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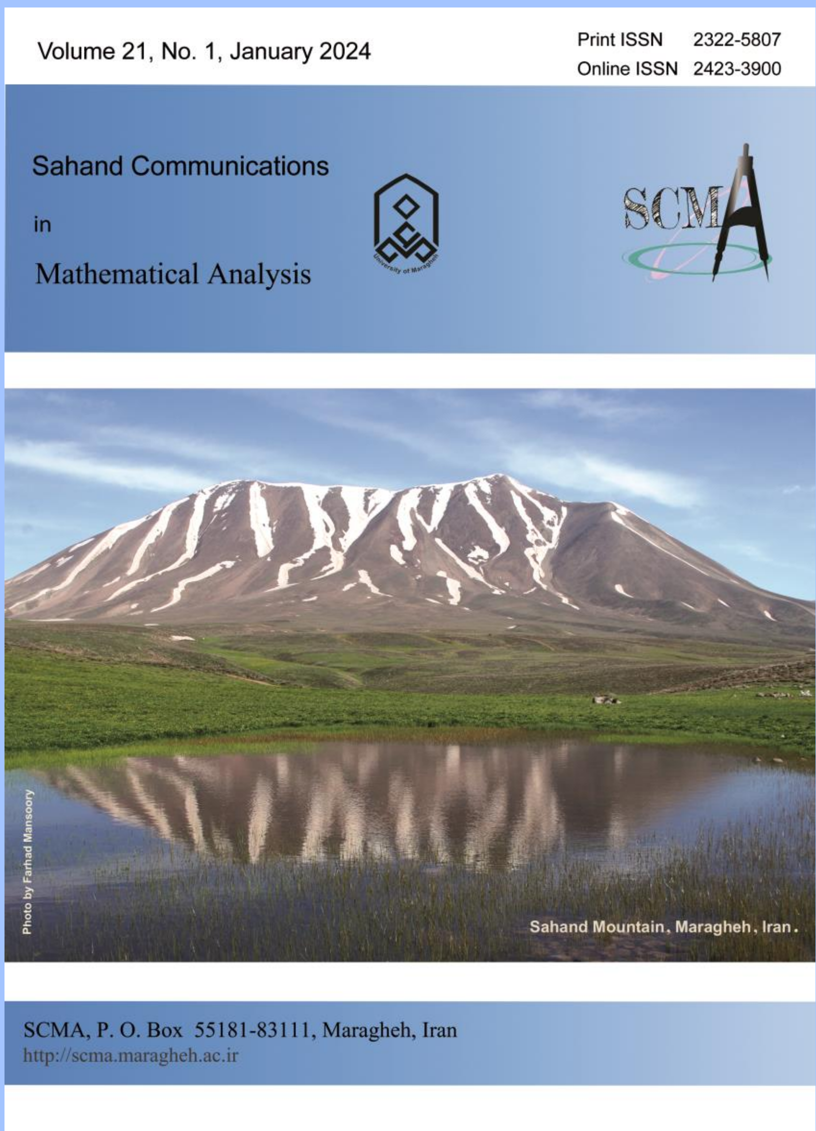

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A Gradient Bound for the Allen-Cahn Equation Under Almost Ricci Solitons

Sakineh Hajiaghasi[1] and Shahroud Azami[2*]

Abstract. In this paper, we consider positive solutions for the Allen-Cahn equation

$$\Delta u+\left(1-u^2\right)u=0,$$

on an almost Ricci soliton without a boundary. Firstly, using volume comparison Theorem and Sobolev inequality, we estimate the upper bound of $|\nabla u|^2$. As one of the applications, we extend this result to a gradient Ricci almost soliton. Finally, we obtain a Liouville-type theorem for almost Ricci solitons.

