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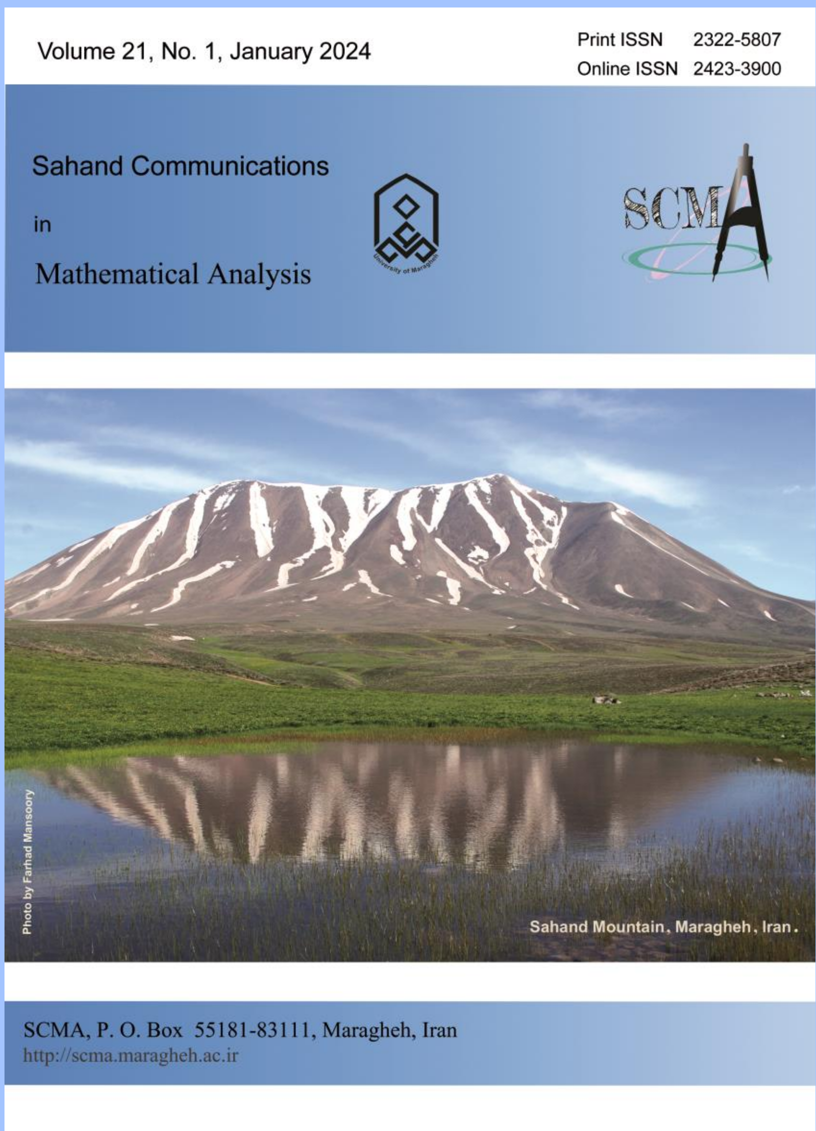
Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 1
Pages: 47-66

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.1971215.1191



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 1 (2024), 47-66
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.1971215.1191

New Fixed Point Results for Some Rational Contraction on (ϕ, ψ)-Metric Spaces

Mohammed M.A.Taleb[1]* and V.C Borkar[2]

ABSTRACT. In this article, we define generalized $(\varphi, \sigma, \gamma)$-rational contraction, generalized $(\alpha\beta, \varphi\theta, F)$-rational contraction and establish some new fixed point results in (ϕ, ψ)-metric space. We also present instances to support our main results. We will use the results we obtained to investigate the existence and uniqueness of solutions to first-order differential equations.

1. Introduction

In metric spaces, fixed point theory is a branch of mathematical analysis intimately linked linked to the existence and uniqueness of integral and differential equation solutions. One of the most important theorems in fixed point theory is the Banach Contraction Principle (see[22]). The notions of metric spaces have been extended in many directions (see[8–10, 18, 19, 28]), for example, controlled metric spaces [21] and double-controlled metric spaces [24]. Bakhtin [15] introduced b-metric spaces as a metric space generalisation. Kamran et al.[25] gave the notion of extended b-metric spaces. $\mathcal{F}$-metric space was introduced by Jalili and Samet [19] as a generalization of metric space. Some researchers introduced an extension (or generalizations) of $\mathcal{F}$-metric space like Chuanxi et al.[7], Kushal Roy et al.[16], and Eskandar Ameer et al.[11] named (ϕ, ψ)-metric space. In this paper, we will present new results for the fixed point theorems in (ϕ, ψ)-metric space under some generalisation rational contractions we defined.

2020 *Mathematics Subject Classification.* 54H25, 47H10.
Key words and phrases. Fixed point, (ϕ, ψ)-metric space, Generalized $(\varphi, \sigma, \gamma)$-rational contraction, Generalized $(\alpha\beta, \varphi\theta, F)$-rational contraction, First order differential equations.
Received: 22 October 2022, Accepted: 08 May 2023.
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2. Preliminaries

Definition 2.1 ([19])**.** Consider the family $\mathcal{F}$ consisting of each functions $\mathfrak{f} : (0, +\infty) \longrightarrow \mathbb{R}$, such that:

($\mathcal{F}_1$) $\mathfrak{f}$ is non decreasing .

($\mathcal{F}_2$) $\forall \, \{\mathfrak{t}_\mathfrak{n}\} \subset (0, +\infty)$, we have

$$\lim_{\mathfrak{n}\to+\infty} \mathfrak{t}_\mathfrak{n} = 0 \quad \Leftrightarrow \quad \lim_{\mathfrak{n}\to+\infty} \mathfrak{f}(\mathfrak{t}_\mathfrak{n}) = -\infty.$$

Definition 2.2 ([19])**.** Let $Z \neq \emptyset$ be a set , and let the function $D_\mathcal{F} : Z \times Z \longrightarrow [0, \infty)$ be a given. If $\exists$ $\mathfrak{f}$ in $\mathcal{F}$, $\tau \in [0, \infty)$, such that,

($\mathfrak{D}_1$) $(\kappa, \nu) \in Z \times Z$, $D_\mathcal{F}(\kappa, \nu) = 0$ if and only if $\kappa = \nu$.

($\mathfrak{D}_2$) $D_\mathcal{F}(\kappa, \nu) = D_\mathcal{F}(\nu, \kappa)$, $\forall \, (\kappa, \nu) \in Z \times Z$.

($\mathfrak{D}_3$) For every $(\kappa, \nu) \in Z \times Z$, $\forall$ N in $\mathbb{N}$ and $N \geq 2$, and also for each $(\mathfrak{v}_\jmath)_{\jmath=1}^N$ in Z with $(\mathfrak{v}_1, \mathfrak{v}_N) = (\kappa, \nu)$, we have

$$D_\mathcal{F}(\kappa, \nu) > 0 \quad \Rightarrow \quad \mathfrak{f}(D_\mathcal{F}(\kappa, \nu)) \leq \mathfrak{f}\left(\sum_{\jmath=1}^{N-1} D_\mathcal{F}(\mathfrak{v}_\jmath, \mathfrak{v}_{\jmath+1})\right) + \tau.$$

Then $(Z, D_\mathcal{F})$ is called $\mathcal{F}$-metric space.

In 2020, Eskander et al.[11], introduced an extension or (generalization) of $\mathcal{F}$-metric spaces named (ϕ, ψ)-metric spaces. This means that (ϕ, ψ)-metric spaces are generalizations of metric spaces and are more significan than $\mathcal{F}$-metric spaces.

