quad + \frac{(\alpha_2 + \alpha\alpha_4 + \alpha_5)^{2n+1}}{(2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4))^{2n}} \\
&\leq \alpha_1 p(s_{2n+1}, s_{2n+2}) + \alpha_2 p(s_{2n}, s_{2n+1}) + \alpha_3 p(s_{2n+1}, s_{2n+1}) \\
&\quad + \alpha_4 \alpha (p(s_{2n}, s_{2n+1}) + p(s_{2n+1}, s_{2n+2})) - \alpha_4 p(s_{2n+1}, s_{2n+1}) \\
&\quad + \alpha_5 p(s_{2n+1}, s_{2n}) + \frac{(\alpha_2 + \alpha\alpha_4 + \alpha_5)^{2n+1}}{(2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4))^{2n}}.
\end{aligned}
$$

On solving, we get

(3.13)

$$
\begin{aligned}
(1 - (\alpha_1 + \alpha\alpha_4))p(s_{2n+1}, s_{2n+2}) \leq\ &(\alpha_2 + \alpha\alpha_4 + \alpha_5)p(s_{2n+1}, s_{2n}) \\
&+ \alpha_3 p(s_{2n+2}, s_{2n+2}) - \alpha_4 p(s_{2n+1}, s_{2n+1}) \\
&+ \frac{(\alpha_2 + \alpha\alpha_4 + \alpha_5)^{2n+1}}{(2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4))^{2n}}.
\end{aligned}
$$

By adding and solving equations (3.12) and (3.13), we get

$$
\begin{aligned}
p(s_{2n+1}, s_{2n+2}) \leq\ &\left(\frac{\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4 + 2\alpha_5}{2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4)}\right) p(s_{2n+1}, s_{2n}) \\
&+ \left(\frac{\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4 + 2\alpha_5}{2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4)}\right)^{2n+1}.
\end{aligned} \tag{3.14}
$$

Therefore,

$$
\begin{aligned}
p(s_n, s_{n+1}) &\leq k p(s_{n-1}, s_n) + k^n \\
&\leq k^2 p(s_{n-2}, s_{n-1}) + k^{n-1} + 2k^n \\
&\ \ \vdots \\
p(s_n, s_{n+1}) &\leq k^n p(s_0, s_1) + nk^n,
\end{aligned}
$$

where

$$
k = \left(\frac{\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4 + 2\alpha_5}{2 - (\alpha_1 + \alpha_2 + \alpha\alpha_3 + \alpha\alpha_4)}\right).
$$

Now,

$$
\begin{aligned}
p(s_n, s_{n+p}) \leq\ &\alpha p(s_n, s_{n+1}) + \alpha^2 p(s_{n+1}, s_{n+2}) + \cdots + \alpha^p p(s_{n+p-1}, s_{n+p}) \\
&- (p(s_{n+1}, s_{n+1} + p(s_{n+2}, s_{n+2}) + \cdots + p(s_{n+p-1}, s_{n+p-1}))
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha[k^n p(s_0,s_1)+nk^n]+\alpha^2[k^{n+1}p(s_0,s_1)+(n+1)k^{n+1}]\\
&\quad+\cdots+\alpha^p[k^{n+p-1}p(s_0,s_1)+(n+p-1)k^{n+p-1}]\\
&=[\alpha k^n+\alpha^2k^{n+1}+\cdots+\alpha^pk^{n+p-1}]p(s_0,s_1)+[\alpha nk^n\\
&\quad+\alpha^2(n+1)k^{n+1}+\cdots+\alpha^p(n+p-1)k^{n+p-1}]\\
&=\left[\sum \alpha k^n\right]p(s_0,s_1)+\left[\sum \alpha nk^n\right].
\end{aligned}$$

Since, $0<k<1$, $\sum \alpha k^n$ and $\sum \alpha nk^n$ have the same radius of convergence. So, $\{s_n\}$ is a Cauchy sequence. Completeness of Q implies that $\exists\ \nu \in Q$ satisfying

$$\begin{aligned}
\lim_{n\to\infty} p(s_n,\nu) &= \lim_{m,n\to\infty} p(s_n,s_m)\\
&= p(\nu,\nu).
\end{aligned} \tag{3.15}$$