1. Introduction

Gradient estimates for the solutions of the Poisson equation and the heat equation are very powerful tools in geometry and analysis, as shown, as instance in [6, 9, 31]. The study of gradient estimates has a long tradition. First in [30], Yau attempted to generalize the theory of the classification of open Riemann surfaces for a higher-dimensional Riemannian manifold and during his studies, he proved a gradient estimate for a C^2-function which is bounded from below (See also [12, 17]). After that, Li [16] obtained a gradient estimate and Harnack inequalities for positive solutions of the nonlinear parabolic equation $\left(\Delta-\dfrac{\partial}{\partial t}\right)u+hu^\alpha=0$, and nonlinear elliptic equation $\Delta u+b\cdot\nabla u+hu^\alpha=0$ on Riemannian manifolds. Then as a result, he proved a Liouville-type theorem for positive solutions of the nonlinear elliptic equation.

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After a while, these works extended for the Schrödinger operator, the heat equation, and nonlinear heat equation in [10, 18, 20, 26, 28, 29] on noncompact Riemannian manifolds.

Lately, Zhang et al. [31] stated elliptic and parabolic gradient estimates for a Riemannian manifold M with Ricci curvature that is bounded from below, and therefore they achieved Gaussian upper and lower bounds for heat kernel and extended the maximum principle, which was stated by Petersen and Wai in [22]. In addition, Bamler in [7] developed a new version of these works and obtained bounds for the heat kernel on a Ricci flow background. In this paper, we consider bounded positive solutions to the Allen-Cahn equation, which is as follows:

$$\Delta u+\left(1-u^{2}\right) u=0. \tag{1.1}$$

The Allen-Cahn equation is a 2^{nd}-order nonlinear parabolic partial differential equation representing on natural physical phenomenon and has its origin in the gradient theory of phase transitions [4]. This equation has been extensively used to study various physical problems, such as crystal growth [23], image segmentation [8], and the motion by mean curvature flows [24]. In particular, it has been employed in material science [25]. Thus, the solutions of this equation have particular significance and have attracted a lot of attention in recent decades.

Recently, Bailesteanu [5] considered Harnack and gradient estimates for parabolic and elliptic Allen-Cahn equations and obtained interesting results. Additionally, in [15], Hou, obtained a gradient estimate for bounded solutions of the Allen-Cahn equation on Riemannian manifolds with bounded Ricci curvature $\mathrm{Ric}>-K(2R)$ for some positive constant $K(2R)>0$. More recently, Abolarinwa et al. [1] proved the gradient estimate for the bounded positive solutions to a certain semilinear elliptic Allen-Cahn equation and obtained conditions under which the gradient estimate gives rise to a Liouville type result with bounded solutions.

In this paper, we will primarily prove the first-order gradient estimate for bounded solutions of the nonlinear equation (1.1) under an almost Ricci soliton with the condition that the Ricci soliton has a non-positive lower bound. Since the present paper is included in Poisson equations, based on [19] it is known that such a solution u for (1.1) exists.

We will begin by reviewing the Volume comparison for an almost Ricci soliton. Moreover, we recall local Sobolev constans in Riemannian manifolds, and we may need another powerful tool named Nash-Morser iteration. Finally, as an application, we generalize the Liouville theorem.

2. Main Results

Before stating our main results, we want to introduce some notations.

We say that a Riemannian manifold (M^n, g) is an almost Ricci soliton if there exist a vector field X and a soliton function $\lambda : M^n \longrightarrow \mathbb{R}$ satisfying

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = -\lambda g,$$

where Ric and $\mathcal{L}$ stand, respectively, for the Ricci tensor and the Lie derivation. It is called expanding, steady or shrinking, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. When the vector field X is a gradient of a differentiable function $h : M^n \longrightarrow \mathbb{R}$, then the manifold called a gradient almost Ricci soliton; in this case the preceding equation turns out to be

$$\mathrm{Ric} + \nabla^2 h = \lambda g,$$

where ∇^2 denotes the Hessian of h. Moreover, when either the vector field X is trivial, or the potential h is constant, the almost Ricci soliton will be called trivial. We notice that when $n \geq 3$ and X is a Killing vector field, an almost Ricci soliton will be simply a Ricci soliton, since in this case we have an Einstein manifold, which implies that λ is a constant.