Definition 2.3 ([11])**.** Let Φ be the class of functions $\phi : (0, \infty) \to (0, \infty)$ such that :

(ϕ_1) ϕ is non-decreasing,

(ϕ_2) for all $\{\mathfrak{t}_\mathfrak{n}\}$ in $(0, \infty)$,

$$\lim_{\mathfrak{n}\to\infty} \phi(\mathfrak{t}_\mathfrak{n}) = 0 \quad \Leftrightarrow \quad \lim_{\mathfrak{n}\to\infty} (\mathfrak{t}_\mathfrak{n}) = 0.$$

Definition 2.4 ([11])**.** Let Ψ be the set of functions $\psi : (0, \infty) \to (0, \infty)$ such that :

(ψ_1) $\kappa < \nu$ implies $\psi(\kappa) \leq \psi(\nu)$,

(ψ_2) $\psi(\mathfrak{t}) \leq \mathfrak{t}$, $\forall$ $\mathfrak{t} > 0$.

They (see[11]) introduced (ϕ, ψ)-metric space as:

Definition 2.5 ([11])**.** Let $Z \neq \emptyset$, $D_\phi : Z \times Z \to [0, \infty)$ be a function. If $\exists$ a functions $\phi \in \Phi$ and $\psi \in \Psi$ such that for each κ ,$\nu \in Z$, the following hold:

($\mathfrak{d}_1$) $D_\phi(\kappa, \nu) = 0 \quad \Leftrightarrow \quad \kappa = \nu$,

($\mathfrak{d}_2$) $D_\phi(\kappa, \nu) = D_\phi(\nu, \kappa)$,

($\mathfrak{d}_3$) for every $(\kappa,\nu) \in Z \times Z$, $\forall\ N$ in $\mathbb{N}$, $N \geq 2$, and for each $(\mathfrak{v}_\jmath)_{\jmath=1}^{N}$ in Z with $(\mathfrak{v}_1, \mathfrak{v}_N) = (\kappa,\nu)$, we have

$$D_\phi(\kappa,\nu) > 0 \quad \Rightarrow \quad \phi\left(D_\phi(\kappa,\nu)\right) \leq \psi\left(\phi\left(\sum_{\jmath=1}^{N-1} D_\phi\left(\mathfrak{v}_\jmath, \mathfrak{v}_{\jmath+1}\right)\right)\right).$$

Then the pair (Z, D_ϕ) is called a (ϕ,ψ)-metric space.

Example 2.6 ([11])**.** Let Z be the set of natural numbers and $D_\phi : Z \times Z \to [0,\infty)$ be the mapping define by

$$D_\phi(\kappa,\nu) = \begin{cases} \frac{(\kappa-\nu)^2}{9}, & \text{if} \quad (\kappa,\nu) \in [0,2]\times[0,2], \\ |\kappa-\nu|, & \text{if} \quad (\kappa,\nu) \notin [0,2]\times[0,2], \end{cases}$$

for all $(\kappa,\nu) \in Z \times Z$. Then D_ϕ is an (ϕ,ψ)-metric on Z.

Definition 2.7 ([11])**.** Let (Z, D_ϕ) be a (ϕ,ψ)-metric space.

(1) Let $\{\kappa_\mathfrak{n}\}$ be a sequence in Z , $\kappa \in Z$. We say that $\{\kappa_\mathfrak{n}\}$ is (ϕ,ψ)-convergent to κ if $\lim\limits_{\mathfrak{n}\to\infty} D_\phi(\kappa_\mathfrak{n},\kappa) = 0$.

(2) $\{\kappa_\mathfrak{n}\}$ is (ϕ,ψ)-Cauchy if $\lim\limits_{\mathfrak{n},\mathfrak{m}\to\infty} D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m}) = 0$.

(3) (Z, D_ϕ) is (ϕ,ψ)-complete, if any (ϕ,ψ)-Cauchy sequence in Z is (ϕ,ψ)-convergent to some element in Z.

Definition 2.8 ([20])**.** If a function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfies the following axioms:

(φ_1) φ is non-decreasing,

(φ_2) $\varphi(\mathfrak{t}) = 0 \quad \Leftrightarrow \quad \mathfrak{t} = 0$,

then is called an altering distance function.

Let Ω be the set of each altering distance function that satisfies the conditions:

(φ_3) $\sum\limits_{\mathfrak{n}=1}^{\infty} \varphi^\mathfrak{n}(\mathfrak{t}) < \infty,\ \forall\ \mathfrak{t} > 0$.

(φ_4) $\varphi(\mathfrak{t}) < \mathfrak{t},\ \forall\ \mathfrak{t} > 0$.

We direct the reader to [2, 6, 7, 13, 17, 26, 27] for more information on the set Ω.

Example 2.9. The below functions $\varphi : [0,+\infty) \to [0,+\infty)$ are elements of Ω for all $0 \leq \mathfrak{t} < \infty$.

(1) $\varphi(\mathfrak{t}) = k\mathfrak{t},\ 0 < k < 1$,

(2) $\varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{1+\mathfrak{t}}$.

Definition 2.10 ([1, 14, 23])**.** The continuous function $F : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies the below conditions:

($\mathfrak{c}_1$) $F(\mathfrak{t},\mathfrak{s}) \leq \mathfrak{t},\ \forall\ \mathfrak{t},\mathfrak{s} \geq 0$

($\mathfrak{c}_2$) $F(\mathfrak{t},\mathfrak{s})=\mathfrak{t}$ implies either $\mathfrak{t}=0$ or $\mathfrak{s}=0,\ \forall\ \mathfrak{t},\mathfrak{s}\geq 0$

($\mathfrak{c}_3$) $F(0,0)=0,\ \forall\ \mathfrak{t},\mathfrak{s}\geq 0,$

is called C-class functions.

The set of all functions F is denoted by C .

Example 2.11. The following functions $F:[0,\infty)\times[0,\infty)\to\mathbb{R}$ are elements of $C,\ \forall\ \mathfrak{s},\mathfrak{t}\in[0,\infty)$

(1) $F(\mathfrak{t},\mathfrak{s})=\mathfrak{t}-\mathfrak{s},$

(2) $F(\mathfrak{t},\mathfrak{s})=k\mathfrak{t},\ k\in(0,1).$

Definition 2.12 ([1])**.** A function $\theta:[0,\infty)\to[0,\infty)$ such that:

(θ_1) θ is continuous,

(θ_2) $\theta(\mathfrak{t})>0,\ \mathfrak{t}>0$ and $\theta(0)\geq 0.$

is called an ultra altering distance function.

Let Θ denote the class of all ultra altering distance function.

The concept of $(\alpha-\varphi)$-contractions and α-admissible mapping was introduced by Samet et al.in 2012 (see[6]). The concept of α-admissible mappings was defined as follows:

Definition 2.13 ([6])**.** Let $H:Z\to Z$ and $\alpha:Z\times Z\to[0,\infty)$ be a mapping. Then H is called α-admissible mapping if:

$$\alpha(\kappa,\nu)\geq 1 \quad\Rightarrow\quad \alpha(H\kappa,H\nu)\geq 1,$$

$\forall\ \kappa,\ \nu\in Z.$

In 2020, Hamed et al. [12] introduced the concepts of twisted (α,β)-admissible in $\mathcal{F}$-metric space, introduced some generalized contractions, and provided new fixed point results.

Definition 2.14 ([12])**.** Let $H:Z\to Z$ and $\alpha,\ \beta:Z\times Z\to[0,+\infty)$. Then H is called twisted (α,β)-admissible if:

$$\begin{cases}\alpha(\kappa,\nu)\geq 1\\ \beta(\kappa,\nu)\geq 1\end{cases} \quad\Rightarrow\quad \begin{cases}\alpha(H\kappa,H\nu)\geq 1\\ \beta(H\kappa,H\nu)\geq 1\end{cases}$$

3. Main Results

In 2021, Bhavana Deshpande et al.[5] established the coincidence point theorem under generalized (φ,σ,γ)-contraction on partially ordered metric spaces. In this section, we present a definition of generalized (φ,σ,γ)-rational contraction and establish a new fixed point theorem in (ϕ,ψ)-metric space.