We shall demonstrate that the common fixed point of $\mathcal{T}$ and $\mathcal{S}$ is ν.

$$\begin{aligned}
p(\nu,\mathcal{S}\nu) &\leq \alpha(p(\nu,s_{2n+2})+p(s_{2n+2},S\nu))-p(s_{2n+2},s_{2n+2})\\
&\leq \alpha(p(\nu,s_{2n+2})+H(s_{2n+2},\mathcal{S}\nu))-p(s_{2n+2},s_{2n+2})\\
&\leq \alpha(p(\nu,s_{2n+2})+H(\mathcal{T}s_{2n+1},S\nu))-p(s_{2n+2},s_{2n+2})\\
&\leq \alpha(p(\nu,s_{2n+2})+\alpha_1p(s_{2n+1},\mathcal{T}s_{2n+1})+\alpha_2p(\nu,\mathcal{S}\nu)\\
&\quad+\alpha_3p(s_{2n+1},\mathcal{S}\nu)+\alpha_4p(\nu,\mathcal{T}s_{2n+1})+\alpha_5p(s_{2n+1},\nu))\\
&\quad-p(s_{2n+2},s_{2n+2})\\
&\leq \alpha(p(\nu,s_{2n+2})+\alpha_1p(s_{2n+1},s_{2n+2})+\alpha_2p(\nu,S\nu)\\
&\quad+\alpha_3p(s_{2n+1},S\nu)+\alpha_4p(\nu,s_{2n+2})+\alpha_5p(s_{2n+1},\nu))\\
&\quad-p(s_{2n+2},s_{2n+2}).
\end{aligned}$$

Letting $n\to\infty$, we get

$$\begin{aligned}
p(\nu,\mathcal{S}\nu) \leq{}& \alpha(p(\nu,\nu)+\alpha_1p(\nu,\nu)+\alpha_2p(\nu,\mathcal{S}\nu)+\alpha_3p(\nu,S\nu)+\alpha_4p(\nu,\nu)\\
&+\alpha_5p(\nu,\nu))-p(\nu,\nu)
\end{aligned} \tag{3.16}$$

or

$$\begin{aligned}
p(\nu,\mathcal{S}\nu)-\alpha\alpha_2p(\nu,\mathcal{S}\nu)-\alpha\alpha_3p(\nu,\mathcal{S}\nu) \leq{}& \alpha(p(\nu,\nu)+\alpha_1p(\nu,\nu)+\alpha_4p(\nu,\nu)\\
&+\alpha_5p(\nu,\nu))-p(\nu,\nu)
\end{aligned} \tag{3.17}$$

or

$$(1-\alpha(\alpha_2+\alpha_3))p(\nu,\mathcal{S}\nu)\leq 0.$$

As $(1-\alpha(\alpha_2+\alpha_3))<0$, we have $\nu\in\mathcal{S}\nu$.

Similarly, we get $\nu\in\mathcal{T}\nu$. Hence, $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point. □

Example 3.4. Let (Q, p) be the complete partial b-metric space, where $Q = [0, 4]$ and $p(s, r) = [\max\{s, r\}]^2 + |s - r|^2$. Suppose the mappings $\mathcal{S}, \mathcal{T} : Q \to CB(Q)$ be defined as

$$\mathcal{T}(s) = \begin{cases} [0, 1], & 0 \leq s \leq 2, \\ \{2\}, & 2 \leq s \leq 3 \end{cases}$$

and

$$\mathcal{S}(s) = \begin{cases} [1, 4], & 0 \leq s \leq 2, \\ \{2\}, & 2 \leq s \leq 3. \end{cases}$$

Taking $\alpha_1 = 0$, $\alpha_2 = \frac{26}{28}$, $\alpha_3 = \frac{1}{28}$, $\alpha_4 = 0$, $\alpha_5 = \frac{1}{28}$, $\alpha = 2^2 > 1$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$, $\alpha_3 \geq \alpha_4$ and $(\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4)(\alpha^2 + \alpha) + 2\alpha\alpha_5 = \frac{54}{28} < 2$. Now, we have the following cases:
Case I: when $s, r \in [0, 2)$,

$$H(\mathcal{T}s, \mathcal{S}r) = 4 \leq 0 + 8\alpha_2 + 8\alpha_3 + 0 + 8\alpha_5 \leq 8.$$