Let M^n be an almost Ricci soliton with following conditions

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \tag{2.1}$$

and

$$|V(y) \leq \frac{K}{d\,(y, O)^{\alpha}}, \tag{2.2}$$

for a smooth function λ with an upper bound N, a smooth vector field V and any $y \in M$. Here $d\,(y, O)$ denotes the distance from O to y, K is a positive constant and $0 \leq \alpha < 1$. In particular, we consider one more condition name the volume non-collapsing condition when $\alpha \neq 0$,

$$Vol\,(B\,(x, 1)) \geq \rho, \tag{2.3}$$

for all $x \in M$ and some constant $\rho > 0$. More precisely, we have the following results:

Theorem 2.1. *Let M^n be a complete compact almost Ricci soliton without boundary. Assume that (2.1), (2.2) and (2.3) hold. Let $\lambda \leq N$ and u as any entire positive solution of (1.1).*

(1) *For $|u| \leq 1$on M and any $q > \dfrac{n}{2}$, we have*

$$\sup_{B\left(x, \frac{1}{2} r\right)} |\nabla u|^2 \leq C(n, N, K, \alpha, \rho) r^{-2}.$$

(2) *For $u>1$ and any $q>n/2$, we have*

$$\sup_{B\left(x,\frac{1}{2}r\right)} |\nabla u|^2 \leq C\left(n,N,K,\alpha,\rho\right)\left[\left(\|u\|^*_{2,B(x,r)}\right)^4 + r^{-2}\left(\|u\|^*_{2,B(x,r)}\right)^2\right],$$

where

$$\|u\|^*_{p,B(x,r)} = \left(\fint_{B(x,r)} |u|^p\right)^{1/p}.$$

As a consequence, motivated by applications to gradient Ricci solitons, we obtain:

Corollary 2.2. *Let the following condition holds for a gradient Ricci almost soliton*

$$\mathrm{Ric} + Hessh \geq -\lambda g,$$

and more over we had two condition for potential function h as follows

$$|h(y)-h(z)| \leq K_1 d(y,z)^\alpha, \qquad \sup_{x\in M, 0\leq r\leq 1}\left(r^\beta \|\nabla h\|^*_{q,B(x,r)}\right) \leq K_2.$$

Then there is a constant $r_0 = r_0(n,N,K_1,K_2,\alpha,\beta)$, such that by the same conditions as Theorem 2.1, the solution of (1.1) with $u>1$ and any $q>\frac{n}{2}$, satisfies

$$\sup_{B\left(x,\frac{r}{2}\right)} |\nabla u|^2 \leq C(n,N,K_1,K_2,\alpha,\beta)\left[r^{-2}\left(\|u\|^*_{2,B(x,r)}\right)^2 + \left(\|u\|^*_{2,B(x,r)}\right)^4\right].$$

Corollary 2.3. *Suppose that all conditions in the Theorem 2.1 hold. Then*

(i) *If $u\leq m$ holds, we obtain $r_0 = r_0(n,N,K,\alpha,\rho,m)$ such that*

$$\sup_{B\left(x,\frac{1}{2}r\right)} |\nabla u|^2 \leq C(n,N,K,\alpha,\rho,m) r^{-2}.$$

(ii) *If $\lambda = 0$, therefore the constant coefficient changes as $C(n,K,\alpha,\rho)$.*

3. Basic Theorems and Proofs of Main Results

To prove our main results, first we need to obtain Sobolev inequalities for almost Ricci solitons and also the volume comparison theorem for an almost Ricci soliton which we state from [3].