3.1. Fixed Point Results for Generalized $(\varphi, \sigma, \gamma)$-Rational Contraction.

Definition 3.1. Let (Z, D_ϕ) be an (ϕ, ψ)-metric space. The mapping $H : Z \to Z$ is said to be a generalized $(\varphi, \sigma, \gamma)$-rational contraction if $\exists$ an upper and a lower semi-continuous functions $\sigma, \gamma : [0, +\infty) \to [0, +\infty)$ respectively, and an altering distance function φ, such that:

(3.1)
$$\begin{aligned}\varphi\left(D_\phi(H\kappa, H\nu)\right) \leq \sigma &\left(\max\left\{D_\phi(\kappa,\nu), \frac{D_\phi(\kappa, H\kappa)D_\phi(\nu, H\nu)}{1 + D_\phi(\kappa,\nu)}\right\}\right) \\ &- \gamma\left(\max\left\{D_\phi(\kappa,\nu), \frac{D_\phi(\kappa, H\kappa)D_\phi(\nu, H\nu)}{1 + D_\phi(\kappa,\nu)}\right\}\right)\end{aligned}$$

$\forall\ \kappa, \nu \in Z$, where $\sigma(0) = \gamma(0) = 0$.

Theorem 3.2. *Let (Z, D_ϕ) be an (ϕ, ψ)-complete (ϕ, ψ)-metric space and $H : Z \to Z$ be a generalized $(\varphi, \sigma, \gamma)$-rational contraction. If the below conditions holds:*

(*i*) $\exists\ \lambda$ *in* $(0, 1)$ *such that:* $\lambda\varphi(\mathfrak{t}) - \sigma(\mathfrak{t}) + \gamma(\mathfrak{t}) > 0$ *for all* $\mathfrak{t} > 0$,
(*i*) $\varphi(\mathfrak{s} + \mathfrak{t}) \leq \varphi(\mathfrak{s}) + \varphi(\mathfrak{t}), \forall\ \mathfrak{s}, \mathfrak{t} \geq 0$.

Then H has a unique fixed point in Z.

Proof. Let $\kappa_0 \in Z$. Define $\{\kappa_\mathfrak{n}\}$ in Z by

$$\kappa_{\mathfrak{n}+1} = H\kappa_\mathfrak{n}, \tag{3.2}$$

where $\mathfrak{n}$ in $\mathbb{N}$. Now if for some $\mathfrak{n}$, $D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}) = 0$, then $\kappa_\mathfrak{n}$ is a fixed point of H, the proof has been completed. Suppose that $D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}) > 0\ \forall \mathfrak{n} \in \mathbb{N}$. Using (3.1), we get

$$\begin{aligned}&\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1}, \kappa_{\mathfrak{n}+2})\right) \\ &\quad= \varphi\left(D_\phi(H\kappa_\mathfrak{n}, H\kappa_{\mathfrak{n}+1})\right) \\ &\quad\leq \sigma\left(\max\left\{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}), \frac{D_\phi(\kappa_\mathfrak{n}, H\kappa_\mathfrak{n})D_\phi(\kappa_{\mathfrak{n}+1}, H\kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}\right\}\right) \\ &\qquad- \gamma\left(\max\left\{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}), \frac{D_\phi(\kappa_\mathfrak{n}, H\kappa_\mathfrak{n})D_\phi(\kappa_{\mathfrak{n}+1}, H\kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}\right\}\right), \\ &\quad\leq \sigma\left(\max\left\{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}), \frac{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1}, \kappa_{\mathfrak{n}+2})}{1 + D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}\right\}\right) \\ &\qquad- \gamma\left(\max\left\{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}), \frac{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1}, \kappa_{\mathfrak{n}+2})}{1 + D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}\right\}\right).\end{aligned}$$

If

$$\max\left\{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}), \frac{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1}, \kappa_{\mathfrak{n}+2})}{1 + D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}\right\}$$

$$= \frac{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})}{1+D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}.$$

Then, we get

$$\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right) \leq \sigma\left(\frac{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})}{1+D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}\right) - \gamma\left(\frac{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})}{1+D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}\right).$$

Since

$$\frac{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})}{1+D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})} < D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2}).$$

Then we have

$$\varphi\big(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\big) \leq \sigma\big(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\big) - \gamma\big(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\big). \tag{3.3}$$

By using (i), we have

$$\lambda\varphi\big(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\big) - \sigma\big(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\big) + \gamma\big(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\big) > 0. \tag{3.4}$$

By (3.3),(3.4), we have

$$\begin{aligned}\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right) &> \lambda\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right)\\ &> \sigma\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right) - \gamma\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right)\\ &\geq \varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right).\end{aligned}$$

Since φ is non-decreasing, we have $D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2}) > D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})$, which is a contradiction, and hence

$$\max\left\{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}), \frac{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})}{1+D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}\right\} = D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}).$$

Then, we get

$$\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right) \leq \sigma\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right) - \gamma\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right). \tag{3.5}$$

By (i), we have

$$\lambda\varphi\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right) - \sigma\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right) + \gamma\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right) > 0. \tag{3.6}$$

Using (3.5),(3.6), we obtain

$$\begin{aligned}\lambda\varphi\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right) &> \sigma\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right) - \gamma\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right)\\ &\geq \varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right).\end{aligned}$$

Then

$$\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})\right) < \lambda\varphi\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right). \tag{3.7}$$

Since φ is non-decreasing, therefore

$$D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2}) < \lambda D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}). \tag{3.8}$$

Similarly by using (3.1)

$$\begin{aligned}
&\varphi\left(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right)\\
&\quad=\varphi\left(D_\phi(H\kappa_{\mathfrak{n}-1},H\kappa_{\mathfrak{n}})\right)\\
&\quad\leq\sigma\left(\max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),\frac{D_\phi(\kappa_{\mathfrak{n}-1},H\kappa_{\mathfrak{n}-1})D_\phi(\kappa_{\mathfrak{n}},H\kappa_{\mathfrak{n}})}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})}\right\}\right)\\
&\qquad-\gamma\left(\max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),\frac{D_\phi(\kappa_{\mathfrak{n}-1},H\kappa_{\mathfrak{n}-1})D_\phi(\kappa_{\mathfrak{n}},H\kappa_{\mathfrak{n}})}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})}\right\}\right),\\
&\quad\leq\sigma\left(\max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),\frac{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})}\right\}\right)\\
&\qquad-\gamma\left(\max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),\frac{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})}\right\}\right).
\end{aligned}$$

Also, we get

$$D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})<\lambda D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}). \tag{3.9}$$

Using (3.8), (3.9) and continuing in this way we get

$$\begin{aligned}
D_\phi(\kappa_{\mathfrak{n}+1},\kappa_{\mathfrak{n}+2})&\leq\lambda D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\\
&\leq\lambda\big(\lambda D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})\big)\\
&\ \ \vdots\\
&\leq\lambda^{\mathfrak{n}}D_\phi(\kappa_0,\kappa_1),
\end{aligned} \tag{3.10}$$

for all $\mathfrak{n}$ in $\mathbb{N}$. Thus

$$\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1}D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\leq\frac{\lambda^{\mathfrak{n}}}{1-\lambda}D_\phi(\kappa_0,\kappa_1),\quad\mathfrak{m}>\mathfrak{n}. \tag{3.11}$$

By (ϕ_1), we have

$$\phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1}D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\right)\leq\phi\left(\frac{\lambda^{\mathfrak{n}}}{1-\lambda}D_\phi(\kappa_0,\kappa_1)\right),\quad\mathfrak{m}>\mathfrak{n}.$$

Using (ψ_1) , (ψ_2), we obtain

$$\begin{aligned}
\psi\left(\phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1}D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\right)\right)&\leq\psi\left(\phi\left(\frac{\lambda^{\mathfrak{n}}}{1-\lambda}D_\phi(\kappa_0,\kappa_1)\right)\right)\\
&\leq\phi\left(\frac{\lambda^{\mathfrak{n}}}{1-\lambda}D_\phi(\kappa_0,\kappa_1)\right),\quad\mathfrak{m}>\mathfrak{n}.
\end{aligned} \tag{3.12}$$

