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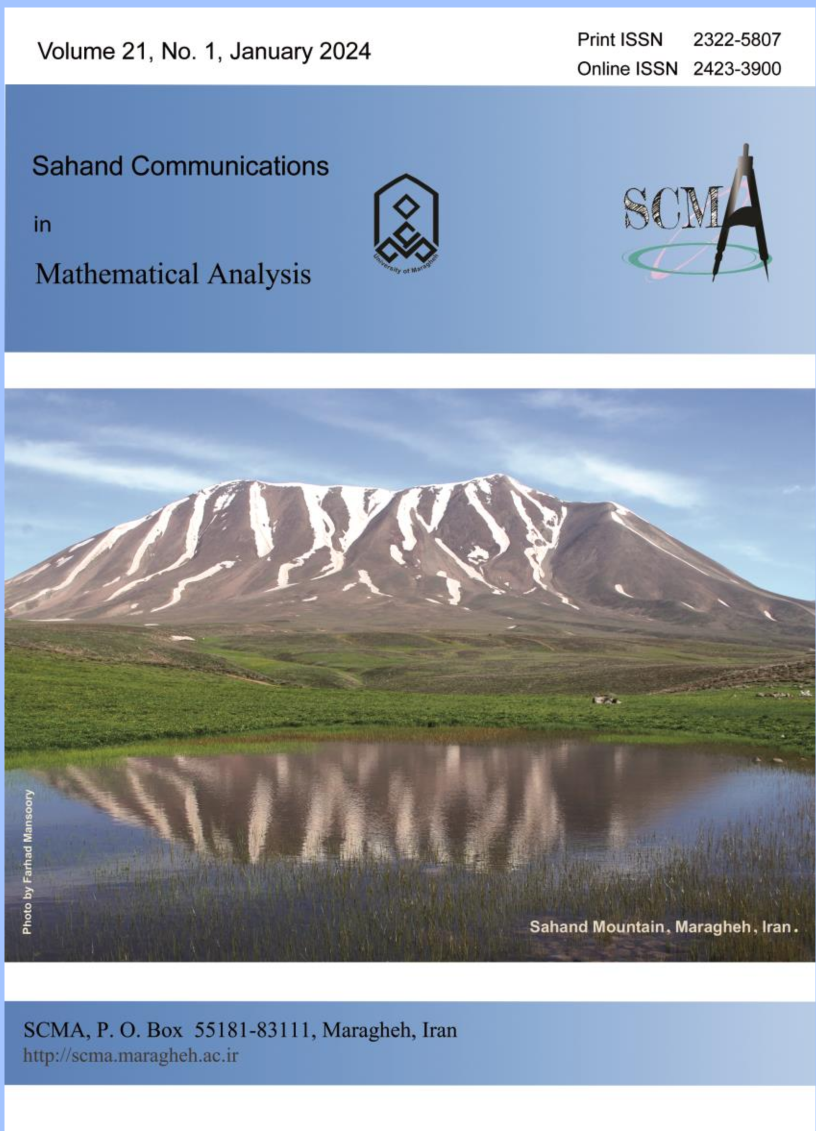

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Quaternion Hankel Transform and its Generalization

Khinal Parmar[1*] and V. R. Lakshmi Gorty[2]

ABSTRACT. In this study, the quaternion Hankel transform is developed. Basic operational properties and inversion formula of quaternion Hankel transform are derived. Parseval's relation for this transform is also established. The generalized quaternion Hankel transform is presented. In the concluding section, we demonstrate the application of the quaternion Hankel transform to Cauchy's problem.

1. INTRODUCTION

The Hankel transform was developed by the mathematician Herman Hankel. The classical Hankel transform and its extensions were studied in [9, 10, 16, 25]. The Hankel transform of distributions and generalized functions were developed in [4, 6]. Sneddon introduced the finite Hankel transform in a finite interval, satisfying Dirichlet's conditions [24] and its extensions were studied in [14, 15]. The fractional Hankel transform was introduced in [17]. The fundamental properties of Fourier-Bessel coefficients and various examples of different boundary conditions are discussed in [7].

In 1853, quaternions were developed by W. R. Hamilton [12]. The operations on three-dimensional vectors include multiplication and division, which necessitates enlarging the operations and leads to the introduction of the four-dimensional algebra of quaternions. Therefore, studying the quaternion Hankel transform becomes important. The quaternion Hankel transform is derived in this study to transfer signals from the real-valued time domain to the quaternion-valued frequency

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domain efficiently. The quaternion Hankel transform is applicable in solving boundary value problems and Euler-Cauchy differential equations of quaternion-valued functions. Authors in [5] investigated the quaternionic extension of the fractional Fourier transform on the real half-line, leading to the fractional Hankel transform. In recent developments, the authors in [18–21] have extended various integral transforms to quaternion-valued functions.

In this study, the authors introduce the quaternion Hankel transform. The authors have analyzed the operational properties of the transform. Parseval's relation and inversion formula are developed for the quaternion Hankel transform. The generalized quaternion Hankel transform is established. In the concluding section, application in Mathematical Physics is demonstrated.

The organization of the paper is as follows. In Section 2, some basic facts of quaternions and quaternion-valued functions are illustrated. In Section 3, we define and develop the quaternion Hankel transform, including its inversion formula and operational properties. In Section 4, the generalization of the quaternion Hankel transform is studied. In Section 5, a demonstration of the quaternion Hankel transform to Cauchy's problem is shown.

2. Preliminary Results

In this section, some basic facts of the quaternion and quaternion-valued functions are illustrated. We will also review the quaternion Fourier transform as defined in [2].