Case II: when $s \in [0, 2)$ and $r \in [2, 4]$,

$$H(\mathcal{T}s, \mathcal{S}r) = 2 \leq 0 + 8\alpha_2 + 8\alpha_3 + 0 + 32\alpha_5 \leq \frac{248}{28}.$$

Case III: when $s \in [2, 4]$ and $r \in [0, 2)$,

$$H(\mathcal{T}s, \mathcal{S}r) = 2 \leq 0 + 32\alpha_2 + 8\alpha_3 + 0 + 32\alpha_5 \leq \frac{872}{28}.$$

Case IV: when $s, r \in [2, 3]$,

$$H(\mathcal{T}s, \mathcal{S}r) = 0 \leq 0 + 8\alpha_2 + 8\alpha_3 + 0 + 32\alpha_5 \leq \frac{248}{28}.$$

Thus, all the hypotheses of the Theorem 3.3 are validated and 1 and 2 are common fixed points of multivalued mappings $\mathcal{T}$ and $\mathcal{S}$. Further, one may verify that $p(1, 1)$ and $p(2, 2)$ are not equal to zero.

Remark 3.5. The uniqueness of fixed points has been a significant area of research for more than a century. However, in the real world, there may arise situations when the fixed point is not necessarily unique. Mappings having non-unique fixed points may fix some geometric figures and find applications in numerous real-life problems. We call such a figure, a fixed figure and it arises naturally. For more work in this direction, we refer to [5, 10–17, 23, 25–30, 33–44], and so on.

Remark 3.6. Noticeably, Theorem 3.1 and Theorem 3.3 are appropriate generalizations of Nadler [24] and many others. Also, it is a generalization, improvement, and extension of some celebrated and recent conclusions in the literature. For instance, Banach [3], Chatterjea [6], Ćirić [7], Kannan [18, 19], Kirk [20], Reich [31], and many others to the set-valued case. Further, on taking distinct values of $\alpha_i,\ i = 1, 2, \ldots, 5$, we achieve some conclusions which are generalizations of some celebrated and recent contractions existing in the literature.

Following, Joshi et al. [13] and Tomar et al. [42] (see also, [25] and [28] for the standard metric version), we define a circle as well as a disc in partial $b-$metric space as:

Definition 3.7. A circle with a centre at the point $s_0 \in Q$ and radius $\mathfrak{r}$ in a partial $b-$metric space (Q, p) is described as

$$C(s_0, \mathfrak{r}) = \{s \in Q : p(s_0, s) = \mathfrak{r} + \rho(s_0, s_0)\}, \quad s_0 \in Q, \mathfrak{r} \in (0, \infty). \tag{3.18}$$

If the sign of 'equality' is replaced by 'less than or equal to sign' in (3.18) then the above definition reduces to that of the disc and we write

$$D(s_0, \mathfrak{r}) = \{s \in Q : p(s_0, s) \leq \mathfrak{r} + \rho(s_0, s_0)\}, \quad s_0 \in Q, \mathfrak{r} \in (0, \infty). \tag{3.19}$$

Geometrically, a circle or a disc in any partial $b-$metric space may not be similar to the circle or a disc described in a Euclidean space.

Theorem 3.8. *Let $C(s_0, \mathfrak{r})/D(s_0, \mathfrak{r})$ be a circle/ disc in a partial $b-$metric space (Q, p). Suppose there exists a multivalued mapping $\mathcal{T}$:$Q \to CB(Q)$ satisfying the subsequent hypotheses:*

(i)

$$\begin{aligned} p(s, \mathcal{T}s) \leq{} & \alpha_1 p(s, \mathcal{T}s) + \alpha_2 p(s_0, \mathcal{T}s_0) + \alpha_3 p(s, \mathcal{T}s_0) + \alpha_4 p(s_0, \mathcal{T}s) \\ & + \alpha_5 p(s, s_0) + \alpha_6 \frac{p(s, \mathcal{T}s)(1 + p(s, s))}{1 + p(s, s_0)}, \quad s \in C(s_0, \mathfrak{r})/D(s_0, \mathfrak{r}); \end{aligned}$$