Since $\lim\limits_{\mathfrak{n}\to\infty}\frac{\lambda^{\mathfrak{n}}}{1-\lambda}D_\phi(\kappa_0,\kappa_1)=0$, then by (ϕ_2), we have

$$\lim_{\mathfrak{n}\to\infty}\phi\left(\frac{\lambda^{\mathfrak{n}}}{1-\lambda}D_\phi(\kappa_0,\kappa_1)\right)=0. \tag{3.13}$$

Using $(\mathfrak{d}_3)$, we get $D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m}) > 0$, $\mathfrak{m} > \mathfrak{n}$ then

$$\begin{aligned}\phi\left(D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})\right) &\leq \psi\left(\phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1} D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\right)\right)\\ &\leq \phi\left(\frac{\lambda^\mathfrak{n}}{1-\lambda} D_\phi(\kappa_0,\kappa_1)\right).\end{aligned}$$

Then implies that

$$\phi\left(D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})\right) \leq \phi\left(\frac{\lambda^\mathfrak{n}}{1-\lambda} D_\phi(\kappa_0,\kappa_1)\right),$$

by (3.13), we get

$$\lim_{\mathfrak{n},\mathfrak{m}\to\infty} \phi\Big(D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})\Big) = 0,$$

and using (ϕ_2), we get,

$$\lim_{\mathfrak{n},\mathfrak{m}\to\infty} D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m}) = 0.$$

This proves that $\{\kappa_\mathfrak{n}\}$ is (ϕ,ψ)-Cauchy in Z. Since Z is (ϕ,ψ)-complete, then there exists a point $\kappa^* \in Z$ such that

$$\lim_{\mathfrak{n}\to\infty} D_\phi(\kappa_\mathfrak{n},\kappa^*) = 0. \tag{3.14}$$

Now we prove that $H\kappa^* = \kappa^*$. We use proof by contradiction. So we will assume that $D_\phi(H\kappa^*,\kappa^*) > 0$. By using $(\mathfrak{d}_3)$, we have

$$\begin{aligned}\phi\left(D_\phi(H\kappa^*,\kappa^*)\right) &\leq \psi\left(\phi\left(D_\phi(H\kappa^*,H\kappa_\mathfrak{n}) + D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right)\right)\\ &\leq \phi\left(D_\phi(H\kappa^*,H\kappa_\mathfrak{n}) + D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right).\end{aligned} \tag{3.15}$$

From (ϕ_1), we get $D_\phi(H\kappa^*,\kappa^*) < D_\phi(H\kappa^*,H\kappa_\mathfrak{n}) + D_\phi(H\kappa_\mathfrak{n},\kappa^*)$. By (φ_1) , (ii), we obtain

$$\begin{aligned}\varphi\left(D_\phi(H\kappa^*,\kappa^*)\right) &\leq \varphi\left(D_\phi(H\kappa^*,H\kappa_\mathfrak{n}) + D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right)\\ &\leq \varphi\left(D_\phi(H\kappa^*,H\kappa_\mathfrak{n})\right) + \varphi\left(D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right),\end{aligned}$$

and using (3.1)

$$\begin{aligned}\varphi\left(D_\phi(H\kappa^*,\kappa^*)\right) &\leq \sigma\left(\max\left\{D_\phi(\kappa^*,\kappa_\mathfrak{n}), \frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_\mathfrak{n},H\kappa_\mathfrak{n})}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})}\right\}\right)\\ &\quad - \gamma\left(\max\left\{D_\phi(\kappa^*,\kappa_\mathfrak{n}), \frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_\mathfrak{n},H\kappa_\mathfrak{n})}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})}\right\}\right)\\ &\quad + \varphi\left(D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right)\\ &\leq \sigma\left(\max\left\{D_\phi(\kappa^*,\kappa_\mathfrak{n}), \frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})}\right\}\right)\\ &\quad - \gamma\left(\max\left\{D_\phi(\kappa^*,\kappa_\mathfrak{n}), \frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})}\right\}\right)\end{aligned}$$

$$+\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa^*)\right).$$

Now if

$$\max\left\{D_\phi(\kappa^*,\kappa_{\mathfrak{n}}),\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_{\mathfrak{n}})}\right\}$$
$$=\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_{\mathfrak{n}})},$$

then we get

$$\varphi\left(D_\phi(H\kappa^*,\kappa^*)\right)$$
$$\leq\sigma\left(\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_{\mathfrak{n}})}\right)-\gamma\left(\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_{\mathfrak{n}})}\right)$$
$$+\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa^*)\right).$$

Taking $\lim_{\mathfrak{n}\to\infty}$, by (3.14) and for $\sigma(0)=\gamma(0)=0$, we get

$$\lim_{\mathfrak{n}\to\infty}\varphi\big(D_\phi(H\kappa^*,\kappa^*)\big)=0\quad\Rightarrow\quad\varphi\big(D_\phi(H\kappa^*,\kappa^*)\big)=0.$$

By (φ_2) we have,

$$D_\phi(H\kappa^*,\kappa^*)=0. \tag{3.16}$$

And this contradicts our assumption that, $D_\phi(H\kappa^*,\kappa^*)>0$, and hence $D_\phi(H\kappa^*,\kappa^*)=0$.

If

$$\max\left\{D_\phi(\kappa^*,\kappa_{\mathfrak{n}}),\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_{\mathfrak{n}})}\right\}=D_\phi(\kappa^*,\kappa_{\mathfrak{n}}),$$

then we get

$$\varphi\left(D_\phi(H\kappa^*,\kappa^*)\right)\leq\sigma\left(D_\phi(\kappa^*,\kappa_{\mathfrak{n}})\right)-\gamma\left(D_\phi(\kappa^*,\kappa_{\mathfrak{n}})\right)+\varphi\left(D_\phi(\kappa_{\mathfrak{n}+1},\kappa^*)\right).$$

Taking $\lim_{\mathfrak{n}\to\infty}$, by (3.14) and for $\sigma(0)=\gamma(0)=0$, we get

$$\lim_{\mathfrak{n}\to\infty}\varphi\left(D_\phi(H\kappa^*,\kappa^*)\right)=0\quad\Rightarrow\quad\varphi\left(D_\phi(H\kappa^*,\kappa^*)\right)=0.$$

By (φ_2) we have,

$$D_\phi(H\kappa^*,\kappa^*)=0. \tag{3.17}$$

Also contradicts our assumption that, $D_\phi(H\kappa^*,\kappa^*)>0$. Therefore $D_\phi(H\kappa^*,\kappa^*)=0 \quad\Rightarrow\quad H\kappa^*=\kappa^*$ i.e. H has a fixed point $\kappa^*\in Z$.

Uniqueness:

Now we prove that κ^* is a unique. Assume that $\exists\,\nu^*\in Z,\ \kappa^*\neq\nu^*$, such that $H\nu^*=\nu^*$. By (3.1) we get

(3.18)

$$\varphi\left(D_\phi(\kappa^*,\nu^*)\right)=\varphi\left(D_\phi(H\kappa^*,H\nu^*)\right)$$

$$\leq \sigma\left(\max\left\{D_\phi(\kappa^*,\nu^*), \frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\nu^*,H\nu^*)}{1+D_\phi(\kappa^*,\nu^*)}\right\}\right) - \gamma\left(\max\left\{D_\phi(\kappa^*,\nu^*), \frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\nu^*,H\nu^*)}{1+D_\phi(\kappa^*,\nu^*)}\right\}\right) \leq \sigma\left(D_\phi(\kappa^*,\nu^*)\right) - \gamma\left(D_\phi(\kappa^*,\nu^*)\right).$$

By (i) we get

$$\lambda\varphi\left(D_\phi(\kappa^*,\nu^*)\right) - \sigma\left(D_\phi(\kappa^*,\nu^*)\right) + \gamma\left(D_\phi(\kappa^*,\nu^*)\right) > 0. \tag{3.19}$$

From (3.18),(3.19), we obtain

$$\begin{aligned}\varphi\left(D_\phi(\kappa^*,\nu^*)\right) &> \lambda\varphi\left(D_\phi(\kappa^*,\nu^*)\right)\\ &> \sigma\left(D_\phi(\kappa^*,\nu^*)\right) - \gamma\left(D_\phi(\kappa^*,\nu^*)\right)\\ &\geq \varphi\left(D_\phi(\kappa^*,\nu^*)\right).\end{aligned} \tag{3.20}$$

