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A Seneta's Conjecture and the Williamson Transform

Edward Omey[1] and Meitner Cadena[2*]

Abstract. Considering slowly varying functions (SVF), Seneta in 2019 conjectured the following implication, for $\alpha \geq 1$,

$$\int_0^x y^{\alpha-1}(1-F(y))dy \text{ is SVF } \Rightarrow \int_0^x y^{\alpha} dF(y) \text{ is SVF, as } x \to \infty,$$

where $F(x)$ is a cumulative distribution function on $[0, \infty)$. By applying the Williamson transform, an extension of this conjecture is proved. Complementary results related to this transform and particular cases of this extended conjecture are discussed.

1. Introduction

A function $f(x)$ is slowly varying (SVF) if for any $t > 0$, $f(tx)/f(x) \to 1$ as $x \to \infty$. If $F(x)$ is a distribution function and $\overline{F}(x) = 1 - F(x)$ is its tail, recently Seneta [9] conjectured the following. Given $\alpha \geq 1$,

$$\int_0^x y^{\alpha-1}\overline{F}(y)dy \text{ is SVF} \quad \Rightarrow \quad \int_0^x y^{\alpha} dF(y) \text{ is SVF, as } x \to \infty. \tag{1.1}$$

Nowadays, Kevei in [5] presented an extension of this conjecture and its proof. In this paper, we also prove an extension of such a conjecture, but unlike Kevei's proofs, ours are based on mainly the Williamson transform. Moreover, the application of this transform to F allows the formulation of another conjecture. Finally, such extended conjecture includes the generalized gamma class.

In the following section, we present our main results. The proofs of these results are presented in Section 3. Complementary results related to the Williamson transform of F and analysis of particular cases of α

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are discussed in Section 4. The last section presents concluding remarks and next steps for research on extensions of the findings shown in this paper.

2. Main Results

2.1. **Some Notation and Transforms.** In what follows, $F(x)$ denotes a distribution function (d.f.) defined on $[0, \infty)$ with $F(0) = 0$, and $\overline{F}(x) = 1 - F(x)$ denotes its tail. The α–th moment of $F(x)$ is denoted by $m(\alpha)$. It is said that $f(x) \sim g(x)$ if $f(x)/g(x) \to 1$ as $x \to \infty$. The case $f(x) \sim 0$ will be understood as $f(x) \to 0$ as $x \to \infty$. The class of regularly varying functions with index α, denoted by RV_α, consists of functions $f(x)$ satisfying, for any $t > 0$, $f(tx) \sim t^\alpha f(x)$. If the index $\alpha = 0$, $f(x)$ is said slowly varying. If $L \in RV_0$, then the de Haan class, denoted by $\Pi_\beta(L)$, consists of functions $f(x)$ satisfying, for any $t > 1$, $(f(tx) - f(x))/L(x) \to \beta \log t$ as $x \to \infty$. The generalized gamma class, denoted by $E\Gamma_\lambda(g, a)$ where g is self-neglecting, i.e., for any $t \in \mathbb{R}$, $g(x + tg(x)) \sim g(x)$, consists of functions $f(x)$ satisfying, for any $t \in \mathbb{R}$, $(f(x + tg(x)) - f(x))/a(x) \to \lambda t$ as $x \to \infty$.

In this paper, we assume that $\alpha > 0$.

We define the following transformations:

$$H_\alpha(x) = \int_0^x y^\alpha dF(y) \quad \text{and} \quad W_\alpha(x) = \int_0^x y^{\alpha-1}\overline{F}(y)dy.$$

Note that Lebesgue's theorem on dominated convergence shows that $\lim_{x\to\infty} x^{-\alpha} H_\alpha(x) = 0$. Hence, we have

$$H_\alpha(x) = \alpha \int_0^x y^{\alpha-1}\overline{F}(y)dy - x^\alpha \overline{F}(x). \tag{2.1}$$

Among the main results proved by Kevei in [5], we have that, for any $\theta \in [0, \alpha)$, $H_\alpha(x) \in RV_\theta$ if and only if $W_\alpha(x) \in RV_\theta$. The proof for the converse of this result mainly lies in Theorem 8.1.2 in [1], see also VIII.9 by [3].

From the definitions given above, we get the following result.

Proposition 2.1. *For $x \geq 0$, we have*

$$\overline{F}(x) = \alpha \int_x^\infty z^{-\alpha-1} H_\alpha(z)dz - x^{-\alpha} H_\alpha(x). \tag{2.2}$$

Proof. Let us consider the integral $\int_x^a z^{-\alpha-1} H_\alpha(z)dz$ for $a \geq x \geq 0$. We have

$$\int_x^a z^{-\alpha-1} H_\alpha(z)dz = \int_x^a z^{-\alpha-1} \left(H_\alpha(x) + \int_x^z y^\alpha dF(y) \right) dz$$

$$
\begin{aligned}
&= \frac{1}{\alpha}(x^{-\alpha} - a^{-\alpha})H_\alpha(x) + \int_x^a \int_y^a z^{-\alpha-1} y^\alpha dz dF(y) \\
&= \frac{1}{\alpha}(x^{-\alpha} - a^{-\alpha})H_\alpha(x) + \frac{1}{\alpha}\int_x^a (y^{-\alpha} - a^{-\alpha}) y^\alpha dF(y) \\
&= \frac{1}{\alpha}(x^{-\alpha} - a^{-\alpha})H_\alpha(x) + \frac{1}{\alpha}(F(a) - F(x)) \\
&\quad - \frac{1}{\alpha} a^{-\alpha}(H_\alpha(a) - H_\alpha(x)) \\
&= \frac{1}{\alpha} x^{-\alpha} H_\alpha(x) + \frac{1}{\alpha}(F(a) - F(x)) - \frac{1}{\alpha} a^{-\alpha} H_\alpha(a).
\end{aligned}
$$