In quaternions, every element is a linear combination of a real scalar and three imaginary units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ with real coefficients. Let

$$\mathbb{H} = \{q = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\}, \tag{2.1}$$

be the division ring of quaternion, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy Hamilton's multiplication rules (see, e.g. [8])

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.2}$$

The quaternion conjugate of q is defined by

$$\bar{q} = x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3; \quad x_0, x_1, x_2, x_3 \in \mathbb{R}. \tag{2.3}$$

The norm of $q \in \mathbb{H}$ is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \tag{2.4}$$

We can also write quaternion as

$$q = \mathrm{Re}(q) + \mathrm{Im}(q), \tag{2.5}$$

where $\mathrm{Re}(q) = x_0$ and $\mathrm{Im}(q) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$.

For $h \in L^2(\mathbb{R};\mathbb{H})$ from [23], the function is expressed as

$$h(x) = h_0(x) + \mathbf{i}h_1(x) + \mathbf{j}h_2(x) + \mathbf{k}h_3(x). \tag{2.6}$$

Alternatively, Quaternions and Matrices of Quaternions by Fuzhen Zhang has mentioned this earlier (1997) in [23] the quaternion is defined as

$$\mathbb{H} = \{q = q_1 + jq_2 : q_1, q_2 \in \mathbb{C}\}, \tag{2.7}$$

where j is the imaginary number satisfying the following conditions: $j^2 = -1,\ jr = rj,\ \forall\, r \in \mathbb{R},\ ji = -ij$, where i is the imaginary number.

Every quaternion number can be uniquely expressed as

$$q = q_1 + jq_2 = (q_1 - iq_2)\, e_1 + (q_1 + iq_2)\, e_2, \tag{2.8}$$

where $e_1 = \dfrac{1+k}{2}, e_2 = \dfrac{1-k}{2}$, $e_1 + e_2 = 1$, $e_1e_2 = 1/2$ and $e_2e_1 = -1/2$.

The auxiliary complex spaces B_1 and B_2 are defined as follows:

$$\begin{aligned} B_1 &= \{w_1 = q_1 - iq_2, \forall\, q_1, q_2 \in \mathbb{C}\}; \\ B_2 &= \{w_2 = q_1 + iq_2, \forall\, q_1, q_2 \in \mathbb{C}\}. \end{aligned} \tag{2.9}$$

A cartesian set $X_1 \times_q X_2$ determined by $X_1 \subseteq B_1$ and $X_2 \subseteq B_2$ and is defined as:

$$X_1 \times_q X_2 = \{q_1 + jq_2 \in \mathbb{H} : q_1 + jq_2 = w_1e_1 + w_2e_2, w_1 \in X_1, w_2 \in X_2\}. \tag{2.10}$$

The projection mappings (2.10) are represented by $\mathcal{P}_1 : \mathbb{H} \to B_1 \subseteq \mathbb{C}, \mathcal{P}_2 : \mathbb{H} \to B_2 \subseteq \mathbb{C}$ as follows:

$$\begin{aligned} \mathcal{P}_1(q_1 + jq_2) &= \mathcal{P}_1[(q_1 - iq_2)\, e_1 + (q_1 + iq_2)\, e_2] \\ &= (q_1 - iq_2) \in B_1, \quad \forall q_1 + jq_2 \in \mathbb{H}, \\ \mathcal{P}_2(q_1 + jq_2) &= \mathcal{P}_2[(q_1 - iq_2)\, e_1 + (q_1 + iq_2)\, e_2] \\ &= (q_1 + iq_2) \in B_2, \quad \forall q_1 + jq_2 \in \mathbb{H}. \end{aligned}$$

Analogous to [1, Theorem 1, p. 2], the convergence of quaternion function for its quaternion component functions can be represented by the following theorem:

Theorem 2.1. *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $\mathcal{D} \subseteq \mathbb{C}$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$, the projections under the functions $\mathcal{P}_1 : \mathcal{D} \to \mathcal{D}_1 \subseteq \mathbb{C}$ and $\mathcal{P}_2 : \mathcal{D} \to \mathcal{D}_2 \subseteq \mathbb{C}$, are convergent in domains $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.*

Definition 2.2. Let $h \in L^2(\mathbb{R}^2, \mathbb{H})$, for

$$x = x_1e_1 + x_2e_2, \qquad w = w_1e_1 + w_2e_2. \tag{2.11}$$

The quaternion Fourier transform of h is defined in [2] as:

$$\mathcal{F}_q\{h\}(w) = \hat{h}(w) \tag{2.12}$$

$$= \int_{\mathbb{R}^2} h\,(x)\, e^{-\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} w.x} d^2x,$$

where $e^{-\left((\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}\,\right)w.x}$ is called the quaternion Fourier kernel. Here $\{e_1, e_2\}$ denote the standard bases of $\mathbb{R}^2$.

The inversion formula of the quaternion Fourier transform is defined in [2] as

$$\begin{aligned}\mathcal{F}_q^{-1}[\mathcal{F}_q\,\{h\}](x) &= h(x)\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\,\{h\}\,(w)\, e^{\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} w.x} d^2w.\end{aligned}$$

3. Quaternion Hankel Transform

In this section, the quaternion Hankel transform is defined and developed its inversion formula and operational properties.

Let h be the quaternion-valued function with quaternion Fourier transform as defined in (2.12). In $\mathbb{R}^2$, let $x = (x_1, x_2)$ and $w = (w_1, w_2)$.