(ii) $\mathfrak{r} + p(s_0, s_0) \leq p(s, \mathcal{T}s)$ *for* $s \neq \mathcal{T}s$,

where, $s_0, s \in Q$ *and* $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha\alpha_4 + \alpha_5 + \alpha_6 < 1$. *Then,* $C(s_0, \mathfrak{r})/D(s_0, \mathfrak{r})$ *is a fixed circle/ fixed disc of* $\mathcal{T}$.

Proof. Suppose $\mathcal{T}s_0 \neq s_0$. Now

$$\begin{aligned} p(s_0, \mathcal{T}s_0) \leq{} & \alpha_1 p(s_0, \mathcal{T}s_0) + \alpha_2 p(s_0, \mathcal{T}s_0) + \alpha_3 p(s_0, \mathcal{T}s_0) + \alpha_4 p(s_0, \mathcal{T}s_0) \\ & + \alpha_5 p(s_0, s_0) + \alpha_6 \frac{p(s_0, \mathcal{T}s_0)(1 + p(s_0, s_0))}{1 + p(s_0, s_0)} \\ \leq{} & (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) p(s_0, \mathcal{T}s_0) \\ <{} & p(s_0, \mathcal{T}s_0), \end{aligned}$$

a contradiction. Hence, $\mathcal{T}s_0 = s_0$.
Again suppose $s \in C(s_0, \mathfrak{r})/D(s_0, \mathfrak{r})$ such that $\mathcal{T}s \neq s$. Now, using (i)

$$\begin{aligned}
p(s,\mathcal{T}s) &\leq \alpha_1 p(s,\mathcal{T}s) + \alpha_2 p(s_0,\mathcal{T}s_0) + \alpha_3 p(s,\mathcal{T}s_0) + \alpha_4 p(s_0,\mathcal{T}s) \\
&\quad + \alpha_5 p(s,s_0) + \alpha_6 \frac{p(s,\mathcal{T}s)(1+p(s,s))}{1+p(s,s_0)} \\
&= \alpha_1 p(s,\mathcal{T}s) + \alpha_2 p(s_0,s_0) + \alpha_3 p(s,s_0) \\
&\quad + \alpha_4 p(s_0,\mathcal{T}s) + \alpha_5 p(s,s_0) + \alpha_6 \frac{p(s,\mathcal{T}s)(1+p(s,s))}{1+p(s,s_0)} \\
&\leq \alpha_1 p(s,\mathcal{T}s) + \alpha_2 p(s_0,s_0) + \alpha_3(\mathfrak{r} + p(s_0,s_0)) \\
&\quad + \alpha_4 \left[\alpha\left(p(s_0,s) + p(s,\mathcal{T}s)\right) - p(s,s)\right] \\
&\quad + \alpha_5(\mathfrak{r} + p(s_0,s_0)) + \alpha_6 p(s,\mathcal{T}s) \\
&\leq (\alpha_2 + \alpha_3 + \alpha\alpha_4 + \alpha_5)(\mathfrak{r} + p(s_0,s_0)) + (\alpha_1 + \alpha\alpha_4 + \alpha_6)p(s,\mathcal{T}s)
\end{aligned}$$

or

$$(1 - \alpha_1 - \alpha\alpha_4 - \alpha_6)p(s,\mathcal{T}s) \leq (\alpha_2 + \alpha_3 + \alpha\alpha_4 + \alpha_5)(\mathfrak{r} + p(s_0,s_0))$$

or

$$p(s,\mathcal{T}s) \leq \frac{\alpha_2 + \alpha_3 + \alpha\alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha\alpha_4 - \alpha_6}\left(\mathfrak{r} + p(s_0,s_0)\right) < \mathfrak{r} + p(s_0,s_0),$$

that is, $p(s,\mathcal{T}s) < \mathfrak{r} + p(s_0,s_0)$, a contradiction. Hence, $\mathcal{T}s = s$, $s \in C(s_0,\mathfrak{r})/D(s_0,\mathfrak{r})$. □