Taking limits as $a \to \infty$, we obtain $a^{-\alpha}H_\alpha(a) \to 0$ and hence

$$\int_x^\infty z^{-\alpha-1} H_\alpha(z) dz = \frac{1}{\alpha} x^{-\alpha} H_\alpha(x) + \frac{1}{\alpha}\overline{F}(x).$$

This proves the result. □

Now, we recall the Williamson transform which is defined by [10],

$$G_\alpha(x) = \int_0^x \left(1 - \left(\frac{t}{x}\right)^\alpha\right) dF(t).$$

Note that

$$
\begin{aligned}
G_\alpha(x) &= \int_0^\infty P\left(Z > \frac{t}{x}\right) dF(t) \\
&= EF(Zx) \\
&= P\left(\frac{X}{Z} \le x\right),
\end{aligned}
$$

where Z is independent of X and $P(Z \le x) = x^\alpha, 0 \le x \le 1$. This shows that $G_\alpha(x)$ is also a d.f. with $G_\alpha(0) = 0$. Also, we have the following result.

Proposition 2.2. *For $x \ge 0$, we have*

(i) $G_\alpha(x) = \alpha x^{-\alpha} \int_0^x t^{\alpha-1} F(t) dt$.
(ii) $F(x) = G_\alpha(x) + \frac{x}{\alpha} G'_\alpha(x)$.
(iii) $\overline{F}(x) = \overline{G}_\alpha(x) - x^{-\alpha} H_\alpha(x)$.
(iv) $\overline{G}_\alpha(x) = \alpha x^{-\alpha} W_\alpha(x)$.

Proof. (i) By using partial integration, we have

$$
\begin{aligned}
G_\alpha(x) &= F(x) - x^{-\alpha} \int_0^x t^\alpha dF(t) \\
&= \alpha x^{-\alpha} \int_0^x t^{\alpha-1} F(t) dt.
\end{aligned}
$$

(ii) Writing the previous result as $x^{\alpha}G_{\alpha}(x) = \alpha \int_0^x t^{\alpha-1}F(t)dt$, deriving this relation, and dividing it by $x^{\alpha-1}/\alpha$ give the result claimed.

(iii) Using $H_{\alpha}(x)$ and (i), we have $G_{\alpha}(x) = F(x) - x^{-\alpha}H_{\alpha}(x)$ and $\overline{G}_{\alpha}(x) = \overline{F}(x) + x^{-\alpha}H_{\alpha}(x)$.

(iv) Using the tail of $G_{\alpha}(x)$, $W_{\alpha}(x)$, and (i), we find

$$\begin{aligned}\overline{G}_{\alpha}(x) &= \alpha x^{-\alpha}\int_0^x t^{\alpha-1}\overline{F}(t)dt\\ &= \alpha x^{-\alpha}W_{\alpha}(x).\end{aligned}$$

□

2.2. **Main Relations.** Our the first main as follows.

Theorem 2.3. *Let $\alpha > 0$ and $0 \leq \theta \leq \alpha$. We have:*

(i) *If $0 < \theta < \alpha$, then the following statements are equivalent:*
  (a) $\overline{F}(x) \in RV_{\theta-\alpha}$;
  (b) $\overline{G}_{\alpha}(x) \in RV_{\theta-\alpha}$;
  (c) $W_{\alpha}(x) \in RV_{\theta}$;
  (d) $\overline{F}(x)/\overline{G}_{\alpha}(x) \to \theta/\alpha$ *as* $x \to \infty$;
  (e) $H_{\alpha}(x) \in RV_{\theta}$;
  (f) $x^{\alpha}\overline{F}(x)/H_{\alpha}(x) \to \theta/(\alpha-\theta)$ *as* $x \to \infty$;
  (g) $x^{\alpha}\overline{G}_{\alpha}(x)/H_{\alpha}(x) \to \alpha/(\alpha-\theta)$ *as* $x \to \infty$; *and,*
  (h) $x^{\alpha}\overline{F}(x)/W_{\alpha}(x) \to \theta$ *as* $x \to \infty$.

(ii) *If $\theta = 0$, then the statements (b), (c), (d), (e), (f), (g) and (h) are equivalent.*

(iii) *If $\theta = \alpha$, then the statements (a), (b) and (c) are equivalent.*

Remarks 2.4.

1) Taking $\theta = 0$, Theorem 2.3(e), (f) and (h) correspond to (17), (18) and (19) in Theorem 3 by [9] , respectively.
2) Theorem 2.3(a), (e) and (f) are proved in VIII.9 by [3], see e.g. Theorem 8.1.2 in [1].
3) The conjecture indicated in (1.1) is contained and extended in the implication, for $0 \leq \theta < \alpha$, Theorem 2.3(c) $\Rightarrow$ Theorem 2.3(e).
4) Theorem 2.3(a), (c), (e), (h) and a combination of (f) and (h) correspond to (1.6), (1.4), (1.5), (1.7) and (1.8) in Theorem 1.1 by [5], respectively.

We continue with results when some moments are finite. They are proved with similar arguments used when proving the results presented in the previous theorem. Therefore, proofs for this theorem are not presented.