Substituting $-\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} = \mu$, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $w_1 = p\cos\alpha$ and $w_2 = p\sin\alpha$ in (2.12) gives the following:

$$\hat{h}\,(p,\alpha) = \int_0^\infty r dr \int_0^{2\pi} h(r,\theta) e^{\mu r p \cos(\theta-\alpha)} d\theta. \tag{3.1}$$

Assuming

$$h(r,\theta) = h(r) e^{-\mu n\theta}, \tag{3.2}$$

further (3.1) can be represented as

$$\hat{h}\,(p,\alpha) = \int_0^\infty r h(r) dr \int_0^{2\pi} e^{\mu(-n\theta + rp\cos(\theta-\alpha))} d\theta. \tag{3.3}$$

Put $\theta - \alpha = -\phi - \frac{\pi}{2}$ in (3.3). Then we get

$$\begin{aligned}\hat{h}\,(p,\alpha) &= \int_0^\infty r h(r) dr \int_0^{2\pi} e^{\mu\left[n\left(\phi-\alpha+\frac{\pi}{2}\right)+rp\cos\left(\phi+\frac{\pi}{2}\right)\right]} d\phi\\ &= \int_0^\infty r h(r) dr e^{\mu n\left(\frac{\pi}{2}-\alpha\right)} \int_0^{2\pi} e^{\mu(n\phi - rp\sin\phi)} d\phi.\end{aligned}$$

Using integral representation of the Bessel function of order n as given in [22], we get

$$\begin{aligned}\hat{h}\,(p,\alpha) &= 2\pi e^{\mu n\left(\frac{\pi}{2}-\alpha\right)} \int_0^\infty r h(r) J_n(pr) dr\\ &= 2\pi e^{\mu n\left(\frac{\pi}{2}-\alpha\right)} \tilde{h}(p),\end{aligned}$$

where $\tilde{h}(p)$ is quaternion Hankel transform of $h(x)$.

Definition 3.1. The quaternion Hankel transform of the function $h(x) \in L^2(\mathbb{R}, \mathbb{H})$ is defined as

$$\begin{aligned} \mathcal{H}_n\{h(x)\}(p) &= \tilde{h}(p) \\ &= \int_0^\infty x h(x) J_n(px) dx, \end{aligned} \tag{3.4}$$

where $J_n(px)$ is the Bessel function of the first kind of order n.

Existence conditions:

1. The quaternion-valued function $h(x)$ must be locally and absolutely integrable on $0 < x < \infty$.
2. Since $xJ_n(x)$ is bounded on the positive real axis, it follows that the Hankel transform exists only if $h(x)$ is defined on the positive real axis.
3. If $h_0(x), h_1(x), h_2(x)$ and $h_3(x)$ are Hankel transformable functions, then every quaternion-valued function can be represented in the form of (2.6).

Convergence: Suppose $\tilde{h}(p) = \tilde{h}_0(p) + \mathbf{i}\tilde{h}_1(p) + \mathbf{j}\tilde{h}_2(p) + \mathbf{k}\tilde{h}_3(p)$, since $\tilde{h}_0(p), \tilde{h}_1(p), \tilde{h}_2(p)$ and $\tilde{h}_3(p)$ are convergent and analytic respectively. Therefore, by theorem 2.1, the quaternion-valued function $\tilde{h}(p)$ is convergent and analytic.

3.1. **Inversion Formula.** Substituting $-\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} = \mu$, $x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ w_1 = p\cos\alpha$ and $w_2 = p\sin\alpha$ in (2.11) yields:

$$h(r,\theta) = \frac{1}{(2\pi)^2} \int_0^\infty p dp \int_0^{2\pi} \hat{h}\,(p,\alpha)\, e^{-\mu r p \cos(\theta-\alpha)} d\alpha. \tag{3.5}$$

Considering (3.2) and $\hat{h}\,(p,\alpha) = 2\pi e^{\mu n\left(\frac{\pi}{2}-\alpha\right)} \tilde{h}(p)$ in (3.5), we obtain:

$$\begin{aligned} h(r)e^{-\mu n\theta} &= \frac{1}{(2\pi)^2} \int_0^\infty p dp \int_0^{2\pi} 2\pi e^{\mu n\left(\frac{\pi}{2}-\alpha\right)} \tilde{h}(p) e^{-\mu r p \cos(\theta-\alpha)} d\alpha \\ &= \frac{1}{2\pi} \int_0^\infty p\tilde{h}(p) dp \int_0^{2\pi} e^{\mu\left[n\left(\frac{\pi}{2}-\alpha\right) - rp\cos(\theta-\alpha)\right]} d\alpha. \end{aligned}$$

By substituting $\theta - \alpha = \psi - \frac{\pi}{2}$, we have

$$h(r)e^{-\mu n\theta} = \frac{1}{2\pi} \int_0^\infty p\tilde{h}(p) dp \int_0^{2\pi} e^{\mu\left[n(\psi-\theta) - rp\cos\left(\frac{\pi}{2}-\psi\right)\right]} d\psi.$$

Thus implies

$$h(r)e^{-\mu n\theta} = \frac{1}{2\pi} \int_0^\infty p\tilde{h}(p) dp e^{-\mu n\theta} \int_0^{2\pi} e^{\mu[n\psi - rp\sin\psi]} d\psi. \tag{3.6}$$

Using integral representation of Bessel function of order n [22] in (3.6), we get

$$h(r)e^{-\mu n\theta} = e^{-\mu n\theta}\int_0^\infty p\tilde{h}(p)J_n(pr)dp, \tag{3.7}$$

further representing (3.7) by

$$h(r) = \int_0^\infty p\tilde{h}(p)J_n(pr)dp, \tag{3.8}$$

which is the inversion formula for the quaternion Hankel transform.