Example 3.9. Let (Q,p) be the partial b-metric space, where $Q = [0,10]$ and $p(s,r) = [\max\{s,r\}]^2 + |s-r|^2$. Suppose the mapping $T : Q \to CB(Q)$ is defined as

$$\mathcal{T}(s) = \begin{cases} [0,6], & 0 \leq s < 7 \\ \\ \{0\}, & 7 \leq s \leq 10. \end{cases}$$

Define a circle $C(2,2)$ having a centre at 2 with a radius of 2 units as

$$\begin{aligned}
C(2,2) &= \{s \in Q : p(2,s) = 2 + p(2,2)\} \\
&= \{s \in Q : [\max\{2,s\}]^2 + |2-s|^2 = 6\}.
\end{aligned}$$

Now, we have the following cases:
Case I: when $s \leq 2$, $4 + (2-s)^2 = 6$, then $-2 + \sqrt{6} \in C(2,2)$.
Case II: when $s > 2$, $s^2 + (2-s)^2 = 6$, then $2 + \sqrt{14} \in Q$,
that is, $C(2,2) = \left\{-2 + \sqrt{6}, 2 + \sqrt{14}\right\}$.

Here, $p(s,\mathcal{T}s) = 7 \leq 2 + p(2,2) = 6$. For $\alpha_1 = \frac{1}{10}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{20}$, $\alpha_4 = 0$, $\alpha_5 = \frac{3}{10}$, $\alpha_6 = \frac{1}{20}$, $\alpha_1 + \alpha_2 + \alpha\alpha_3 + 2\alpha\alpha_4 + \alpha_5 + \alpha_6 = \frac{7}{10} < 1$.

Thus, all the hypotheses of the Theorem 3.8 are validated and $C(2,2)$ is a fixed circle of a discontinuous multivalued mapping $\mathcal{T}$. Further, one may verify that $D(2,2) = [-2+\sqrt{6}, 2+\sqrt{14}]$ is a fixed disc of $\mathcal{T}$.

Remark 3.10. It is significant to notice that (see, Theorem 3.8 and Example 3.9) if a multivalued mapping fixes a disc, it also fixes a circle. But its reverse implication may only sometimes be accurate. Also, a fixed circle of a multivalued mapping is only sometimes unique (see, Example 3.9) and the discs lying in the interior of a fixed disc are also fixed discs.

4. Application to Elliptic Boundary Value Problem

Now, we exploit Theorem 3.3 to solve the following pair of elliptic boundary value problems:

$$-\frac{d^2\tilde{p}}{d\tilde{z}^2} = \mathcal{F}(\tilde{z}, \tilde{p}(\tilde{z})), \quad \tilde{z} \in [0,1]$$
$$\tilde{p}(0) = \tilde{p}(1) = 0.$$

and

$$-\frac{d^2\tilde{q}}{d\tilde{z}^2} = \mathcal{K}(\tilde{z}, \tilde{q}(\tilde{z})), \quad \tilde{z} \in [0,1]$$
$$\tilde{q}(0) = \tilde{q}(1) = 0, \tag{4.1}$$

where, $C(I)$ is the space of continuous functions on $I = [0,1]$ and functions $\mathcal{F}, \mathcal{K} : C(I) \times C(I) \to R$ are continuous. Then, the corresponding Green function linked to the underlying pair of elliptic boundary value problems (4.1) is:

$$G(\tilde{z}, \tilde{\nu}) = \begin{cases} \tilde{\nu}(1-\tilde{z}), & 0 \le \tilde{\nu} \le \tilde{z} \le 1, \\ \tilde{z}(1-\tilde{\nu}), & 0 \le \tilde{z} \le \tilde{\nu} \le 1. \end{cases} \tag{4.2}$$

Define $p : C(I) \times C(I) \to [0, \infty)$ as $p(\tilde{p}, \tilde{q}) = \sup_{\tilde{z} \in I} |\tilde{p}(\tilde{z}) - \tilde{q}(\tilde{z})|^2 + k$, $\tilde{p}, \tilde{q} \in C(I)$ and $k > 0$. Here, $(C(I), p)$ is a partial $b-$metric space for $s = 4$, which is complete.