Theorem 3.1 (Volume comparison [3])**.** *Assume that for an n-dimension almost Ricci soliton (2.1) and (2.2) hold. Moreover consider a positive constant N as an upper bound for λ. Suppose in addition that the volume non-collapsing condition (2.3) holds for positive constants $\rho > 0$, $K \geq 0$ and $0 \leq \alpha < 1$, then for any $0 < r_1 < r_2 \leq 1$, we have the volume ratio bound as follows*

$$\frac{Vol\left(B\left(x,r_2\right)\right)}{r_2^n} \leq e^{C(n,N,K,\alpha,\rho)\left[N(r_2^2-r_1^2)+K(r_2-r_1)^{1-\alpha}\right]}.\frac{Vol\left(B\left(x,r_1\right)\right)}{r_1^n}.$$

In particular, this result are true by considering the gradient soliton vector field $V = \nabla f$, where $f : M \to \mathbb{R}$.

Now, we will use the volume comparison result and follow the techniques and arguments in [13] to prove the Sobolove inequality on manifolds under the almost Ricci soliton condition.

Theorem 3.2 (Sobolev inequality)**.** *Under the same conditions as in the above theorem for an n-dimensional almost Ricci soliton, we have the following Sobolev inequalities:*

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}}\, dg\right)^{\frac{n-1}{n}} \leq C(n) r \oint_{B(x,r)} |\nabla f|\, dg,$$

and

$$\left(\oint_{B(x,r)} |f|^{\frac{2n}{n-2}}\, dg\right)^{\frac{n-2}{n}} \leq C(n) r^2 \oint_{B(x,r)} |\nabla f|^2\, dg. \tag{3.1}$$

Moreover, for the case that $X = \nabla f$ for some smooth function f, we get

$$\left(\oint_{B(x,r)} |f|^{\frac{n}{n-1}}\, dg\right)^{\frac{n-1}{n}} \leq C(n) r \oint_{B(x,r)} |\nabla f|\, dg.$$

Proof. Because of the similarity of the method for proving this theorem to the case of integral Ricci curvature [13] and also under considering Bakry-Émery Ricci condition [31], we will only describe the general path of the proof here.

First, let (2.1), (2.2) and (2.3) hold for an almost Ricci soliton M^n. It follows from above volume comparison theorem, that for $r_0 = r_0(n, N, K, \alpha, \rho) < 1$, we have

$$e^{C(n,N,K,\alpha,\rho)\left(Nr_0^2+Kr_0^{1-\alpha}\right)} \leq \frac{3}{2},$$

so for any $x \in M^n$ and $0 < r_1 < r_2 \leq r_0$, one has

$$\frac{Vol\left(B\left(x, r_1\right)\right)}{Vol\left(B\left(x, r_2\right)\right)} \geq \frac{2}{3} \frac{r_1^n}{r_2^n}.$$

It is clear that we could have the following just like in [31, Proposition 3.1],

$$\frac{Vol\left(B\left(x, \delta r\right)\right)}{Vol\left(B\left(x, r\right)\right)} \leq \frac{1}{2},$$

for $\delta = \delta(n)$ and $r \leq r_0$. Now, Let H be any hypersurface dividing M into two parts M_1 and M_2, $B(x, r)$ be the geodesic ball which is divided equally by H, then we infer

$$Vol\left(B\left(x, r\right)\right) \leq 2^{n+3} r Vol\left(H \cap B\left(x, 2r\right)\right).$$

Following the proof of [13, Theorem 1.1], we get isoperimetric inequality as follows

$$ID_n^*(B(x, r)) \leq C(n)r,$$

for any $r \leq r_0$. Here $r_0 = r_0(n, N, K, \alpha, \rho)$. This is equivalent to the Sobolev inequality stated in theorem. □

Now we prove our main results:

Proof of Theorem 2.1. Let $v = |\nabla u|^2 + \left\|u^6\right\|_{q,B(x,r)}^*$, then the Bochner formula gives

$$\begin{aligned}
\Delta v &= 2\left|\nabla^2 u\right|^2 + 2\left\langle \nabla u, \nabla \Delta u\right\rangle + 2\mathrm{Ric}(\nabla u, \nabla u) \\
&\geq -2\left\langle \nabla\left(\left(1-u^2\right)u\right), \nabla u\right\rangle - 2\lambda v - \left(\mathcal{L}_V g\right)_{ij} u_i u_j \\
&= 6u^2 v - 2\left(1+\lambda\right)v - \left(\mathcal{L}_V g\right)_{ij} u_i u_j.
\end{aligned}$$