Then by (φ_1), we have $D_\phi(\kappa^*,\nu^*) < D_\phi(\kappa^*,\nu^*)$, which is a contradiction, and hence $\kappa^* = \nu^*$. □

Example 3.3. Let $Z = [0,1]$. Defined $D_\phi : Z \times Z \to [0,\infty)$ as

$$D_\phi(\kappa,\nu) = \left(\frac{\kappa-\nu}{6}\right)^2$$

then D_ϕ is a (ϕ,ψ)-metric on Z with $\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{36}$ and $\phi(\mathfrak{t}) = \mathfrak{t}$. Define $H : Z \to Z$ by $H\kappa = \frac{\kappa}{2}$ and take $\sigma(\mathfrak{t}) = \mathfrak{t}$, $\varphi(\mathfrak{t}) = \mathfrak{t}$, and $\gamma(\mathfrak{t}) = \frac{2\mathfrak{t}}{3}$ for $\mathfrak{t} \geq 0$.
Clearly, H is a generalized (φ,σ,γ)-rational contraction, and all condition in theorem (3.2) are satisfied with $\frac{1}{3} < \lambda < 1$. Hence $0 \in Z$ is a unique fixed point of H .

3.2. **Fixed Point Results for Generalized $(\alpha\beta,\varphi\theta,F)$-Rational Contraction.** In b-metric spaces, several researchers proved fixed point results for C-class functions, and they introduced the definition of generalized (φ,θ,F)-contraction, where φ is the altering distance function, θ is the ultra altering distance function (see[3, 4, 14, 23, 29]).
In this section, we define the concept of generalized $(\alpha\beta,\varphi\theta,F)$-rational contraction in (ϕ,ψ)-metric space, where $\varphi \in \Omega$ and θ is the ultra altering distance function, and provenew fixed point theorems in (ϕ,ψ)-metric space.

Definition 3.4. Let (Z,D_ϕ) be a (ϕ,ψ)-metric space. The mapping $H : Z \to Z$ is called a generalized $(\alpha\beta,\varphi\theta,F)$-rational contraction if $\exists$ a functions $\varphi \in \Omega$, $\theta \in \Theta$, $F \in C$ and $\alpha,\ \beta : Z \times Z \to [0,+\infty)$, such that

$$\alpha(\kappa,\nu)\beta(\kappa,\nu)D_\phi(H\kappa,H\nu) \leq F\left(\varphi(\mathcal{B}(\kappa,\nu)),\theta\left(\mathcal{B}(\kappa,\nu)\right)\right), \tag{3.21}$$

where

$$\mathcal{B}(\kappa,\nu) = \max\left\{D_\phi(\kappa,\nu), \frac{D_\phi(\kappa,H\kappa)D_\phi(\nu,H\nu)}{1+D_\phi(\kappa,\nu)}, \frac{D_\phi(\nu,H\nu)\left[1+D_\phi(\kappa,H\kappa)\right]}{1+D_\phi(\kappa,\nu)}\right\},$$

for $\kappa,\nu \in Z$.

Theorem 3.5. *Let (Z, D_ϕ) be a (ϕ,ψ)-metric space and $H : Z \to Z$ be both generalized $(\alpha\beta,\varphi\theta,F)$-rational contraction and twisted (α,β)-admissible. If the below hypotheses are satisfied:*

(a) *(Z, D_ϕ) is (ϕ,ψ)-complete,*
(b) *$\exists\ \kappa_0 \in Z$ such that $\alpha(\kappa_0, H\kappa_0) \geq 1$ and $\beta(\kappa_0, H\kappa_0) \geq 1$,*
(c) *if $\{\kappa_\mathfrak{n}\}$ is a sequence in Z such that $\alpha(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1}) \geq 1$ and $\beta(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1}) \geq 1$ for all $\mathfrak{n}$, and $\kappa_\mathfrak{n} \to \kappa^* \in Z$ as $\mathfrak{n} \to \infty$, then $\alpha(\kappa_\mathfrak{n},\kappa^*) \geq 1$ and $\beta(\kappa_\mathfrak{n},\kappa^*) \geq 1$, $\forall\ \mathfrak{n} \in \mathbb{N}$.*

Then there exists a fixed point $\kappa^* \in Z$ such $H\kappa^* = \kappa^*$.

Proof. Let $\kappa_0 \in Z$ such that $\alpha(\kappa_0, H\kappa_0) \geq 1$ and $\beta(\kappa_0, H\kappa_0) \geq 1$. Defined a sequence $\{\kappa_\mathfrak{n}\}$ in Z by

$$\kappa_{\mathfrak{n}+1} = H\kappa_\mathfrak{n},\ \forall\ \mathfrak{n} \in \mathbb{N}. \tag{3.22}$$

If $\kappa_{\mathfrak{n}+1} = \kappa_\mathfrak{n}$ for some $\mathfrak{n}$ in $\mathbb{N}$ then $\kappa_\mathfrak{n}$ is a fixed point of H. As a result, the proof is complete. So suppose that $\kappa_{\mathfrak{n}+1} \neq \kappa_\mathfrak{n}$ $\forall$ $\mathfrak{n}$ in $\mathbb{N}$. Since H is twisted (α,β)-admissible, we have

$\alpha(\kappa_0,\kappa_1) = \alpha(\kappa_0,H\kappa_0) \geq 1 \quad\Rightarrow\quad \alpha(\kappa_1,\kappa_2) = \alpha(H\kappa_0,H\kappa_1) \geq 1,$
and
$\beta(\kappa_0,\kappa_1) = \beta(\kappa_0,H\kappa_0) \geq 1 \quad\Rightarrow\quad \beta(\kappa_1,\kappa_2) = \beta(H\kappa_0,H\kappa_1) \geq 1.$

By induction, we get, $\alpha(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1}) \geq 1$ and $\beta(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1}) \geq 1$ implies $\alpha(H\kappa_{\mathfrak{n}-1},H\kappa_\mathfrak{n}) \geq 1$ and $\beta(H\kappa_{\mathfrak{n}-1},H\kappa_\mathfrak{n}) \geq 1$, $\forall\ \mathfrak{n} \in \mathbb{N}$. By inequality (3.21) with $\kappa = \kappa_{\mathfrak{n}-1}$ and $\nu = \kappa_\mathfrak{n}$, we have

$$\begin{aligned} D_\phi(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1}) &= D_\phi(H\kappa_{\mathfrak{n}-1},H\kappa_\mathfrak{n}) \\ &\leq \alpha(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})\beta(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})D_\phi(H\kappa_{\mathfrak{n}-1},H\kappa_\mathfrak{n}) \\ &\leq F\big(\varphi(\mathcal{B}(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})),\theta(\mathcal{B}(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n}))\big), \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \mathcal{B}(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n}) = \max\Bigg\{ & D_\phi(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n}), \frac{D_\phi(\kappa_{\mathfrak{n}-1},H\kappa_{\mathfrak{n}-1})D_\phi(\kappa_\mathfrak{n},H\kappa_\mathfrak{n})}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})}, \\ & \frac{D_\phi(\kappa_\mathfrak{n},H\kappa_\mathfrak{n})\left[1+D_\phi(\kappa_{\mathfrak{n}-1},H\kappa_{\mathfrak{n}-1})\right]}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})}\Bigg\} \\ = \max\Bigg\{ & D_\phi(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n}), \frac{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})D_\phi(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_\mathfrak{n})}, \end{aligned}$$

$$\left.\frac{D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\left[1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})\right]}{1+D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})}\right\}$$
$$\leq \max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right\}.$$