Theorem 4.1. *Suppose $\mathcal{S}, \mathcal{T} : C(I) \to C(I)$ are operators which are described as:*

$$\mathcal{T}\tilde{p}(\tilde{z}) = \int_0^1 G(\tilde{z}, \tilde{\nu})\mathcal{F}(\tilde{\nu}, \tilde{p}(\tilde{\nu}))d\tilde{\nu}$$

and

$$\mathcal{S}\tilde{p}(\tilde{z}) = \int_0^1 G(\tilde{z}, \tilde{\nu})\mathcal{K}(\tilde{\nu}, \tilde{q}(\tilde{\nu}))d\tilde{\nu}, \tag{4.3}$$

for all $\tilde{z} \in I$. Suppose $\mathcal{F}, \mathcal{K} : C(I) \times C(I) \to R$ are continuous functions satisfying

$$|\mathcal{F}(\tilde{z}, \tilde{p}) - \mathcal{K}(\tilde{z}, \tilde{q})|^2 \le 64.\Theta(\tilde{p}, \tilde{q}), \tag{4.4}$$

$\tilde{p}, \tilde{q} \in C(I), \tilde{z} \in I$, where $\Theta(\tilde{p}, \tilde{q})$ is right-hand side of (3.5). Then, a pair of elliptic boundary value problem (4.1) has at least one solution $\tilde{p}^* \in C(I)$.

Proof. Noticeably, $\tilde{p}^* \in C(I)$ is a solution of (4.1) iff $\tilde{p}^* \in C(I)$ is a common fixed point of operators $\mathcal{S}$ and $\mathcal{T}$ given by (4.3).
Suppose $\tilde{p}, \tilde{q} \in C(I)$. Using (4.3), we get

$$\begin{aligned}
|\mathcal{T}\tilde{p}(\tilde{z}) - \mathcal{S}\tilde{q}(\tilde{z})|^2 &= \left| \int_0^1 G\left(\tilde{z}, \tilde{\nu}\right) \left[\mathcal{F}(\tilde{\nu}, \tilde{p}(\tilde{\nu})) - \mathcal{K}(\tilde{\nu}, \tilde{q}(\tilde{\nu}))\right] d\tilde{\nu} \right|^2 \\
&\leq \left[\int_0^1 G(\tilde{z}, \tilde{\nu}) |\mathcal{F}(\tilde{\nu}, \tilde{p}(\tilde{\nu})) - \mathcal{K}(\tilde{\nu}, \tilde{q}(\tilde{\nu}))| d\tilde{\nu} \right]^2 \\
&= 64.\Theta(\tilde{p}, \tilde{q}) \left(\sup_{\tilde{z} \in I} \left[\int_0^1 G(\tilde{z}, \tilde{\nu}) d\tilde{\nu} \right]^2 \right).
\end{aligned}$$

Since $\int_0^1 G(\tilde{z}, \tilde{\nu}) d\tilde{\nu} = \frac{-\tilde{z}^2}{2} + \frac{\tilde{z}}{2}$ for all $\tilde{z} \in I$, then we have

$$\sup_{\tilde{z} \in I} \left(\int_0^1 G(\tilde{z}, \tilde{\nu}) d\tilde{\nu} \right)^2 = \frac{1}{64},$$

which implies that

$$p(\mathcal{T}s, \mathcal{S}\tilde{z}) \leq \Theta(\tilde{p}, \tilde{q}).$$

Therefore, all the hypotheses of Theorem 3.3 are verified. Thus $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point $\tilde{p}^* \in C(I)$, which is a solution of equation (4.1). □

5. Conclusion

We have given a method to establish a fixed and common fixed points utilizing a partial b-metric for discontinuous multivalued mappings. Also, we have explored the geometry of the multivalued fixed points in a partial b-metric space, utilized novel contractions to establish a multivalued fixed circle and disc, and furnished elucidatory examples to establish the validity of the hypotheses. Further, we have exploited our result to solve a pair of elliptic boundary value problems. Our investigation of multivalued fixed points (or common fixed) and their geometry will be a fascinating area for future study and contributes to the expansion of fixed point theory.