Property 3.2 (Linearity property)**.** *Let $g, h \in L^2(\mathbb{R}, \mathbb{H})$ and $l_1, l_2 \in \mathbb{R}$. Then*

$$\mathcal{H}_n\left[l_1 g(x) + l_2 h(x)\right] = l_1\mathcal{H}_n[g(x)] + l_2\mathcal{H}_n[h(x)]. \tag{3.9}$$

Proof. For $g, h \in L^2(\mathbb{R}, \mathbb{H})$ and $l_1, l_2 \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{H}_n\left[l_1 g(x) + l_2 h(x)\right] &= \int_0^\infty \left(l_1 g(x) + l_2 h(x)\right) xJ_n(px)dx \\ &= l_1\int_0^\infty g(x)xJ_n(px)dx + l_2\int_0^\infty h(x)xJ_n(px)dx \\ &= l_1\mathcal{H}_n[g(x)] + l_2\mathcal{H}_n[h(x)].\end{aligned}$$

Hence the proof. □

Property 3.3 (Scaling Property)**.** *Let $h(x) \in L^2(\mathbb{R}, \mathbb{H})$. If $\mathcal{H}_n\{h(x)\} = \tilde{h}(p)$, then*

$$\mathcal{H}_n\{h(ax)\} = \frac{1}{a^2}\tilde{h}\left(\frac{p}{a}\right), \quad a \in \mathbb{R}. \tag{3.10}$$

Proof. For $h(x) \in L^2(\mathbb{R}, \mathbb{H})$,

$$\mathcal{H}_n\{h(ax)\} = \int_0^\infty xh(ax)J_n(px)dx.$$

By substituting $ax = t$, we have

$$\begin{aligned}\mathcal{H}_n\{h(ax)\} &= \int_0^\infty \frac{t}{a}h(t)J_n\left(\frac{p}{a}t\right)\frac{dt}{a} \\ &= \frac{1}{a^2}\int_0^\infty t\, h(t)J_n\left(\frac{p}{a}t\right)dt \\ &= \frac{1}{a^2}\tilde{h}\left(\frac{p}{a}\right).\end{aligned}$$

Hence the proof. □

Example 3.4. Find quaternion Hankel transform for the quaternion-valued function $h(x) = 1 + \mathbf{i}e^{-x} + \mathbf{j}x + \mathbf{k}\dfrac{e^{-x}}{x}$ of order zero.

Solution:

$$\begin{aligned}\mathcal{H}_n\{h(x)\} &= \int_0^\infty \left(1+\mathbf{i}e^{-x}+\mathbf{j}x+\mathbf{k}\frac{e^{-x}}{x}\right)xJ_0(px)dx\\ &= \int_0^\infty xJ_0(px)dx+\mathbf{i}\int_0^\infty xe^{-x}J_0(px)dx\\ &\quad +\mathbf{j}\int_0^\infty x^2J_0(px)dx+\mathbf{k}\int_0^\infty e^{-x}J_0(px)dx\\ &= \frac{2}{p^2}+\mathbf{i}\frac{1}{(1+p^2)^{3/2}}-\mathbf{j}\frac{1}{p^3}+\mathbf{k}\frac{1}{(1+p^2)^{1/2}}.\end{aligned}$$

Theorem 3.5. *Let $\mathcal{H}_n\{h(x)\}$ be the quaternion Hankel transform of order n for a quaternion-valued function $h(x)$. Then*

$$\mathcal{H}_n\left\{\frac{dh(x)}{dx}\right\}=\frac{p}{2}\mathcal{H}_{n+1}\{h(x)\}-\frac{p}{2}\mathcal{H}_{n-1}\{h(x)\}-\mathcal{H}_n\left\{\frac{h(x)}{x}\right\}. \tag{3.11}$$

Proof. The quaternion Hankel transform of order n for a quaternion-valued function $h(x)$ is given by

$$\mathcal{H}_n\{h(x)\}=\int_0^\infty xh(x)J_n(px)dx.$$

The quaternion Hankel transform of $\frac{dh}{dx}$ is

$$\mathcal{H}_n\left\{\frac{dh(x)}{dx}\right\}=\int_0^\infty x\frac{dh}{dx}J_n(px)dx.$$

Integrating by parts and assuming $xh(x)\to 0$ as $x\to 0$ and $x\to\infty$, we get

$$\begin{aligned}\mathcal{H}_n\left\{\frac{dh(x)}{dx}\right\} &= -\int_0^\infty h(x)\frac{d}{dx}\{xJ_n(px)\}dx\\ &= -\int_0^\infty h(x)\left\{J_n(px)+pxJ_n'(px)\right\}dx\\ &= -\int_0^\infty h(x)\left\{J_n(px)+\frac{px}{2}J_{n-1}(px)-\frac{px}{2}J_{n+1}(px)\right\}dx\\ &= -\int_0^\infty h(x)J_n(px)dx-\frac{p}{2}\int_0^\infty xf(x)J_{n-1}(px)dx\\ &\quad +\frac{p}{2}\int_0^\infty xh(x)J_{n+1}(px)dx\\ &= -\mathcal{H}_n\left\{\frac{h(x)}{x}\right\}-\frac{p}{2}\mathcal{H}_{n-1}\{h(x)\}+\frac{p}{2}\mathcal{H}_{n+1}\{h(x)\}.\end{aligned}$$

Hence the proof. □

Theorem 3.6. *Let $\mathcal{H}_n\{h(x)\}$ be the quaternion Hankel transform of order n for a quaternion-valued function $h(x)$. Then*

$$\mathcal{H}_n\left\{\frac{d^2h(x)}{dx^2}\right\} = p\mathcal{H}_{n-1}\left\{\frac{h(x)}{x}\right\} - p\mathcal{H}_{n+1}\left\{\frac{h(x)}{x}\right\} - \frac{p^2}{2}\mathcal{H}_n\{h(x)\} + \frac{p^2}{4}\mathcal{H}_{n+2}\{h(x)\} + \frac{p^2}{4}\mathcal{H}_{n-2}\{h(x)\}.$$

Proof. From theorem 3.5, we have

$$\mathcal{H}_n\left\{\frac{dh(x)}{dx}\right\} = \frac{p}{2}\mathcal{H}_{n+1}\{h(x)\} - \frac{p}{2}\mathcal{H}_{n-1}\{h(x)\} - \mathcal{H}_n\left\{\frac{h(x)}{x}\right\}. \tag{3.12}$$