For any $p > 0$, we get

(3.2)

$$\begin{aligned}
\Delta v^p &= p v^{p-1}\Delta v + p(p-1)v^{p-2}\left|\nabla v\right|^2 \\
&\geq p v^{p-1}\left(6u^2 v - 2\left(1+\lambda\right)v - \left(\mathcal{L}_V g\right)_{ij} u_i u_j\right) + \frac{p-1}{p} v^{-p}\left|\nabla v^p\right|^2.
\end{aligned}$$

Let $B_x(R)$ be the geodesic ball with radius R around $x \in M$, then it follows from (3.2), that

$$\text{(3.3)} \quad \int_B \left|\nabla(\eta v^p)\right|^2 = \int_B \left|\eta \nabla v^p + v^p \nabla \eta\right|^2$$

$$
\begin{aligned}
&= \int_B \eta^2 |\nabla v^p|^2 + v^{2p} |\nabla \eta|^2 + 2\eta v^p \langle \nabla v^p, \nabla \eta \rangle \\
&= \int_B v^{2p} |\nabla \eta|^2 - \eta^2 v^p \Delta v^p \\
&\leq \int_B v^{2p} |\nabla \eta|^2 \\
&\quad - p\eta^2 v^{2p-1} \Big(6u^2 v - 2(1+\lambda) v - (\mathcal{L}_V g)_{ij} u_i u_j \Big).
\end{aligned}
$$

for any $\eta \in C_0^\infty(B_x(1))$ and $p \geq 1$. We know that $(\mathcal{L}_V g)_{ij} = \nabla_i V_j + \nabla_j V_i$ thus we get

$$
\begin{aligned}
&\frac{1}{2}\int_B \eta^2 v^{2p-1} (\mathcal{L}_V g)_{ij} u_i u_j \\
&\qquad = -\int_B 2\eta v^{2p-1} \eta_j V_i u_i u_j + (2p-1)\eta^2 v^{2p-2} v_j V_i u_i u_j \\
&\qquad\quad + \eta^2 v^{2p-1} V_i u_{ij} u_j - \eta^2 v^{2p-1} V_i u_i (1-u^2) u.
\end{aligned} \tag{3.4}
$$

As we know $v_j = 2u_{jj}u_j$, using Cauchy-Schwarz inequality, (3.4) becomes

$$
\begin{aligned}
&\frac{1}{2}\int_B \eta^2 v^{2p-1} (\mathcal{L}_V g)_{ij} u_i u_j \\
&\quad \leq \int_B v^{2p} |\nabla \eta|^2 + \eta^2 v^{2p-2} |V|^2 |\nabla u|^4 \\
&\quad\quad - \frac{2p-1}{p} \eta v^{p-1} V_i u_i u_j \left[(\eta v^p)_j - v^p \eta_j \right] \\
&\quad\quad - \frac{1}{2}\eta^2 v^{2p-1} V_i v_i + \frac{1}{2}\eta^2 v^{2p-2} \left((1-u^2)u\right)^2 |\nabla u|^2 + \frac{1}{2}\eta^2 v^{2p} |V|^2 .
\end{aligned}
$$

From the definition of v, its obvious that $|\nabla u|^4 \leq v^2$, therefore

$$
\begin{aligned}
&\frac{1}{2}\int_B \eta^2 v^{2p-1} (\mathcal{L}_V g)_{ij} u_i u_j \\
&\qquad \leq \int_B \frac{8p-1}{4p} v^{2p} |\nabla \eta|^2 + \frac{2(2p-1)^2 + 5p}{2p} \eta^2 v^{2p} |V|^2 \\
&\qquad\quad + \frac{1}{2p} |\nabla(\eta v^p)|^2 + \frac{1}{2}\eta^2 v^{2p-1} \left((1-u^2)u\right)^2 .
\end{aligned} \tag{3.5}
$$