Theorem 2.5. *Let $\alpha > 0$ and $\theta > \alpha$. Assume $m(\alpha) < \infty$. Set $\overline{W}_\alpha(x) = W_\alpha(\infty) - W_\alpha(x)$. The following statements are equivalent:*

(a) $\overline{F}(x) \in RV_{-\theta}$;
(b) $\overline{W}_\alpha(x) \in RV_{\alpha-\theta}$;
(c) $x^{-\alpha}m(\alpha) - \overline{G}_\alpha(x) \in RV_{-\theta}$;
(d) $\overline{W}_\alpha(x)\big/(x^\alpha \overline{F}(x)) \to 1\big/(\theta-\alpha)$ *as* $x \to \infty$;
(e) $(x^{-\alpha}m(\alpha) - \overline{G}_\alpha(x))\big/\overline{F}(x) \to \alpha\big/(\theta-\alpha)$ *as* $x \to \infty$;
(f) $m(\alpha) - H_\alpha(x) \in RV_{\alpha-\theta}$; *and,*
(g) $(m(\alpha) - H_\alpha(x))\big/(x^\alpha \overline{F}(x)) \to \theta\big/(\theta-\alpha)$ *as* $x \to \infty$.

Next, results involving the de Haan class follow.

Theorem 2.6. *Let $\alpha, \beta > 0$. Let $L(x) \in RV_0$. The following statements are equivalent:*

(a) $x^\alpha \overline{G}_\alpha(x) \in \Pi_\beta(L)$;
(b) $x^\alpha \overline{F}(x)\big/L(x) \to \beta\big/\alpha$ *as* $x \to \infty$; *and,*
(c) $W_\alpha(x) \in \Pi_{\beta/\alpha}(L)$

Remark 2.7. The relation (b) $\Leftrightarrow$ (c) was identified in Theorem 1.1 by [5].

Next, results involving the extended gamma class follow.

Theorem 2.8. *Let $\lambda, \alpha > 0$. The following statements are equivalent:*

(a) $x^\alpha \overline{G}_\alpha(x) \in E\Gamma_\lambda(g, 1)$;
(b) $g(x)x^{\alpha-1}\overline{F}(x) \to \lambda\big/\alpha$ *as* $x \to \infty$; *and,*
(c) $W_\alpha(x) \in E\Gamma_{\lambda/\alpha}(g, 1)$.

3. Proofs

To prove the theorems indicated in the previous section, the following well-known result is used, see [4] and e.g. [2, Theorem 1.2.1]:

Proposition 3.1. *Suppose $U : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-summable on finite intervals. Then*

$$U(x) \in RV_\alpha,\ \alpha > -1 \quad \textit{iff} \quad xU(x) \sim (\alpha+1)\int_0^x U(t)dt; \tag{3.1}$$

$$U(x) \in RV_\alpha,\ \alpha < -1 \quad \textit{iff} \quad xU(x) \sim -(\alpha+1)\int_x^\infty U(t)dt. \tag{3.2}$$

The following lemmas are also used to prove those theorems.

Lemma 3.2. *Let $\alpha > 0$. Assume that there exist $A(x)$ (> 0), $B(x)$, and $C(x)$ satisfying, for any $z > 1$,*

$$\frac{W_\alpha(zx) - W_\alpha(x)}{A(x)} \to B(z), \quad \textit{and} \quad \frac{W_\alpha(zx) - W_\alpha(x)}{A(zx)} \to C(z),$$

as $x \to \infty$, such that, for some $\xi \geq 0$, $B(z)/(z^{\alpha}-1) \to \xi/\alpha$ and $C(z)/(1-z^{-\alpha}) \to \xi/\alpha$ as $z \downarrow 1$. Then, we have

$$\frac{\overline{F}(x)x^{\alpha}}{A(x)} \to \xi, \quad \text{as } x \to \infty.$$

Proof. For $z > 1$, we have $W_{\alpha}(zx) - W_{\alpha}(x) = \int_x^{zx} y^{\alpha-1}\overline{F}(y)dy$. Since $\overline{F}(x)$ is nonincreasing, we find

$$\begin{aligned}\frac{1}{\alpha}\overline{F}(xz)x^{\alpha}(z^{\alpha}-1) &\leq W_{\alpha}(zx) - W_{\alpha}(x)\\ &\leq \frac{1}{\alpha}\overline{F}(x)x^{\alpha}(z^{\alpha}-1),\end{aligned}$$

then,

$$\frac{W_{\alpha}(zx) - W_{\alpha}(x)}{A(x)} \leq \frac{x^{\alpha}\overline{F}(x)}{A(x)}\frac{1}{\alpha}(z^{\alpha}-1),$$

and

$$\frac{(xz)^{\alpha}\overline{F}(xz)}{A(xz)}\frac{1}{\alpha}z^{-\alpha}(z^{\alpha}-1) \leq \frac{W_{\alpha}(zx) - W_{\alpha}(x)}{A(zx)}.$$

Taking limits, we obtain

$$\alpha\frac{B(z)}{z^{\alpha}-1} \leq \liminf_{x\to\infty}\frac{\overline{F}(x)x^{\alpha}}{A(x)}, \quad \text{and} \quad \limsup_{x\to\infty}\frac{(xz)^{\alpha}\overline{F}(xz)}{A(xz)} \leq \alpha\frac{C(z)}{1-z^{-\alpha}},$$

or

$$\begin{aligned}\alpha\frac{B(z)}{z^{\alpha}-1} &\leq \lim\begin{pmatrix}\sup\\ \inf\end{pmatrix}\frac{\overline{F}(x)x^{\alpha}}{A(x)}\\ &\leq \alpha\frac{C(z)}{1-z^{-\alpha}}.\end{aligned}$$