Now if $\max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right\} = D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})$, from (3.23) we get

$$\begin{aligned}D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}) &\leq F\left(\varphi(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})),\theta(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}))\right)\\ &\leq \varphi(D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})), \quad (by\ c_1)\\ &< D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}), \quad (by\ \varphi_4)\end{aligned}$$

which is a contradiction, and hence $\max\left\{D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}),D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1})\right\} = D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})$, we have,

$$\begin{aligned}D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}) &\leq F\left(\varphi(D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})),\theta(D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}}))\right)\\ &\leq \varphi(D_\phi(\kappa_{\mathfrak{n}-1},\kappa_{\mathfrak{n}})). \quad (by\ c_1)\end{aligned} \tag{3.24}$$

Consequently, we get

$$D_\phi(\kappa_{\mathfrak{n}},\kappa_{\mathfrak{n}+1}) \leq \varphi^{\mathfrak{n}}(D_\phi(\kappa_0,\kappa_1)). \tag{3.25}$$

Thus

$$\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1} D_\phi(\kappa_\jmath,\kappa_{\jmath+1}) \leq \sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1} \varphi^\jmath(D_\phi(\kappa_0,\kappa_1)), \quad \mathfrak{m}>\mathfrak{n}. \tag{3.26}$$

Let $\varepsilon>0$ be fixed. By (ϕ_2), $\exists\ \delta>0$ such that

$$0<\mathfrak{t}<\delta \quad \Rightarrow \quad \phi(\mathfrak{t})<\phi(\varepsilon). \tag{3.27}$$

Let $\mathfrak{n}(\varepsilon)\in\mathbb{N}$ such that

$$0<\sum_{\mathfrak{n}\geq\mathfrak{n}(\varepsilon)} \varphi^{\mathfrak{n}}(D_\phi(\kappa_0,\kappa_1))<\delta. \tag{3.28}$$

By (3.26),(3.27) and (ϕ_1), we have

$$\begin{aligned}\phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1} D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\right) &\leq \phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1} \varphi^\jmath(D_\phi(\kappa_0,\kappa_1))\right)\\ &\leq \phi\left(\sum_{\mathfrak{n}\geq\mathfrak{n}(\varepsilon)} \varphi^{\mathfrak{n}}(D_\phi(\kappa_0,\kappa_1))\right)\\ &< \phi(\varepsilon), \quad \mathfrak{m}>\mathfrak{n}\geq\mathfrak{n}(\varepsilon).\end{aligned} \tag{3.29}$$

By ψ_1 and (ψ_2), we obtain

$$\psi\left(\phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1} D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\right)\right) \leq \psi\left(\phi(\varepsilon)\right) \leq \phi(\varepsilon), \quad \mathfrak{m}>\mathfrak{n}\geq\mathfrak{n}(\varepsilon). \tag{3.30}$$

Using $(\mathfrak{d}_3)$ and (3.30), we get $D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})>0$ then

$$\phi\left(D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})\right)\le\psi\left(\phi\left(\sum_{\jmath=\mathfrak{n}}^{\mathfrak{m}-1}D_\phi(\kappa_\jmath,\kappa_{\jmath+1})\right)\right)$$
$$\le\phi(\varepsilon)$$

then

$$\phi\left(D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})\right)\le\phi(\varepsilon),\quad \mathfrak{m}>\mathfrak{n}\ge\mathfrak{n}(\varepsilon). \tag{3.31}$$

By (ϕ_1), we get $D_\phi(\kappa_\mathfrak{n},\kappa_\mathfrak{m})<\varepsilon$, $\mathfrak{m}>\mathfrak{n}\ge\mathfrak{n}(\varepsilon)$. This shows that $\{\kappa_\mathfrak{n}\}$ is (ϕ,ψ)-Cauchy sequence. Since (Z,D_ϕ) is (ϕ,ψ)-complete, $\exists\ \kappa^*\in Z$ such that $\{\kappa_\mathfrak{n}\}$ is (ϕ,ψ)-convergent to κ^* i.e.

$$\lim_{\mathfrak{n}\to\infty}D_\phi(\kappa_\mathfrak{n},\kappa^*)=0. \tag{3.32}$$

Since $\kappa_\mathfrak{n}\to\kappa^*$, by (c) $\alpha(\kappa_\mathfrak{n},\kappa^*)\ge 1$ and $\beta(\kappa_\mathfrak{n},\kappa^*)\ge 1$, $\forall\ \mathfrak{n}\in\mathbb{N}$. Since H is twisted (α,β)-admissible, we get, $\alpha(H\kappa_\mathfrak{n},H\kappa^*)\ge 1$ and $\beta(H\kappa_\mathfrak{n},H\kappa^*)\ge 1$. Now we prove that $H\kappa^*=\kappa^*$. We prove by contradiction. Assume that $D_\phi(H\kappa^*,\kappa^*)>0$ and inequality (3.21) holed. By $(\mathfrak{d}_3)$ and (ϕ_1), we have

(3.33)

$$\begin{aligned}\phi\left(D_\phi(H\kappa^*,\kappa^*)\right)&\le\psi\left(\phi\left(D_\phi(H\kappa^*,H\kappa_\mathfrak{n})+D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right)\right)\\&\le\phi\left(D_\phi(H\kappa^*,H\kappa_\mathfrak{n})+D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right)\quad (by\ \psi_2)\\&\le\phi\left(\alpha(\kappa^*,\kappa_\mathfrak{n})\beta(\kappa^*,\kappa_\mathfrak{n})D_\phi(H\kappa^*,H\kappa_\mathfrak{n})+D_\phi(H\kappa_\mathfrak{n},\kappa^*)\right)\\&\le\phi\left(F\left(\varphi\left(\mathcal{B}(\kappa^*,\kappa_\mathfrak{n})\right),\theta\left(\mathcal{B}(\kappa^*,\kappa_\mathfrak{n})\right)\right)+D_\phi(\kappa_{\mathfrak{n}+1},\kappa^*)\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{B}(\kappa^*,\kappa_\mathfrak{n})&=\max\Bigg\{D_\phi(\kappa^*,\kappa_\mathfrak{n}),\\&\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_\mathfrak{n},H\kappa_\mathfrak{n})}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})},\frac{D_\phi(\kappa_\mathfrak{n},H\kappa_\mathfrak{n})\left[1+D_\phi(\kappa^*,H\kappa^*)\right]}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})}\Bigg\}\\&=\max\Bigg\{D_\phi(\kappa^*,\kappa_\mathfrak{n}),\\&\frac{D_\phi(\kappa^*,H\kappa^*)D_\phi(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1})}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})},\frac{D_\phi(\kappa_\mathfrak{n},\kappa_{\mathfrak{n}+1})\left[1+D_\phi(\kappa^*,H\kappa^*)\right]}{1+D_\phi(\kappa^*,\kappa_\mathfrak{n})}\Bigg\}.\end{aligned}$$

Now we have three cases:

(1) If

$$\mathcal{B}(\kappa^*,\kappa_\mathfrak{n})=D_\phi(\kappa^*,\kappa_\mathfrak{n}),$$

then from (3.33), we get

$$\begin{aligned}\phi\left(D_\phi(H\kappa^*,\kappa^*)\right)&\le\phi\left(F\left(\varphi\left(D_\phi(\kappa^*,\kappa_\mathfrak{n})\right),\theta\left(D_\phi(\kappa^*,\kappa_\mathfrak{n})\right)\right)+D_\phi(\kappa_{\mathfrak{n}+1},\kappa^*)\right)\\&\le\phi\left(\varphi\left(D_\phi(\kappa^*,\kappa_\mathfrak{n})\right)+D_\phi(\kappa_{\mathfrak{n}+1},\kappa^*)\right)\quad (by\ c_1)\end{aligned}$$

$$< \phi\left(D_\phi(\kappa^*, \kappa_\mathfrak{n}) + D_\phi(\kappa_{\mathfrak{n}+1}, \kappa^*)\right). \quad (by\ \varphi_4)$$