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References

1. M. Abtahi, *Cauchy sequences in fuzzy metric spaces and fixed point theorems*, Sahand Commun. Math. Anal., 20(1) (2023), pp. 137-152.
2. H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric space*, Topology Appl., 159 (2012), pp. 3234-3242.
3. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux l'équations intégrales*, Fundam. Inform., 3 (1922), pp. 133-181.
4. I.A. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Funct. Anal., 30 (1989), pp. 26-37.
5. U. Çelik and N.Y. Özgür, *On the fixed-circle problem*, Facta Univ. Ser. Math. Inform., 35(5) (2020), pp. 1273-1290.
6. S.K. Chatterjea, *Fixed point theorem*, C. R. Acad, Bulgar Sci., 25(6) (1972), pp. 727-730.
7. Lj. B Ćirić, *A generalization of Banach's contraction principle*, Proc. Am. Math. Soc., 45(2) (1974), pp. 267-273.
8. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., 1 (1993), pp. 5-11.
9. P. Debnath, *Results on discontinuity at fixed point for a new class of F-contractive mappings*, Sahand Commun. Math. Anal., (2023) https://doi.org/10.22130/scma.2023.560141.1161.
10. G.Z. Erçınar, *Some geometric properties of fixed points*, Ph.D. Thesis, Eskişehir Osmangazi University, Eskişehir, 2020.
11. M. Joshi, A. Tomar, H.A. Nabwey and R. George, *On unique and non-unique Fixed Points and Fixed Circles in M_v^b-metric space and application to cantilever beam problem*, J. Funct. Spaces, 2021 (2021), Article ID 6681044.
12. M. Joshi, A. Tomar and S.K. Padaliya, *On geometric properties of nonunique fixed points in b-metric spaces*, Fixed Point Theory and its Applications to Real World Problem, Nova Science Publishers, New York, USA, 2021 (2021), pp. 34-50.
13. M. Joshi, A. Tomar and S.K. Padaliya, *Fixed point to fixed disc and application in partial metric spaces*, Fixed Point Theory and its Applications to Real World Problem Nova Science Publishers, New York, USA, 2021 (2021), pp. 391-406.
14. M. Joshi, A. Tomar and S.K. Padaliya, *Fixed point to fixed ellipse in metric spaces and discontinuous activation function*, Appl. Math. E-Notes, 21 (2021), pp. 225-237.
15. M. Joshi and A. Tomar, *On unique and non-unique fixed points in metric spaces and application to chemical sciences*, J. Funct.

Spaces, 2021 (2021), Article ID 5525472.

16. M. Joshi and A. Tomar, *Near fixed point, near fixed interval circle and their equivalence classes in a b-interval metric space*, Int. J. Nonlinear Anal. Appl., 13(1) (2022), pp. 1999-2014.
17. M. Joshi, A. Tomar and T. Abdeljawad, *On Fixed Point, its geometry and application to satellite web coupling problem in S-metric spaces*, AIMS Math., 8(2) (2023), pp. 4407-4441.
18. R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., 60 (1968), pp. 71-76.
19. R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly, 76(4) (1969), pp. 405-408.
20. W.A. Kirk, P.S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, 4(1) (2003), pp. 79-89.
21. A. Malhotra and D. Kumar, *Fixed point results for multivalued mapping in R-metric space*, Sahand Commun. Math. Anal., 20(2) (2023), pp. 109-121.
22. S.G. Matthews, *Partial metric topology*, in: Proc. 8th Summer Conference on General Topology and application, Ann. New York Acad. Sci., 728 (1994), pp. 183-197.
23. N. Mlaiki, N.Y. Özgür and N. Taş, *New fixed-circle results related to F_c-contractive and F_c-expanding mappings on metric spaces*, 2021 (2021), arXiv:2101.10770.
24. S.B. Nadler, *Multivalued contraction mappings*, Pac. J. Math., 30(2) (1969), pp. 475-488.
25. N.Y. Özgür and N. Taş, *Some fixed-circle theorems on metric spaces*, Bull. Malays. Math. Sci. Soc., 42(4) (2019), pp. 1433-1449.
26. N.Y. Özgür and N. Taş, *Some fixed-circle theorems and discontinuity at fixed circle*, AIP Conf. Proc., 1926(1) (2018), 020048.
27. N.Y. Özgür and N. Taş, *Geometric properties of fixed points and simulation functions*, (2021), arXiv:2102.05417.
28. N.Y. Özgür, *Fixed-disc results via simulation functions*, Turk. J. Math., 43(6) (2019), pp. 2794-2805.
29. N. Taş, N. Mlaiki, A. Hassen and N. Özgür, *Fixed-disc results on metric spaces*, Filomat, 35(2) (2021), pp. 447-457.
30. S. Petwal, A. Tomar and M. Joshi, *On unique and non-unique fixed point in parametric metric spaces with application*, Acta Univ. Sapientiae, Math., 14(2) (2022), pp. 68-97.
31. S. Reich, *Some remarks concerning contraction mappings*, Can. Math. Bull., 14(1) (1971), pp. 121-124.
32. S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math., 11 (2014), pp. 703-711.