Now, let $z \downarrow 1$ to find, by hypothesis, that $x^{\alpha}\overline{F}(x)/A(x) \to \xi$ as $x \to \infty$. □

Lemma 3.3. *Let $\alpha > 0$. If $\overline{F}(x)/\overline{G}_{\alpha}(x) \to \lambda$ as $x \to \infty$, $0 \leq \lambda < 1$, then $\overline{G}_{\alpha} \in RV_{\alpha(\lambda-1)}$.*

Proof. Assume that $\overline{F}(x)/\overline{G}_{\alpha}(x) \to \lambda$ as $x \to \infty$, $0 \leq \lambda < 1$. By Proposition 2.2 (ii), we have that $\overline{F}(x) = \overline{G}_{\alpha}(x) - xG'_{\alpha}(x)/\alpha$. Then, we get

$$\begin{aligned}\frac{xG'_{\alpha}(x)}{\overline{G}_{\alpha}(x)} &= \alpha\frac{\overline{G}_{\alpha}(x) - \overline{F}(x)}{\overline{G}_{\alpha}(x)}\\ &\to \alpha(1-\lambda), \quad \text{as } x \to \infty.\end{aligned}$$

Hence, by applying L'Hopital rule, we have

$$\frac{\overline{G}_\alpha(x)}{\int_x^\infty y^{-1}\overline{G}_\alpha(y)dy} \to \alpha(1-\lambda), \quad \text{as } x \to \infty.$$

Then, noting that $-\alpha(1-\lambda)-1 < -1$, by applying (3.2), we have $x^{-1}\overline{G}_\alpha(x) \in RV_{-\alpha(1-\lambda)-1}$. This result is equivalent to $\overline{G}_\alpha(x) \in RV_{\alpha(\lambda-1)}$. □

Lemma 3.4. *Let $\alpha > 0$. If $W_\alpha \in E\Gamma_\lambda(g,1)$, then we have*

$$g(x)\overline{F}(x)x^{\alpha-1} \to \lambda, \quad \textit{as } x \to \infty.$$

Proof. Let $y \in \mathbb{R}\setminus\{0\}$. We have $W_\alpha(x+yg(x)) - W_\alpha(x) = \int_x^{x+yg(x)} t^{\alpha-1}$ $\overline{F}(t)dt$. Since $\overline{F}(x)$ is nonincreasing, we find

$$\begin{aligned}\frac{1}{\alpha}\overline{F}(x+yg(x))x^\alpha((1+yg(x)/x)^\alpha-1) &\le W_\alpha(x+yg(x)) - W_\alpha(x)\\ &\le \frac{1}{\alpha}\overline{F}(x)x^\alpha((1+yg(x)/x)^\alpha-1),\end{aligned}$$

then, using the binomial expansion formulae,

$$\begin{aligned}W_\alpha(x+yg(x)) - W_\alpha(x) &\le x^\alpha\overline{F}(x)\frac{1}{\alpha}\left(\alpha y\frac{g(x)}{x} + o\left(y\frac{g(x)}{x}\right)\right)\\ &= yg(x)x^{\alpha-1}\overline{F}(x)\left(1+o\left(1\right)\right),\end{aligned}$$

and

$$\begin{aligned}W_\alpha(x+yg(x)) - W_\alpha(x) &\ge \frac{1}{\alpha}\overline{F}(x+yg(x))x^\alpha\left(\alpha y\frac{g(x)}{x} + o\left(y\frac{g(x)}{x}\right)\right)\\ &\ge yg(x+yg(x))(x+yg(x))^{\alpha-1}\overline{F}(x+yg(x))\\ &\quad\times\left(\frac{x}{x+yg(x)}\right)^{\alpha-1}\frac{g(x)}{g(x+yg(x))}\left(1+o\left(1\right)\right).\end{aligned}$$

Taking limits, we obtain

$$\lambda y \le \liminf_{x\to\infty} yg(x)x^{\alpha-1}\overline{F}(x)$$

and

$$\limsup_{x\to\infty} yg(x+yg(x))(x+yg(x))^{\alpha-1}\overline{F}(x+yg(x)) \le \lambda y,$$

or

$$\begin{aligned}\lambda y &\le \lim\binom{\sup}{\inf} yg(x)x^{\alpha-1}\overline{F}(x)\\ &\le \lambda y.\end{aligned}$$

This implies that

$$\lim_{x\to\infty} yg(x)x^{\alpha-1}\overline{F}(x) = \lambda y,$$

i.e.

$$\lim_{x\to\infty} g(x)x^{\alpha-1}\overline{F}(x) = \lambda. \qquad \square$$

Proof of Theorem 2.3: Let $\alpha > 0$.

Let us prove (b) $\Leftrightarrow$ (c) in Theorem 2.3 (i), (ii), and (iii) when $0 \le \theta \le \alpha$. From Proposition 2.2 (iv) we have $\overline{G}_\alpha(x) \in RV_{\theta-\alpha}$ iff $W_\alpha(x) \in RV_\theta$ because.

Let us prove (a) $\Leftrightarrow$ (b) in Theorem 2.3 (i) and (iii) when $0 < \theta \le \alpha$. If $\overline{F}(x) \in RV_{\theta-\alpha}$, noting that $\alpha - 1 + \theta - \alpha = \theta - 1 > -1$, by applying Proposition 2.2 iv and (3.1), $\overline{G}_\alpha(x) = \alpha x^{-\alpha}\int_0^x y^{\alpha-1}\overline{F}(y)dy \sim \alpha\overline{F}(x)/\theta$. Hence, $\overline{G}_\alpha(x) \in RV_{\theta-\alpha}$. Conversely, if $\overline{G}_\alpha(x) \in RV_{\theta-\alpha}$, from Proposition 2.2 (iv), we have $W_\alpha(x) \in RV_\theta$. Hence, by applying Lemma 3.2 with $A(x) = W_\alpha(x)$, $B(x) = x^\theta - 1$, $C(x) = 1 - x^{-\theta}$, and $\xi = \theta$, we find that $x^\alpha\overline{F}(x)/W_\alpha(x) \to \theta$ as $x \to \infty$ and, hence, $\overline{F}(x) \in RV_{\theta-\alpha}$.