Taking the $\lim\limits_{\mathfrak{n}\to\infty}$ and using (3.32), (ϕ_2), we get

$$\begin{aligned}\lim_{\mathfrak{n}\to\infty} \phi\left(D_\phi(H\kappa^*, \kappa^*)\right) &\leq \lim_{\mathfrak{n}\to\infty} \phi\left(D_\phi(\kappa^*, \kappa_\mathfrak{n}) + D_\phi(\kappa_{\mathfrak{n}+1}, \kappa^*)\right)\\ &= 0,\end{aligned}$$

then we have $D_\phi(H\kappa^*, \kappa^*) = 0$.
(2) If

$$\mathcal{B}(\kappa^*, \kappa_\mathfrak{n}) = \frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})}$$

then from (3.33), we get

$$\begin{aligned}\phi\left(D_\phi(H\kappa^*, \kappa^*)\right) \leq \phi\Bigg(F\Bigg(\varphi\left(\frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})}\right),\\ \theta\left(\frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})}\right)\Bigg) + D_\phi(\kappa_{\mathfrak{n}+1}, \kappa^*)\Bigg)\\ \leq \phi\left(\varphi\left(\frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})}\right) + D_\phi(\kappa_{\mathfrak{n}+1}, \kappa^*)\right)\\ < \phi\left(\frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})} + D_\phi(\kappa_{\mathfrak{n}+1}, \kappa^*)\right).\end{aligned}$$

Taking the $\lim\limits_{\mathfrak{n}\to\infty}$ and using (3.32), (ϕ_2), we get

$$\begin{aligned}&\lim_{\mathfrak{n}\to\infty} \phi\left(D_\phi(H\kappa^*, \kappa^*)\right)\\ &\quad\leq \lim_{\mathfrak{n}\to\infty} \phi\left(\frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})} + D_\phi(\kappa_{\mathfrak{n}+1}, \kappa^*)\right)\\ &\quad= 0,\end{aligned}$$

then we have $D_\phi(H\kappa^*, \kappa^*) = 0$.
(3) If

$$\mathcal{B}(\kappa^*, \kappa_\mathfrak{n}) = \frac{D_\phi(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1})\left[1 + D_\phi(\kappa^*, H\kappa^*)\right]}{1 + D_\phi(\kappa^*, \kappa_\mathfrak{n})}$$

also we get $D_\phi(H\kappa^*, \kappa^*) = 0$. In all cases (1,2,3) we got a contradiction to our assumption that $D_\phi(H\kappa^*, \kappa^*) > 0$, hence, $D_\phi(H\kappa^*, \kappa^*) = 0$ i.e. $H\kappa^* = \kappa^*$. □

Uniqueness. We now show that κ^* is a unique fixed point of H. So, we take the following property:
($\mathcal{P}$) $\alpha(\kappa, \nu) \geq 1$ and $\beta(\kappa, \nu) \geq 1$ for all fixed point $\kappa, \nu \in Z$.

Theorem 3.6. *Consider the hypotheses of theorem* (3.5)*, and let the property* ($\mathcal{P}$) *satisfied, then* H *has unique fixed point.*

Proof. Let $\kappa^*, \nu^* \in Z$ be such that $H\kappa^* = \kappa^*$ and $H\nu^* = \nu^*$, $\kappa^* \neq \nu^*$. Then by $(\mathcal{P})$, we have $\alpha(\kappa^*, \nu^*) \geq 1$ and $\beta(\kappa^*, \nu^*) \geq 1$. By inequality (3.21) with $\kappa = \kappa^*$ and $\nu = \nu^*$, we have

$$\begin{aligned} D_\phi(\kappa^*, \nu^*) = D_\phi(H\kappa^*, H\nu^*) &\leq \alpha(\kappa^*, \nu^*)\beta(\kappa^*, \nu^*) D_\phi(H\kappa^*, H\nu^*) \\ &\leq F\big(\varphi(\mathcal{B}(\kappa^*, \nu^*)), \theta(\mathcal{B}(\kappa^*, \nu^*))\big), \\ &\leq \varphi(\mathcal{B}(\kappa^*, \nu^*)), \end{aligned}$$

where

$$\begin{aligned} \mathcal{B}(\kappa^*, \nu^*) = \max\bigg\{ & D_\phi(\kappa^*, \nu^*), \\ & \frac{D_\phi(\kappa^*, H\kappa^*) D_\phi(\nu^*, H\nu^*)}{1 + D_\phi(\kappa^*, \nu^*)}, \frac{D_\phi(\nu^*, H\nu^*)[1 + D_\phi(\kappa^*, H\kappa^*)]}{1 + D_\phi(\kappa^*, \nu^*)} \bigg\} \\ = {} & D_\phi(\kappa^*, \nu^*). \end{aligned}$$

Then

$$D_\phi(\kappa^*, \nu^*) \leq \varphi(\mathcal{B}(\kappa^*, \nu^*)) = \varphi(D_\phi(\kappa^*, \nu^*)) < D_\phi(\kappa^*, \nu^*),$$

which is a contradiction and hence $D_\phi(\kappa^*, \nu^*) = 0 \quad \Rightarrow \quad \kappa^* = \nu^*$. □

Example 3.7. Let $Z = [0, 1]$. Defined $D_\phi : Z \times Z \to [0, \infty)$ as

$$D_\phi(\kappa, \nu) = \left(\frac{\kappa - \nu}{6}\right)^2,$$

then D_ϕ is a (ϕ, ψ)-metric on Z with $\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{36}$ and $\phi(\mathfrak{t}) = \mathfrak{t}$. Define $H : Z \to Z$ by

$$H\kappa = \begin{cases} \frac{\kappa}{2}, & \kappa \geq 0 \\ 0, & \kappa < 0. \end{cases}$$

Now define $\alpha,\ \beta : Z \times Z \to [0, \infty)$ by

$$\alpha(\kappa, \nu)\beta(\kappa, \nu) = \begin{cases} 1, & \text{if} \quad \kappa, \nu \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Take $F(\mathfrak{s}, \mathfrak{t}) = k\mathfrak{s},\ k \in [\frac{1}{2}, 1),\ \varphi(\mathfrak{t}) = \frac{\mathfrak{t}}{2}$ and $\theta(\mathfrak{t}) = \mathfrak{t}$.

Clearly, H is a generalized $(\alpha\beta, \varphi\theta, F)$-rational contraction, and all conditions in theorems (3.5) and (3.6) are satisfied. Hence H has unique fixed point $0 \in Z$.

Corollary 3.8. *Let (Z, D_ϕ) be a (ϕ, ψ)-metric space and $H : Z \to Z$ be twisted (α, β)-admissible, such that*

$$\alpha(\kappa, \nu)\beta(\kappa, \nu) D_\phi(H\kappa, H\nu) \leq \varphi(\mathcal{B}(\kappa, \nu)) - \theta(\mathcal{B}(\kappa, \nu)), \tag{3.34}$$

where

$$\mathcal{B}(\kappa,\nu) = \max\left\{D_\phi(\kappa,\nu), \frac{D_\phi(\kappa,H\kappa)D_\phi(\nu,H\nu)}{1+D_\phi(\kappa,\nu)}, \frac{D_\phi(\nu,H\nu)[1+D_\phi(\kappa,H\kappa)]}{1+D_\phi(\kappa,\nu)}\right\},$$

for $\kappa,\nu \in Z$. *Suppose that the hypotheses below are satisfied:*

- (a) (Z, D_ϕ) *is* (ϕ,ψ)*-complete,*
- (b) $\exists\ \kappa_0 \in Z$ *such that* $\alpha(\kappa_0, H\kappa_0) \geq 1$ *and* $\beta(\kappa_0, H\kappa_0) \geq 1$,
- (c) *if* $\{\kappa_\mathfrak{n}\}$ *is a sequence in* Z *such that* $\alpha(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}) \geq 1$ *and* $\beta(\kappa_\mathfrak{n}, \kappa_{\mathfrak{n}+1}) \geq 1$ *for all* $\mathfrak{n}$, *and* $\kappa_\mathfrak{n} \to \kappa^* \in Z$ *as* $\mathfrak{n} \to \infty$, *then* $\alpha(\kappa_\mathfrak{n}, \kappa^*) \geq 1$ *and* $\beta(\kappa_\mathfrak{n}, \kappa^*) \geq 1$, $\forall\ \mathfrak{n} \in \mathbb{N}$.