By substituting $\frac{dh}{dx}$ in place of $h(x)$ in (3.12), we get

$$\mathcal{H}_n\left\{\frac{d^2h(x)}{dx^2}\right\} = -\mathcal{H}_n\left\{\frac{1}{x}\frac{dh(x)}{dx}\right\} - \frac{p}{2}\mathcal{H}_{n-1}\left\{\frac{dh(x)}{dx}\right\} + \frac{p}{2}\mathcal{H}_{n+1}\left\{\frac{dh(x)}{dx}\right\}.$$

Further implies

$$\begin{aligned}\mathcal{H}_n\left\{\frac{d^2h(x)}{dx^2}\right\} = {} & -\mathcal{H}_n\left\{\frac{1}{x}\frac{dh(x)}{dx}\right\} \\ & - \frac{p}{2}\left(-\mathcal{H}_{n-1}\left\{\frac{h(x)}{x}\right\} - \frac{p}{2}\mathcal{H}_{n-2}\{h(x)\} + \frac{p}{2}\mathcal{H}_n\{h(x)\}\right) \\ & + \frac{p}{2}\left(-\mathcal{H}_{n+1}\left\{\frac{h(x)}{x}\right\} - \frac{p}{2}\mathcal{H}_n\{h(x)\} + \frac{p}{2}\mathcal{H}_{n+2}\{h(x)\}\right).\end{aligned} \tag{3.13}$$

Now

$$\begin{aligned}\mathcal{H}_n\left\{\frac{1}{x}\frac{dh(x)}{dx}\right\} &= -\int_0^\infty h(x)\left(pJ_n'(px)\right)dx \\ &= -\int_0^\infty h(x)\left(\frac{p}{2}J_{n-1}(px) - \frac{p}{2}J_{n+1}(px)\right)dx \\ &= -\frac{p}{2}\int_0^\infty h(x)J_{n-1}(px) + \frac{p}{2}\int_0^\infty h(x)J_{n+1}(px)dx.\end{aligned}$$

Thus can be represented as

$$\mathcal{H}_n\left\{\frac{1}{x}\frac{dh(x)}{dx}\right\} = -\frac{p}{2}\mathcal{H}_{n-1}\left\{\frac{h(x)}{x}\right\} + \frac{p}{2}\mathcal{H}_{n+1}\left\{\frac{h(x)}{x}\right\}. \tag{3.14}$$

By substituting (3.14) in (3.13), we get

$$\begin{aligned}\mathcal{H}_n\left\{\frac{d^2h(x)}{dx^2}\right\} = {} & -\left(-\frac{p}{2}\mathcal{H}_{n-1}\left\{\frac{h(x)}{x}\right\} + \frac{p}{2}\mathcal{H}_{n+1}\left\{\frac{h(x)}{x}\right\}\right) \\ & + \frac{p}{2}\mathcal{H}_{n-1}\left\{\frac{h(x)}{x}\right\} + \frac{p^2}{4}\mathcal{H}_{n-2}\{h(x)\} - \frac{p^2}{4}\mathcal{H}_n\{h(x)\}\end{aligned}$$

$$
\begin{aligned}
&- \frac{p}{2}\mathcal{H}_{n+1}\left\{\frac{h(x)}{x}\right\} - \frac{p^2}{4}\mathcal{H}_n\left\{h(x)\right\} + \frac{p^2}{4}\mathcal{H}_{n+2}\left\{h(x)\right\} \\
&= p\mathcal{H}_{n-1}\left\{\frac{h(x)}{x}\right\} - p\mathcal{H}_{n+1}\left\{\frac{h(x)}{x}\right\} - \frac{p^2}{2}\mathcal{H}_n\left\{h(x)\right\} \\
&+ \frac{p^2}{4}\mathcal{H}_{n+2}\left\{h(x)\right\} + \frac{p^2}{4}\mathcal{H}_{n-2}\left\{h(x)\right\}.
\end{aligned}
$$

Hence the proof. □

Property 3.7. *(Parseval's Relation) If* $\tilde{g}(p) = \mathcal{H}_n\left\{g(x)\right\}$ *and* $\tilde{h}(p) = \mathcal{H}_n\left\{h(x)\right\}$*, then*

$$
\int_0^\infty p\tilde{g}(p)\tilde{h}(p)dp = \int_0^\infty xg(x)h(x)dx. \tag{3.15}
$$

Proof. For $\tilde{g}(p) = \mathcal{H}_n\left\{g(x)\right\}$ and $\tilde{h}(p) = \mathcal{H}_n\left\{h(x)\right\}$,

$$
\begin{aligned}
\int_0^\infty p\tilde{g}(p)\tilde{h}(p)dp &= \int_0^\infty p\tilde{g}(p)dp \int_0^\infty xJ_n(px)h(x)dx \\
&= \int_0^\infty xh(x)dx \int_0^\infty pJ_n(px)\tilde{g}(p)dp \\
&= \int_0^\infty xg(x)h(x)dx.
\end{aligned}
$$

Thus Parseval's relation holds. □

3.2. **Operational Calculus.** Let $\triangle_n h = \frac{1}{x}\frac{d}{dx}\left(x\frac{dh}{dx}\right) - \frac{n^2}{x^2}h$. Consider quaternion Hankel transform of $\triangle_n h$ using (3.4) is stated as

$$
\begin{aligned}
\mathcal{H}_n\left\{\triangle_n h\right\} &= \int_0^\infty x \triangle_n hJ_n(px)dx \\
&= \int_0^\infty \left(\frac{d}{dx}\left(x\frac{dh}{dx}\right) - \frac{n^2}{x}h\right) J_n(px)dx.
\end{aligned} \tag{3.16}
$$