Let us prove (e) $\Leftrightarrow$ (f) in Theorem 2.3 (i) and (ii) when $0 \le \theta < \alpha$. If $H_\alpha(x) \in RV_\theta$, we have $x^{-\alpha-1}H(x) \in RV_{\theta-\alpha-1}$ and, since $\theta-\alpha-1 < -1$, (3.2) gives

$$\frac{x^{-\alpha}H_\alpha(x)}{\int_x^\infty y^{-\alpha-1}H_\alpha(y)dy} \to \alpha - \theta, \quad \text{as } x \to \infty.$$

On the other hand, from (2.2), it follows that, as $x \to \infty$,

$$\begin{aligned}\frac{\overline{F}(x)}{x^{-\alpha}H_\alpha(x)} &\to \frac{\alpha}{\alpha-\theta} - 1\\ &= \frac{\theta}{\alpha-\theta}.\end{aligned}$$

Conversely, if $x^\alpha\overline{F}(x)/H_\alpha(x) \to \theta/(\alpha-\theta)$ as $x \to \infty$, from (2.2) we have as $x \to \infty$,

$$\begin{aligned}\frac{\alpha\int_x^\infty z^{-\alpha-1}H_\alpha(z)dz}{x^{-\alpha}H_\alpha(x)} &\to \frac{\theta}{\alpha-\theta} + 1\\ &= \frac{\alpha}{\alpha-\theta}.\end{aligned}$$

Noting that $\theta - \alpha - 1 < -1$, we have, by applying (3.2), $x^{-\alpha-1}H_\alpha(x) \in RV_{\theta-\alpha-1}$. This leads to $H_\alpha(x) \in RV_\theta$.

Let us prove (c) $\Leftrightarrow$ (h) in Theorem 2.3 (i) and (ii) when $0 \le \theta < \alpha$. If $W_\alpha(x) \in RV_\theta$, by applying Lemma 3.2 with $A(x) = W_\alpha(x)$, $B(x) = x^\theta - 1$, $C(x) = 1 - x^{-\theta}$, and $\xi = \theta$ we have $x^\alpha\overline{F}(x)/W_\alpha(x) \to \beta$ as $x \to \infty$. Conversely, assume that $x^\alpha\overline{F}(x)/W_\alpha(x) \to \beta$ as $x \to \infty$. By Proposition 2.2 (iv), we have $\overline{F}(x)/\overline{G}_\alpha(x) \to \beta/\alpha$ as $x \to \infty$. Hence, taking $0 \le \lambda := \beta/\alpha < 1$, Lemma 3.3 implies that $\overline{G}_\alpha(x) \in RV_{\alpha(\beta/\alpha-1)}$.

Then, by the equivalence (b) $\Leftrightarrow$ (c) proved above and from $\alpha(\beta/\alpha-1) = \beta - \alpha$, we have $W_\alpha(x) \in RV_\theta$.

Let us prove (d) $\Leftrightarrow$ (h) in Theorem 2.3 (i) and (ii) when $0 \le \theta < \alpha$. This result holds from Proposition 2.2 (iv).

Let us prove (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (g) in Theorem 2.3 (i) and (ii) when $0 \le \theta < \alpha$. Assume that $H_\alpha(x) \in RV_\theta$, $0 \le \theta < \alpha$. By Proposition 2.2 (iii), we have $x^\alpha \overline{G}_\alpha(x) = x^\alpha \overline{F}(x) + H_\alpha(x)$. Hence and using of the equivalence (e) $\Leftrightarrow$ (f) proved above, we find that, as $x \to \infty$,

$$\begin{aligned}\frac{x^\alpha \overline{G}_\alpha(x)}{H_\alpha(x)} &= \frac{x^\alpha \overline{F}(x)}{H_\alpha(x)} + 1 \\ &\to \frac{\theta}{\alpha-\theta} + 1 \\ &= \frac{\alpha}{\alpha-\theta}.\end{aligned}$$

This fact implies that $\overline{G}_\alpha(x) \in RV_{\theta-\alpha}$. Thus, it is proved that (e) $\Rightarrow$ (b). Now, suppose that $\overline{G}_\alpha(x) \in RV_{\theta-\alpha}$. Using (2.2) gives

$$\int_x^\infty y^{-\alpha-1} H_\alpha(y)dy = \frac{H_\alpha(x)x^{-\alpha}}{\alpha} + \frac{1}{\alpha}\overline{F}(x),$$

and, by Proposition 2.2 (iv), we have $\overline{G}_\alpha(x) = \alpha \int_x^\infty y^{-\alpha-1} H_\alpha(y)dy$. Hence, we get

$$\frac{H_\alpha(x)x^{-\alpha}}{\overline{G}_\alpha(x)} = 1 - \frac{\overline{F}(x)}{\overline{G}_\alpha(x)}.$$