We consider two cases as follows:

(1) For $u \leq 1$, since $0 \leq 1 - u^2 \leq 1$, it is easy to see that

$$
\begin{aligned}
&\frac{1}{2}\int_B \eta^2 v^{2p-1} (\mathcal{L}_V g)_{ij} u_i u_j \\
&\qquad \leq \int_B \frac{8p-1}{4p} v^{2p} |\nabla \eta|^2 + \frac{2(2p-1)^2 + 5p}{2p} \eta^2 v^{2p} |V|^2
\end{aligned}
$$

$$+\frac{1}{2p}\left|\nabla(\eta v^p)\right|^2+\frac{1}{2}\eta^2 v^{2p-1}.$$

(2) For $u>1$, we know $1-u^2\leq 0$ so $(1-u^2)^2\leq u^4$, then it follows from (3.5) that

$$\begin{aligned}\frac{1}{2}\int_B \eta^2 v^{2p-1}(\mathcal{L}_V g)_{ij}u_i u_j &\\ \leq \int_B \frac{8p-1}{4p}v^{2p}\left|\nabla\eta\right|^2+\frac{2(2p-1)^2+5p}{2p}\eta^2 v^{2p}\left|V\right|^2 &\\ +\frac{1}{2p}\left|\nabla(\eta v^p)\right|^2+\frac{1}{2}\eta^2 v^{2p-1}u^6. &\end{aligned}\tag{3.6}$$

We only prove the second case. The proof of the first case is similar. By (3.6), (3.3) becomes

$$\begin{aligned}2\int_B\left|\nabla(\eta v^p)\right|^2\leq \int_B & 4v^{2p}\left|\nabla\eta\right|^2+8(1+\lambda)p\eta^2v^{2p}+(8p-1)v^{2p}\left|\nabla\eta\right|^2\\ &+\left(4(2p-1)^2+10p\right)\eta^2v^{2p}\left|V\right|^2-6p\eta^2v^{2p}u^2\\ &+p\eta^2v^{2p-1}u^6.\end{aligned}$$

Define cut function $\varphi_i(s)$ so that $\eta_i(y)=\varphi_i(s)$, such that for $r_i=\left(\frac{1}{2},\frac{1}{2^{i+2}}\right)$, $i=0,1,2,\ldots$, $\varphi_i(t)\equiv 1$ for $t\in[0,r_{i+1}]$, $supp\varphi_i\subseteq[0,r_i]$ and $-\frac{52^i}{r}\leq\varphi_i^{'}\leq 0$. so

$$\begin{aligned}\fint_{B(x,r_i)}\left|\nabla(\eta_i v^p)\right|^2\leq\fint_{B(x,r_i)} & 8(1+\lambda)p\eta_i^2v^{2p}+6pv^{2p}\left|\nabla\eta_i\right|^2\\ &-6p\eta_i^2v^{2p}u^2+30p^2\eta_i^2v^{2p}\left|V\right|^2+p\eta_i^2v^{2p-1}u^6.\end{aligned}\tag{3.7}$$

Using Young's inequality

$$xy\leq\epsilon x^\gamma+\epsilon^{-\frac{\gamma^*}{\gamma}}y^{\gamma^*},\qquad\forall x,y>0,\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\gamma^*}=1,$$

and volume comparison Theorem 3.1, for $\frac{r}{2}\leq r_i\leq\frac{3r}{4}$, we have

$$\begin{aligned}p\fint_{B(x,r_i)} & \eta_i^2v^{2p-1}u^6\\ &\leq\frac{p}{\left\|u^6\right\|^*_{q,B(x,r)}}\fint_{B(x,r_i)}\eta_i^2v^{2p}u^6\end{aligned}$$