Then there exists a fixed point $\kappa^* \in Z$ *such* $H\kappa^* = \kappa^*$.

Proof. We get the required conclusion by using $F(\mathfrak{t},\mathfrak{s}) = \mathfrak{t}-\mathfrak{s}$ in theorem (3.5). □

4. Applications

We will apply our results to solve the first-order periodic boundary value problem:

$$\begin{cases} \kappa'(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t},\kappa(\mathfrak{t})), & \mathfrak{t} \in [0,T] = I \\ \kappa(0) = \kappa(T). \end{cases} \tag{4.1}$$

Where $\mathfrak{f} : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function on I and $T > 0$. Problem (4.1), can be written as:

$$\begin{cases} \kappa'(\mathfrak{t}) + \mu\kappa(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t},\kappa(\mathfrak{t})) + \mu\kappa(\mathfrak{t}), & \mathfrak{t} \in [0,T] = I \\ \kappa(0) = \kappa(T). \end{cases} \tag{4.2}$$

The integral equation below is equivalent to the problem (4.2)

$$\kappa(\mathfrak{t}) = \int_0^T G(\mathfrak{t},\mathfrak{s})\big(\mathfrak{f}(\mathfrak{s},\kappa(\mathfrak{s}) + \mu\kappa(\mathfrak{s}))\big)d\mathfrak{s}, \tag{4.3}$$

where G is given by

$$G(\mathfrak{t},\mathfrak{s}) = \begin{cases} \frac{e^{\mu(T+\mathfrak{s}-\mathfrak{t})}}{e^{\mu T}-1}, & 0 \leq \mathfrak{s} \leq \mathfrak{t} \leq T, \\ \frac{e^{\mu(\mathfrak{s}-\mathfrak{t})}}{e^{\mu T}-1}, & 0 \leq \mathfrak{t} \leq \mathfrak{s} \leq T. \end{cases}$$

Then we see that

$$\int_0^T G(\mathfrak{t},\mathfrak{s})d\mathfrak{s} = \frac{1}{\mu}.$$

Let $C(I) = S$ be the family of each continuous functions defined in I. Define $D_\phi : S \times S \to [0, \infty)$ by

$$D_\phi(\kappa, \nu) = \left(\frac{1}{6} \sup_{\mathfrak{t} \in I} |\kappa(\mathfrak{t}) - \nu(\mathfrak{t})|\right)^2, \quad \kappa, \nu \in S. \tag{4.4}$$

Then (S, D_ϕ) is a (ϕ, ψ)-complete (ϕ, ψ)-metric space with $\phi(\mathfrak{t}) = \mathfrak{t}$ and $\psi(\mathfrak{t}) = \frac{t}{6^2}$. Define the function $H : S \to S$ by

$$H\kappa(\mathfrak{t}) = \int_0^T G(\mathfrak{t}, \mathfrak{s})\big(\mathfrak{f}(\mathfrak{s}, \kappa(\mathfrak{s}) + \mu\kappa(\mathfrak{s}))\big)d\mathfrak{s}. \tag{4.5}$$

Now we will use theorem (3.2) to prove that H has a unique fixed point, which solve problem (4.1).

Theorem 4.1. *Suppose that $\exists\ \mu > 0$ such that, for all $\kappa, \nu \in S$ and $\mathfrak{s} \in I$,*

$$\begin{aligned} &|\mathfrak{f}(\mathfrak{t}, \kappa(\mathfrak{t})) + \mu\kappa(\mathfrak{t}) - \mathfrak{f}(\mathfrak{t}, \nu(\mathfrak{t})) - \mu\nu(\mathfrak{t})| \\ &\quad \le \left((6\mu)^2\left(\left(\frac{1}{6}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))\right)^2 \log\left(\frac{1}{36}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))^2 + 1\right)\right.\right. \\ &\qquad \left.\left. - \left(\frac{1}{6}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))\right)^2\right)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.6}$$

Then the problem (4.1) *has unique solution in S.*

Proof. Let D_ϕ be a function given by (4.4), H be the operator function given by (4.5).

$$\begin{aligned} &D_\phi(H\kappa(\mathfrak{t}), H\nu(\mathfrak{t})) \\ &\quad = \left(\frac{1}{6}\sup_{\mathfrak{t}\in I} |H\kappa(\mathfrak{t}) - H\nu(\mathfrak{t})|\right)^2 \\ &\quad = \frac{1}{36}\left(\sup_{\mathfrak{t}\in I}\left|\int_0^T G(\mathfrak{t}, \mathfrak{s})\left[\mathfrak{f}(\mathfrak{s}, \kappa(\mathfrak{s})) + \mu\kappa(\mathfrak{s}) - \mathfrak{f}(\mathfrak{s}, \nu(\mathfrak{s})) - \mu\nu(\mathfrak{s})\right] d\mathfrak{s}\right|\right)^2 \\ &\quad \le \frac{1}{36}\left(\sup_{\mathfrak{t}\in I}\int_0^T G(\mathfrak{t}, \mathfrak{s})d\mathfrak{s}\right)^2 \left(\left((6\mu)^2\left(\left(\frac{1}{6}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))\right)^2\right.\right.\right. \\ &\qquad \left.\left.\left.\log\left(\frac{1}{36}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))^2 + 1\right) - \left(\frac{1}{6}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))\right)^2\right)\right)^{\frac{1}{2}}\right)^2 \\ &\quad \le \frac{1}{36}\left(\frac{1}{\mu^2}\right)\left((6\mu)^2\left(\left(\frac{1}{6}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))\right)^2\right.\right. \\ &\qquad \left.\left.\log\left(\frac{1}{36}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))^2 + 1\right) - \left(\frac{1}{6}(\kappa(\mathfrak{t}) - \nu(\mathfrak{t}))\right)^2\right)\right) \end{aligned}$$

$$\leq \left(D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t}))\log\left(D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t}))+1\right)-D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t}))\right)$$

we get

$$D_\phi(H\kappa(\mathfrak{t}),H\nu(\mathfrak{t})) \leq D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t})\log(D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t})))+1) - D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t})). \tag{4.7}$$

Assuming

$$\varphi(\mathfrak{t})=\mathfrak{t}, \qquad \sigma(\mathfrak{t})=\mathfrak{t}\log(\mathfrak{t}+1), \qquad \gamma(\mathfrak{t})=\mathfrak{t}. \tag{4.8}$$

By (4.7),(4.8), we obtain

$$\begin{aligned}\varphi\left(D_\phi(H\kappa(\mathfrak{t}),H\nu(\mathfrak{t}))\right) &\leq \sigma\left(D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t}))\right)-\gamma\left(D_\phi(\kappa(\mathfrak{t}),\nu(\mathfrak{t}))\right)\\ &\leq \sigma\left(\max\left\{D_\phi(\kappa,\nu),\frac{D_\phi(\kappa,H\kappa)D_\phi(\nu,H\nu)}{1+D_\phi(\kappa,\nu)}\right\}\right)\\ &\quad -\gamma\left(\max\left\{D_\phi(\kappa,\nu),\frac{D_\phi(\kappa,H\kappa)D_\phi(\nu,H\nu)}{1+D_\phi(\kappa,\nu)}\right\}\right).\end{aligned}$$

Then H is a generalized (φ,σ,γ)-rational contraction, with $\sigma(0)=\gamma(0)=0$, all conditions in theorem (3.2) are satisfied. Hence H has unique fixed point in S which solve problem (4.1). □

5. Conclusions

In this article, we provided definitions, generalized (φ,σ,γ)-rational contraction, and generalized $(\alpha\beta,\varphi\theta,F)$-rational contraction in (ϕ,ψ)-metric space, and we established and proved some new fixed point results. We supported our results with examples, and we used our results to prove the existence and uniqueness of a solution to a first-order periodic boundary value problem (4.1). Our results have improved, developed, and generalized some results in metric space, b-metric space, and $\mathcal{F}$-metric space.

Acknowledgment. The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and fruitful comments to improve this manuscript.

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