Further, the equivalences (b) $\Leftrightarrow$ (c), (c) $\Leftrightarrow$ (h), and (d) $\Leftrightarrow$ (h) proved above imply that (b) $\Leftrightarrow$ (d). Then, we deduce that, as $x \to \infty$,

$$\begin{aligned}\frac{H_\alpha(x)x^{-\alpha}}{\overline{G}_\alpha(x)} &\to 1 - \frac{\theta}{\alpha} \\ &= \frac{\alpha-\theta}{\alpha}.\end{aligned}$$

This proves that (b) $\Rightarrow$ (g). Now, assume that $x^\alpha \overline{G}_\alpha(x)/H_\alpha(x) \to \alpha/(\alpha-\theta)$ as $x \to \infty$. Using again Proposition 2.2 (iii), i.e. $x^\alpha \overline{G}_\alpha(x) = x^\alpha \overline{F}(x) + H_\alpha(x)$, we have, as $x \to \infty$,

$$\begin{aligned}\frac{x^\alpha \overline{F}(x)}{H_\alpha(x)} &\to \frac{\alpha}{\alpha-\theta} - 1 \\ &= \frac{\theta}{\alpha-\theta}.\end{aligned}$$

Hence, we use the equivalence (e) $\Leftrightarrow$ (f) proved above for concluding that $H_\alpha(x) \in RV_\theta$. This proves that (g) $\Rightarrow$ (e). $\square$

Proof of Theorem 2.6: Let $\alpha, \beta > 0$, and $L(x) \in RV_0$.

Let us prove (a) $\Leftrightarrow$ (c). The reason for this is $(zx)^\alpha \overline{G}_\alpha(zx) - x^\alpha \overline{G}_\alpha(x) = \alpha\left(W_\alpha(zx) - W_\alpha(x)\right)$.

Let us prove (a) $\Rightarrow$ (b). Assume that $x^\alpha \overline{G}_\alpha(x) \in \Pi_\beta(L)$. Noting that, for any $z > 0$, $(zx)^\alpha \overline{G}_\alpha(zx) - x^\alpha \overline{G}_\alpha(x) = \alpha\left(W_\alpha(zx) - W_\alpha(x)\right)$ and $L(zx)/L(x) \to 1$ as $x \to \infty$, by applying Lemma 3.2 with $A(x) = L(x)$, $B(x) = C(x) = \frac{\beta}{\alpha}\log x$, and $\xi = \beta/\alpha$, we find that $x^\alpha \overline{F}(x)/L(x) \to \beta/\alpha$ as $x \to \infty$.

Let us prove (b) $\Rightarrow$ (a). Assume that $x^\alpha \overline{F}(x)/L(x) \to \beta/\alpha$ as $x \to \infty$. This means that $\overline{F}(x) \in RV_{-\alpha}$. Using $W_\alpha(x)$, because $\overline{F}(x)$ is decreasing, for $z > 1$, we have

$$W_\alpha(zx) - W_\alpha(x) = \int_x^{zx} y^{\alpha-1}\overline{F}(y)dy.$$

Then, by using $L(zx)/L(x) \to 1$ as $x \to \infty$ and the dominated convergence theorem, as $x \to \infty$, we obtain

$$\begin{aligned}\frac{W_\alpha(zx) - W_\alpha(x)}{L(x)} &= \int_x^{zx} y^{-1}\frac{y^\alpha \overline{F}(y)}{L(x)}dy \\ &= \int_1^{z} y^{-1}\frac{(xy)^\alpha \overline{F}(xy)}{L(xy)}\frac{L(xy)}{L(x)}dy \\ &\to \frac{\beta}{\alpha}\log z.\end{aligned}$$

It follows that $W_\alpha(x) \in \Pi_{\beta/\alpha}(L)$. Hence, we deduce that $x^\alpha \overline{G}_\alpha(x) \in \Pi_\beta(L)$ by applying the equivalence (a) $\Leftrightarrow$ (c) proved above. $\square$

Proof of Theorem 2.8: Let us prove (a) $\Leftrightarrow$ (c). This follows immediately from Proposition 2.2 (iv).

Let us prove (a) $\Rightarrow$ (b). Assume that $x^\alpha \overline{G}_\alpha(x) \in E\Gamma_\lambda(g,1)$. Noting that, for any $y \in \mathbb{R}$, $(x + yg(x))^\alpha \overline{G}_\alpha(x + yg(x)) - x^\alpha \overline{G}_\alpha(x) = \alpha\left(W_\alpha(x + yg(x)) - W_\alpha(x)\right)$, we have $\alpha W_\alpha \in E\Gamma_{\lambda/\alpha}(g,1)$. Next, by applying Lemma 3.4, we find $g(x)x^{\alpha-1}\overline{F}(x) \to \lambda/\alpha$ as $x \to \infty$.

Let us prove (b) $\Rightarrow$ (a). Assume that $g(x)x^{\alpha-1}\overline{F}(x) \to \lambda/\alpha$ as $x \to \infty$. Note that, for $y \in \mathbb{R}$,

$$W_\alpha(x + yg(x)) - W_\alpha(x) = \int_x^{x+yg(x)} t^{\alpha-1}\overline{F}(t)dt.$$

Then, by using the change of variable $t = x + ug(x)$ and the dominated convergence theorem, as $x \to \infty$, we obtain

$$W_\alpha(x + yg(x)) - W_\alpha(x) = \int_0^y (x + ug(x))^{\alpha-1}\overline{F}(x + ug(x))g(x)du$$

$$\begin{aligned}
&= \int_0^y g(x+ug(x))(x+ug(x))^{\alpha-1} \\
&\quad \overline{F}(x+ug(x))\frac{g(x)}{g(x+ug(x))}du \\
&\to \frac{\lambda}{\alpha}\int_0^y du \\
&= \frac{\lambda}{\alpha}y.
\end{aligned}$$