$$\leq C(n,N,K,\alpha,\rho)p\left(\oint_{B(x,r_i)}(\eta_i v^p)^{\frac{2q}{q-1}}\right)^{\frac{q-1}{q}}$$

$$\leq C(n,N,K,\alpha,\rho)p^2\left(\oint_{B(x,r_i)}(\eta_i v^p)^{a.\frac{2q}{q-1}.b}\right)^{\frac{q-1}{q^b}}$$

$$\times\left(\oint_{B(x,r_i)}(\eta_i v^p)^{(1-a).\frac{2q}{q-1}.\frac{b}{b-1}}\right)^{\frac{(q-1)(b-1)}{q^b}}$$

$$\leq \epsilon\left(\oint_{B(x,r_i)}(\eta_i v^p)^{a.\frac{2q}{q-1}.b}\right)^{\frac{q-1}{q^{ba}}}$$

$$+\,\epsilon^{-\frac{a}{1-a}}C^{\frac{1}{1-a}}p^{\frac{1}{1-a}}\left(\oint_{B(x,r_i)}(\eta_i v^p)^{(1-a).\frac{2qb}{a_1}}\right)^{\frac{a_1}{q^{b(1-a)}}}$$

where $a_1=(q-1)(b-1)$. Since $q>\frac{n}{2}$, choosing $a=\frac{n}{2q}$ and $b=\frac{2q-2}{n-2}$, if follows

$$p\oint_{B(x,r_i)}\eta_i^2 v^{2p-1}u^6 \leq \epsilon\left(\oint_{B(x,r_i)}(\eta_i v^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \epsilon^{-\frac{a}{1-a}}C^{\frac{1}{1-a}}p^{\frac{1}{1-a}}\oint_{B(x,r_i)}\eta_i^2 v^{2p}. \tag{3.8}$$

By the same way with $\left\|u^2\right\|^*_{q,B(x,r)}\leq\|u^6\|^*_{q,B(x,r)}$, we get

$$-6p\oint_{B(x,r_i)}\eta_i^2 v^{2p}u^2 \leq -6p\oint_{B(x,r_i)}\eta_i^2 v^{2p-1}u^2 \tag{3.9}$$

$$\leq \epsilon\left(\oint_{B(x,r_i)}(\eta_i v^p)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}$$

$$+\epsilon^{-\frac{a}{1-a}} C^{\frac{1}{1-a}} p^{\frac{1}{1-a}} \fint_{B(x,r_i)} \eta_i^2 v^{2p}.$$

Again from volume comparison theorems in [3], If $q \in \left(\frac{n}{2}, \frac{n}{2\alpha}\right)$, then

$$\begin{aligned}
(3.10) \qquad & 30p^2 \fint_{B(x,r_i)} \eta_i^2 v^{2p} |V|^2 \\
& \leq 30p^2 \left(\fint_{B(x,r_i)} (\eta_i v^p)^{\frac{2q}{q-1}} \right)^{\frac{q-1}{q}} \cdot \left(\fint_{B(x,r_i)} |V|^{2q} \right)^{\frac{1}{q}} \\
& \leq p^2 C(n,N,K,\alpha,\rho) r_i^{-2\alpha} \left(\fint_{B(x,r_i)} (\eta_i v^p)^{\frac{2q}{q-1}} \right)^{\frac{q-1}{q}} \\
& \leq \epsilon r_i^{-2\alpha} \left(\fint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\
& \quad + \epsilon^{-\frac{a}{1-a}} p^{\frac{2}{1-a}} C^{\frac{1}{1-a}} r_i^{-2\alpha} \fint_{B(x,r_i)} \eta_i^2 v^{2p}.
\end{aligned}$$

In last three inequalities $C = C(n,N,K,\alpha,\rho)$ denotes constant depending on parameters n, N, K, α, ρ.