Using integration by parts in (3.16) as in [3], we obtain

$$
\begin{aligned}
\mathcal{H}_n\left\{\triangle_n h\right\} &= \left[\left(x\frac{dh}{dx}\right) J_n(px)\right]_0^\infty - \int_0^\infty \left(x\frac{dh}{dx}pJ_n'(px) + \frac{n^2}{x}hJ_n(px)\right) dx \\
&= -\left[pxhJ_n'(px)\right]_0^\infty + \int_0^\infty \frac{d}{dx}\left[pxJ_n'(px)\right] hdx \\
&\quad - \int_0^\infty \frac{n^2}{x}hJ_n(px)dx \\
&= -\int_0^\infty \left(p^2 - \frac{n^2}{x^2}\right) xhJ_n(px)dx - \int_0^\infty \frac{n^2}{x}hJ_n(px)dx
\end{aligned}
$$

$$= -p^2 \int_0^\infty xhJ_n(px)dx$$
$$= -p^2\mathcal{H}_n\{h\}.$$

Thus

$$\mathcal{H}_n\{\triangle_n h\} = -p^2\mathcal{H}_n\{h(x)\}. \tag{3.17}$$

A similar argument can be used to prove the general result

$$\mathcal{H}_n\{\triangle_n^m h\} = (-1)^m p^{2m}\mathcal{H}_n\{h(x)\}. \tag{3.18}$$

4. Generalization

In this section, the generalization of the quaternion Hankel transform is studied.

4.1. **Testing Function Space.** Let a be a positive real number and $n \in \mathbb{R}$. For each pair of a and n as given in [11], we define $\mathscr{T}_{n,\,a}$ as a space of testing functions $\phi(x)$, well-defined and smooth on $0 < x < \infty$ and satisfy

$$\tau_k^{n,\,a}(\phi) = \sup_{0<x<\infty}\left|e^{-ax}x^{-n-1/2}\xi_{n,\,x}^k\phi(x)\right| < \infty,$$

where $\xi_{n,\,x}^k = \left(x^{-n-1/2}Dx^{2n+1}Dx^{-n-1/2}\right)^k$.

$\mathscr{T}_{n,\,a}$ is a linear space and each $\tau_k^{n,\,a}$ is a seminorm on $\mathscr{T}_{n,\,a}$. The collection $\{\tau_k^{n,\,a}\}$ is a multinorm. The topology of $\mathscr{T}_{n,\,a}$ is generated by $\{\tau_k^{n,\,a}\}$. Let $\mathscr{T}_n(\sigma) = \bigcup_{i=1}^{\infty}\mathscr{T}_{n,\,a_i}$ denote the countable-union space generated by $\{\mathscr{T}_{n,\,a_i}\}$, where σ is the limit of monotonically increasing sequence $\{a_i\}$. $\mathscr{T}_{n,\,a}$ is complete and therefore, a Fréchet space. $\mathscr{T}'_{n,\,a}$ and $\mathscr{T}'_n(\sigma)$ denote the dual of $\mathscr{T}_{n,\,a}$ and $\mathscr{T}_n(\sigma)$ respectively. $\mathscr{T}'_{n,\,a}$ is also complete. The members of $\mathscr{T}_{n,\,a}$ are the generalized functions on which the quaternion Hankel transform is defined.

Let the quaternion-valued fuction $h(x)$ be locally integrable on $0 < x < \infty$ such that $\int_0^\infty\left|h(x)e^{ax}x^{n+\frac{1}{2}}\right|dx < \infty$. Then $h(x)$ generates a regular generalized function in $\mathscr{T}'_{n,\,a}$ if $n \geq -\frac{1}{2}$ and is defined as

$$\langle h,\ \phi\rangle = \int_0^\infty h(x)\phi(x)dx, \quad \phi \in \mathscr{T}_{n,\,a}. \tag{4.1}$$

Definition 4.1. Let n be restricted to $n \geq -\frac{1}{2}$ and for every $h(x) \in \mathscr{T}'_{n,\,a}$, there exists a unique real number σ_h such that the generalized quaternion Hankel transform of h is defined as

$$H(p) = \mathscr{H}_n\{h(x)\}(p) = \langle h(x),\ xJ_n(px)\rangle, \tag{4.2}$$

where p is a quaternion parameter belonging to the strip $\Omega_h = \{p : |Im(p)| < \sigma_h; p \neq 0 \text{ or a negative number}\}$.

Theorem 4.2 (Analyticity theorem)**.** *For $p > 0$, let $H(p)$ be the generalized quaternion Hankel transform of h. Then $H(p)$ is differentiable and is denoted by*

$$D_p H = \left\langle h(x), \ \frac{\partial}{\partial p} x J_n(px) \right\rangle. \tag{4.3}$$

Proof. Let p be an arbitrary fixed point. Construct two concentric circles of radii r and r_1 with p as the center such that $r < r_1$. Let q be denoted as a small increment in p such that $|q| < r$.

Consider

$$\frac{H(p+q) - H(p)}{q} - \left\langle h(x), \ \frac{\partial}{\partial p} x J_n(px) \right\rangle = \langle h(x), \ \theta_q(x) \rangle \tag{4.4}$$

where

$$\theta_q(x) = \frac{x J_n(x(p+q)) - x J_n(px)}{q} - \frac{\partial}{\partial p} x J_n(px). \tag{4.5}$$

For any non-negative integer k, we have

$$\xi^k_{n,\,x}(x J_n(px)) = (-1)^k p^{2k} x J_n(px). \tag{4.6}$$

Using (4.5), further (4.6) can be represented as

$$\xi^k_{n,\,x}\theta_q(x) = \frac{(-1)^k}{2\pi i} \int_C \frac{q}{(\eta - p)^2(\eta - p - q)} \eta^{2k} x J_n(\eta x) d\eta,$$

where $\xi^k_{n,\,x} = \left(x^{-n-1/2} D x^{2n+1} D x^{-n-1/2}\right)^k$ and C is the circle of radius r_1. Analogous to [11, lemma 4], let A be a boundary of $e^{-ax} x^{-n+\frac{1}{2}} J_n(\eta x)$ for $0 < x < \infty$ and all $\eta \in C$.