It follows that $W_\alpha(x) \in E\Gamma_{\lambda/\alpha}(g,1)$. Hence, we deduce that $x^\alpha \overline{G}_\alpha(x) \in E\Gamma_\alpha(g,1)$ by applying the equivalence (a) $\Leftrightarrow$ (c) proved above. $\square$

4. Some Complements

4.1. **Derivatives.** We briefly discuss statements about the derivatives $G'_\alpha(x)$ and $G''_\alpha(x)$. To this end, we consider the inversion formula, see Proposition 2.2 (ii),

$$F(x) = G_\alpha(x) + \frac{x}{\alpha}G'_\alpha(x).$$

If F has a probability density function (p.d.f.) f, we also have

$$f(x) = \left(1+\frac{1}{\alpha}\right)G'_\alpha(x) + \frac{x}{\alpha}G''_\alpha(x).$$

Now, suppose that $H_\alpha(x) \in RV_\theta$, $0 \le \theta < \alpha$. By Theorem 2.3 (i) and (ii), we proved that

$$\frac{x^\alpha \overline{G}_\alpha(x)}{H_\alpha(x)} \to \frac{\alpha}{\alpha-\theta}, \quad \text{and} \quad \frac{x^\alpha \overline{F}(x)}{H_\alpha(x)} \to \frac{\theta}{\alpha-\theta},$$

as $x \to \infty$, and it follows, by using the previous inversion formula,

$$\frac{x^{1+\alpha}G'_\alpha(x)}{H_\alpha(x)} \to \alpha, \quad \text{as } x \to \infty,$$

so that $G'_\alpha(x) \in RV_{\theta-\alpha-1}$.

Also, if $F(x)$ has a p.d.f. $f(x)$, we find

$$x^{2+\alpha}G''_\alpha(x) = \alpha x^{1+\alpha}f(x) - (\alpha+1)x^{1+\alpha}G'_\alpha(x).$$

Moreover, if $xf(x)/\overline{F}(x) \to \alpha-\theta$ as $x \to \infty$, we have $x^{1+\alpha}f(x) \sim (\alpha-\theta)x^\alpha\overline{F}(x) \sim \theta H_\alpha(x)$, and it follows that, as $x \to \infty$,

$$\begin{aligned}
\frac{x^{2+\alpha}G''_\alpha(x)}{H_\alpha(x)} &= \frac{x^{1+\alpha}f(x)}{H_\alpha(x)} - \frac{\alpha+1}{\alpha}\frac{x^{1+\alpha}G'_\alpha(x)}{H_\alpha(x)} \\
&\to \theta-\alpha-1.
\end{aligned}$$

In the special case that $m(\alpha) < \infty$, we have the following corollary.

Corollary 4.1. *Let $\alpha > 0$. Suppose that $m(\alpha) < \infty$. We have $x^{\alpha}\overline{F}(x) \to 0$, $x^{\alpha}\overline{G}_{\alpha}(x) \to m(\alpha)$, and $x^{1+\alpha}G'_{\alpha}(x) \to \alpha m(\alpha)$ as $x \to \infty$. If F has a p.d.f. $f(x)$, and also, for some $\delta \in \mathbb{R}$, $x^{1+\delta}f(x) = o(1)$ or $xf(x) = O(\overline{F}(x))$, then $x^{2+\delta}G''_{\alpha}(x) \to -\alpha(\alpha+1)m(\alpha)$ as $x \to \infty$.*

4.2. **The Case $\alpha = 2$.** If $\alpha = 2$, we have $H_2(x) = \int_0^x y^2 dF(y)$ which is the truncated second moment function. Note that regular variation of $H_2(x)$ appears when we discuss domains of attraction. As an example: $F(x)$ is in the domain of attraction of a normal law if and only if $H_2(x) \in RV_0$. Then, we have the following obtained result when it is fixed $\alpha = 2$ in Theorems 2.3 and 2.6.

Theorem 4.2.

(i) *Suppose that $0 < \theta < 2$. We have $H_2(x) \in RV_{\theta} \Leftrightarrow \overline{G}_2(x) \in RV_{\theta-2} \Leftrightarrow \overline{F}(x) \in RV_{\theta-2} \Leftrightarrow W_2(x) \in RV_{\theta}$. Each of these statements implies that $x^2\overline{F}(x)/H_2(x) \to \theta/(2-\theta)$, $x^2\overline{G}_2(x)/H_2(x) \to 2/(2-\theta)$, and $W_2(x)/H_2(x) \to 1/(2-\theta)$ as $x \to \infty$.*

(ii) *Suppose that $0 \leq \theta < 2$. We have $H_2(x) \in RV_{\theta} \Leftrightarrow x^2\overline{F}(x)/H_2(x) \to \theta/(2-\theta)$ as $x \to \infty \Leftrightarrow x^2\overline{G}_2(x)/H_2(x) \to 2/(2-\theta)$ as $x \to \infty$.*

(iii) *Suppose that $L(x) \in RV_0$ and $\delta = 2\lambda$ for $\lambda, \delta > 0$. We have $H_2(x) \in \Pi_{\lambda}(L) \Leftrightarrow W_2(x) \in \Pi_{\delta}(L) \Leftrightarrow x^2\overline{F}(x)/L(x) \to \lambda$ as $x \to \infty$.*

Furthermore, if $H_2(\infty) = m(2) < \infty$, we have the following result. It is obtained by applying Theorem 2.5 when taking $\alpha = 2$. Note that $\overline{H}_2(x) = m(2) - H_2(x)$.