33. N. Taş and N.Y. Özgür, *New fixed-figure results on metric spaces, Fixed point theory and fractional calculus-recent advances and applications*, Forum Interdiscip. Math., Springer, Singapore, 2022 (2022), pp. 33-62.
34. N. Taş and N.Y. Özgür, *New multivalued contractions and the fixed-circle problem*, 2021 (2021), arXiv:1911.02939.
35. A. Tomar, R. Sharma and A.H. Ansari, *Strict coincidence and common strict fixed point of a faintly compatible hybrid pair of maps via C-class function and applications*, Palest. J. Math., 9(1) (2020), pp. 274-288.
36. A. Tomar, R. Sharma and S. Upadhyay, *Some applications of existence of common fixed and common stationary point of a hybrid pair*, Bull. Int. Math. Virtual Inst., 8 (2018), pp. 181-198.
37. A. Tomar, S. Beloul, R. Sharma and S. Upadhyay, *Common fixed point theorems via generalized condition (B) in quasi-partial metric space and applications*, Demonstr. Math., 50(1) (2017), pp. 278-298.
38. A. Tomar, S. Beloul, S. Upadhyay and R. Sharma, *Strict coincidence and strict common fixed point via strongly tangential property with an application*, Electron. J. Math. Anal. Appl., 7(1) (2019), pp. 82-94.
39. A. Tomar, S. Upadhyay and R. Sharma, *On existence of strict coincidence and common strict fixed point of a faintly compatible hybrid pair of maps*, Electron. J. Math. Anal. Appl., 5(2) (2017), pp. 298-305.
40. A. Tomar, S. Upadhyay and R. Sharma, *Strict coincidence and common strict fixed point of hybrid pairs of self-map with an application*, Math. Sci. Appl. E-notes, 5(2) (2017), pp. 51-59.
41. A. Tomar, S. Upadhyay and R. Sharma, *Common Fixed Point Theorems with an Application*, Recent Advances in Fixed Point Theory and Applications, Nova Science Publishers, 2017 (2017), pp. 157-169.
42. A. Tomar, M. Joshi and S.K. Padaliya, *Fixed point to fixed circle and activation function in partial metric space*, J. Appl. Anal., 28(1) (2022), pp. 57-66.
43. A. Tomar, U.S. Rana and V. Kumar, *Fixed point, its geometry and application via ω-interpolative contraction of Suzuki type mapping*, Math. Meth. Appl. Sci. (Special Issue), 2022 (2022) pp. 1-22.
44. A. Tomar, N. Taş and M. Joshi, *On interpolative type non-unique fixed points, their geometry and applications on S-metric spaces*, Appl. Math. E-Notes., 2022(2022) (In press).
45. I. Yildirim, *Fixed point results for F-Hardy-Rogers contractions via Mann's iteration process in complete convex b-metric spaces*,

Sahand Commun. Math. Anal., 19(2) (2022), pp. 15-32.

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