Then, we have

$$\begin{aligned} \left| e^{-ax} x^{-n-\frac{1}{2}} \xi^k_{n,\,x} \theta_q(x) \right| &\leq \frac{|q|}{2\pi} \int_C \frac{|\eta|^{2k} |e^{-ax} x^{-n+\frac{1}{2}} J_n(\eta x)|}{(\eta - p)^2(\eta - p - q)} d\eta \\ &\leq \frac{|q| A}{r_1^2 (r_1 - r)} \sup_{\eta \in C} |\eta|^{2k}. \end{aligned}$$

Thus, as $q \to 0$, $\tau^{n,\,a}_k(\theta_q(x)) \to 0$. Hence we conclude that (4.4) also tends to zero.

The proof is complete. □

Theorem 4.3 (Boundedness theorem)**.** *Let $H(p)$ be the generalized quaternion Hankel transform of $h \in \mathscr{T}'_{n,\,a}$. Then $H(p)$ is bounded on any strip $\{p : |Im(p)| \leq a < \sigma_h; p \neq 0$ or a negative number$\}$.*

Proof. Let $h \in \mathscr{T}'_{n,\,a}$ where $0 < a < \sigma$. By the boundedness property of generalized functions [11, note viii], there exists a constant T and a non-negative integer s such that

$$\begin{aligned}|H(p)| &\le T \max_{0\le k\le s} \sup_{0<x<\infty} \left| e^{-ax} x^{-n-\frac{1}{2}} \xi^k_{n,\,x} x J_n(px) \right| \\ &\le T \max_{0\le k\le s} \sup_{0<x<\infty} \left| e^{-ax} x^{-n+\frac{1}{2}} p^{2k} J_n(px) \right|.\end{aligned}$$

According to [11, lemma 4], $e^{-ax}x^{-n+\frac{1}{2}}J_n(px)$ is bounded by an arbitrary constant and hence the proof. $\square$

Theorem 4.4 (Inversion theorem)**.** *Let $h \in \mathscr{T}'_{n,\,a}$ and $H(p)$ be the generalized quaternion Hankel transform of h. Then*

$$h(x) = \lim_{r\to\infty} \int_0^r pH(p)J_n(px)dp. \tag{4.7}$$

Proof. We will show that for any function $\phi(x) \in \mathscr{T}_{n,\,a}$, the expression $\left\langle \int_0^r H(p)pJ_n(px)dp,\ \phi(x) \right\rangle$ tends to $\langle h(x),\ \phi(x)\rangle$ as r tends to ∞.

By the definition of the inner product, we have

$$\begin{aligned}\left\langle \int_0^r H(p)pJ_n(px)dp,\ \phi(x) \right\rangle &= \int_0^\infty \phi(x) \int_0^r H(p)pJ_n(px)dpdx \\ &= \int_0^r \langle h(t),\ tJ_n(pt)\rangle \int_0^\infty \phi(x)pJ_n(px)dxdp \\ &= \left\langle h(t),\ \int_0^r tJ_n(pt) \int_0^\infty \phi(x)pJ_n(px)dxdp \right\rangle.\end{aligned}$$

By using the formula from [11],

$$\int_0^r pJ_n(pt)J_n(px)dp = \frac{r}{x^2-t^2}\left\{xJ_{n+1}(xr)J_n(tr) - tJ_n(xr)J_{n+1}(tr)\right\}, \tag{4.8}$$

and the asymptotic representation of Bessel function, it can be shown that $\int_0^r tJ_n(pt)\int_0^\infty \phi(x)pJ_n(px)dxdp$ converges to $\phi(t)$ as $r\to\infty$.

Thus follows

$$\left\langle \int_0^r H(p)pJ_n(px)dp,\ \phi(x) \right\rangle \to \langle h(x),\ \phi(x)\rangle, \tag{4.9}$$

as r tends to ∞. Hence the proof. $\square$

Theorem 4.5 (Uniqueness theorem)**.** *Let $H(p) = \mathscr{H}_n\{h(x)\}$ for $p \in \Omega_h$ and $G(p) = \mathscr{H}_n\{g(x)\}$ for $p \in \Omega_g$. If $H(p) = G(p)$ on $\Omega_h \cap \Omega_g = \{p : |Im(p)| < \min(\sigma_h,\ \sigma_g); p \neq 0$ or a negative number$\}$, then $h = g$ in the sense of equality in $\mathscr{T}'_{n,\,a}$.*

Proof. By the inversion theorem, we get

$$h - g = \lim_{r\to\infty} \int_0^r p\left[H(p) - G(p)\right] J_n(px)dp = 0. \tag{4.10}$$

Thus, $h = g$. Hence the proof. □

5. Application

In [13], the generalized Hankel transform is used to solve the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\mu+1}{x}\frac{\partial u}{\partial x} - \frac{\nu^2-\mu^2}{x^2}u = \lambda\frac{\partial u}{\partial t}, \tag{5.1}$$

with initial condition $u(x,\ t) \to f(x)$ in $D'(I)$, where $f \in \mathbb{H}'_{\mu,\ \nu}(\sigma)$ for some $\sigma > 0$ as $t \to 0^+$. The notations and terminologies used in the (5.1) are as defined in [26].
Analogous to [1], Cauchy problem of quaternion-valued function can be represented as

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{n^2}{x^2}u = \lambda\frac{\partial u}{\partial t}, \tag{5.2}$$

with the initial conditions mentioned in (5.1).