Lemma 4.3. *Suppose that $\theta > 2$ and $m(2) < \infty$. The following are equivalent:*

(i) *$\overline{F}(x) \in RV_{-\theta}$;*
(ii) *$\overline{H}_2(x) = \int_x^{\infty} y^2 dF(y) \in RV_{2-\theta}$; and,*
(iii) *$\int_x^{\infty} y^2 dF(y)/(x^2\overline{F}(x)) \to \theta/(\theta-2)$ as $x \to \infty$.*

Also, we have the following result. It is obtained by applying Theorem 2.5 when taking $\alpha = 2$.

Lemma 4.4. *Suppose that $\theta > 2$ and $m(2) < \infty$. The following are equivalent:*

(i) *$\overline{F}(x) \in RV_{-\theta}$;*
(ii) *$\overline{W}_2(x) = \int_x^{\infty} y\overline{F}(y)dy \in RV_{2-\theta}$;*
(iii) *$x^{-2}m(2) - \overline{G}_2(x) \in RV_{-\theta}$;*
(iv) *$\frac{\int_x^{\infty} y\overline{F}(y)dy}{x^2\overline{F}(x)} \to \frac{1}{\theta-2}$ as $x \to \infty$; and,*
(v) *$\frac{x^{-2}m(2)-\overline{G}_2(x)}{\overline{F}(x)} \to \frac{2}{\theta-2}$ as $x \to \infty$.*

4.3. **The Case $\alpha = 1$.** Now we have $H_1(x) = \int_0^x ydF(y)$, the truncated first moment.

Fixing $\alpha = 1$ in Theorems 2.3 and 2.6, we have the following results.

Theorem 4.5.

(i) *Suppose that* $0 < \theta < 1$. *We have* $H_1(x) \in RV_\theta \Leftrightarrow \overline{G}_1(x) \in RV_{\theta-1} \Leftrightarrow \overline{F}(x) \in RV_{\theta-1} \Leftrightarrow W_1(x) \in RV_\theta$. *Each of these statements implies that* $x\overline{F}(x)/H_1(x) \to \theta/(1-\theta)$, $x\overline{G}_1(x)/H_1(x) \to 1/(1-\theta)$, *and* $W_1(x)/H_1(x) \to 1/(1-\theta)$ *as* $x \to \infty$.

(ii) *Suppose that* $0 \le \theta < 1$. *We have* $H_1(x) \in RV_\theta \Leftrightarrow x\overline{F}(x)/H_1(x) \to \theta/(1-\theta)$ *as* $x \to \infty \Leftrightarrow x\overline{G}_1(x)/H_1(x) \to 1/(1-\theta)$ *as* $x \to \infty$.

(iii) *Suppose that* $L(x) \in RV_0$. *Then* $H_1(x) \in \Pi_\lambda(L) \Leftrightarrow W_1(x) \in \Pi_\delta(L) \Leftrightarrow x\overline{F}(x)/L(x) \to \delta$ *as* $x \to \infty$. *The relation between* λ *and* δ *is given by* $\lambda = \delta$.

Furthermore, if $m(1) < \infty$, we have the following result. It is obtained by applying Theorem 2.5 when taking $\alpha = 1$. Note that $\overline{H}_1(x) = m(1) - H_1(x)$.

Lemma 4.6. *Suppose that* $\theta > 1$ *and* $m(1) < \infty$. *The following are equivalent:*

(i) $\overline{F}(x) \in RV_{-\theta}$;

(ii) $\overline{H}_1(x) = \int_x^\infty ydF(y) \in RV_{1-\theta}$; *and,*

(iii) $\int_x^\infty ydF(y)/(x\overline{F}(x)) \to \theta/(\theta-1)$ *as* $x \to \infty$.

Moreover, we have that the following result, which is obtained by applying Theorem 2.5 when taking $\alpha = 1$.

Lemma 4.7. *Suppose that* $\theta > 1$ *and* $m(1) < \infty$. *The following are equivalent:*

(i) $\overline{F}(x) \in RV_{-\theta}$;

(ii) $\overline{W}_1(x) = \int_x^\infty \overline{F}(y)dy \in RV_{1-\theta}$;

(iii) $x^{-1}m(1) - \overline{G}_1(x) \in RV_{-\theta}$;

(iv) $\frac{\int_x^\infty \overline{F}(y)dy}{x\overline{F}(x)} \to \frac{1}{\theta-1}$ *as* $x \to \infty$; *and,*

(v) $\frac{x^{-1}m(1)-\overline{G}_1(x)}{\overline{F}(x)} \to \frac{1}{\theta-1}$ *as* $x \to \infty$.

Remark 4.8. The equivalence between $H_1(x) \sim l_1(x)$ and $W_1(x) \sim l_2(x)$ for some $l_1(x), l_2(x) \in RV_0$, was proved by [8], see e.g. (i) and (ii) in Theorem 8.8.1 in [1].

5. Concluding Remarks

We have extended and proved the conjecture formulated by [9] on distribution functions (d.f.s) by applying the Williamson transform to such a d.f.. Interestingly, that application became a new function as part of that conjecture.

This novel application and the procedures used for proving the extended conjecture have motivated us to explore asymptotic behaviors when considering other functions than regularly varying functions. For instance, long tailed functions and functions belonging to the class $\Gamma_\lambda(g)$, see e.g. [6]. This class consists of functions f satisfying, for all $y \in \mathbb{R}$, $f(x+yg(x))/f(x) \to \exp(\alpha y)$ as $x \to \infty$. These studies may involve convergence rates as the ones presented in [7]. In a forthcoming paper, we will present the findings of this analysis.

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