Applying quaternion Hankel transform to (5.2) and using (3.17), we get

$$\begin{gathered}
-p^2\mathcal{H}_n\left\{u(x,\ t)\right\} = \lambda\frac{\partial}{\partial t}\mathcal{H}_n\left\{u(x,\ t)\right\},\\
\frac{\partial}{\partial t}\mathcal{H}_n\left\{u(x,\ t)\right\} + \frac{p^2}{\lambda}\mathcal{H}_n\left\{u(x,\ t)\right\} = 0.
\end{gathered}$$

Further solving, we have

$$\mathcal{H}_n\left\{u(x,\ t)\right\} = \mathcal{H}_n\left\{f(x)\right\} e^{\frac{-p^2}{\lambda}t}. \tag{5.3}$$

Using the inversion formula of the quaternion Hankel transform of (5.3), we obtain the required solution:

$$u(x,\ t) = \int_0^\infty \mathcal{H}_n\left\{f(x)\right\} e^{\frac{-p^2}{\lambda}t} pJ_n(pr)dp. \tag{5.4}$$

6. Conclusion

In this paper, the authors have presented the quaternion Hankel transform and its inversion. Some basic properties of the quaternion Hankel transform are derived. Parseval's relation and operational calculus are also developed. Generalized quaternion Hankel transform is established using testing function space. In the concluding section, the application of the quaternion Hankel transform is given.

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References

1. R. Agarwal, M.P. Goswami and R.P. Agarwal, *Hankel transform in bicomplex space and applications*, TJMM, 8 (1) (2016), pp. 1-14.
2. M. Bahri, R. Ashino and R. Vaillancourt, *Continuous quaternion Fourier and wavelet transform*, Int. J. Wavelets Multiresolution Inf. Process., 12 (4) (2014), 1460003.
3. B. Davies, *Integral transforms and their Applications*, Springer, 1978.
4. L. S. Dube and J. N. Pandey, *On the Hankel transform of distributions*, Tohoku Math. J., 27 (3) (1975), pp. 337-354.
5. A. Elkachkouri, A. Ghanmi and A. Hafoud, *Bargmann's versus of the quaternionic fractional Hankel transform*, arXiv preprint arXiv:2003.05552, 2020.
6. I. M. Gelfand and G.E. Shilov, *Generalized Functions*, Academic Press, New York, 1967.
7. A. Ghaani Farashahi and G.S. Chirikjian, *Fourier-Bessel series of compactly supported convolutions on disks*, Anal. Appl., 20 (2) (2022), pp. 171-192.
8. E. Hitzer, *Quaternion Fourier Transform on Quaternion Fields and Generalizations*, Adv. Appl. Clifford Algebr., 17 (3) (2007), pp. 497-517.
9. F.H. Kerr, *Fractional powers of Hankel transforms in the Zemanian spaces*, J. Math. Anal. Appl., 166(1) (1992), pp. 65-83.
10. E.L. Koh and C.K. Li, *On the inverse of the Hankel transform*, Integral Transform Spec. Funct., 2(4) (1994), pp. 279-282.
11. E.L. Koh and A.H. Zemanian, *The complex Hankel and I-transformations of generalized functions*, SIAM J. Appl. Math., 16 (5) (1968), pp. 945-957.
12. A.C. Lewis, Chapter 35 - William Rown Hamilton, Lectures on quaternions (1853), Landmark Writings in Western Mathematics, Elsevier Science, 2005.
13. S.P. Malgonde and L. Debnath, *On Hankel type integral transform of generalized functions*, Integral Transform Spec. Funct., 15 (5) (2004), pp. 421-430.

14. S.P. Malgonde and V.R. Lakshmi Gorty, *Orthogonal series expansions of generalized functions and the finite generalized Hankel-Clifford transformation of distributions*, Rev. Acad. Canar. Cienc., XX (1-2) (2008), pp. 49-61.
15. J.M. Méndez, *The finite Hankel-Schwartz transform*, J. Korean Math. Soc., 26 (1) (1989), pp. 43-55.
16. J.M. Méndez, *On the Bessel transformation of arbitrary order*, Math. Nachr., 136 (1) (1988), pp. 233-239.
17. V. Namias, *Fractionalization of Hankel transforms*, IMA J. Appl. Math., 26 (2) (1980), pp. 187-197.
18. K. Parmar and V.R. Lakshmi Gorty, *One-Dimensional Quaternion Mellin Transform and its applications*, Proc. Jangjeon Math. Soc., 24 (1) (2021), pp. 99-112.
19. K. Parmar and V.R. Lakshmi Gorty, *Application and graphical interpretation of a new two-dimensional quaternion fractional Fourier transform*, Int. J. Anal. Appl., 19 (4) (2021), pp. 561-575.
20. K. Parmar and V.R. Lakshmi Gorty, *Quaternion Stieltjes Transform and Quaternion Laplace-Stieltjes Transform*, Commun. Math. Appl., 12 (3) (2021), pp. 633-643.
21. K. Parmar and V.R. Lakshmi Gorty, *Numerical Computation of Finite Quaternion Mellin Transform Using a New Algorithm*, International Conference on Advances in Computing and Data Sciences, (2021), pp. 172-182.
22. E.D. Rainville, *Special functions*, The Macmillan Company, New York, 1960.
23. R. Roopkumar, Quaternionic one-dimensional fractional Fourier transform, Optik, 127 (24) (2016), pp. 11657-11661.
24. I.N. Sneddon, *III. Finite Hankel transform*, Philos. Mag., 37 (264) (1946), pp. 17-25.
25. A. Torre, Hankel-type integral transforms and their fractionalization: a note, Integral Transform Spec. Funct., 19 (4) (2008), pp. 277-292.
26. A.H. Zemanian, *Generalized Integral Transformation*, John Wiley Sons Inc., New York, 1969.

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