Now for any $\epsilon > 0$ and $a = \frac{n}{2q}$, substituting (3.8), (3.9) and (3.10) in (3.7), gives

$$\begin{aligned}
& \fint_{B(x,r_i)} |\nabla(\eta_i v^p)|^2 \\
& \leq \fint_{B(x,r_i)} (8+8\lambda) p \eta_i^2 v^{2p} + 6pv^{2p} |\nabla \eta_i|^2 \\
& \quad + \epsilon \left(\fint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\
& \quad + p^{\frac{2q}{2q-n}} \epsilon^{-\frac{n}{2q-n}} C^{\frac{2q}{2q-n}} \fint_{B(x,r_i)} \eta_i^2 v^{2p}
\end{aligned}$$

$$+ \epsilon r_i^{-2\alpha} \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$+ p^{\frac{4q}{2q-n}} \epsilon^{-\frac{n}{2q-n}} C^{\frac{2q}{2q-n}} r_i^{-2\alpha} \oint_{B(x,r_i)} \eta_i^2 v^{2p}.$$

Since $r_i \le r \le 1$ and $\alpha < 1$, using Sobolev inequality (3.1) and choosing ϵ small enough, the above inequality becomes

$$\left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\le C(n, N, K, \alpha, \rho) r_i^2 \oint_{B(x,r_i)} p v^{2p} |\nabla \eta_i|^2 + p^{\frac{4q}{2q-n}} \eta_i^2 v^{2p}.$$

From the volume comparison theorem, we infer

$$\left(\oint_{B(x,r_{i+1})} (v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\le C(n, N, K, \alpha, \rho) \left(\oint_{B(x,r_i)} (\eta_i v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\le C(n, N, K, \alpha, \rho) \oint_{B(x,r_i)} 2^{2i} p v^{2p} + p^{\frac{4q}{2q-n}} v^{2p}.$$

Now, take $\mu = \frac{n}{n-2}$ and $p = \frac{\mu^i}{2}$ for $i = 1, 2, \cdots$, therefore

$$\left(\oint_{B(x,r_{i+1})} v^{\mu^{i+1}} \right)^{\frac{n-2}{n}}$$

$$= \left(\oint_{B(x,r_{i+1})} (v^p)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\leq C(n,N,K,\alpha,\rho)\left(2^{2i}\mu^{i}+\mu^{\dfrac{4qi}{2q-n}}\right)\oint_{B(x,r_i)} v^{\mu^i}$$
$$\leq C(n,N,K,\alpha,\rho)4^{2qi/(2q-n)}\oint_{B(x,r_i)} v^{\mu^i}.$$

Hence

$$\|V\|^*_{\mu^{i+1},B(x,r_{i+1})}\leq C^{\mu^{-i}}\left(4^{2qi/(2q-n)}\right)^{\mu^{-i}}\|v\|^*_{\mu^i,B(x,r_i)}.$$

By Morser's iteration, we get

$$\sup_{B\left(x,\frac{1}{2}r\right)} v\leq C^{\Sigma_i\mu^{-i}}\left(4^{2qi/(2q-n)}\right)^{\Sigma_i\mu^{-i}}\|v\|^*_{1,B\left(x,\frac{3}{4}r\right)} \tag{3.11}$$
$$\leq C(n,N,K,\alpha,\rho,C)\,\|v\|^*_{1,B\left(x,\frac{3}{4}r\right)}.$$

Since

$$\int_{B(x,r)}\eta^2|\nabla u|^2=\int_{B(x,r)}-\eta^2u^2(1-u^2)-2\eta u\nabla_i u\nabla_i\eta$$
$$\leq\int_{B(x,r)}\frac{1}{2}u^4\eta^2+\frac{1}{2}(1-u^2)^2\eta^2+\frac{1}{2}\eta^2|\nabla u|^2+2u^2|\nabla\eta|^2.$$

Due to the definition of η and the fact that $(1-u^2)^2\leq u^4$, we infer

$$\int_{B(x,r)}\eta^2|\nabla u|^2\leq 4\int_{B(x,r)}u^4\eta^2+u^2|\nabla\eta|^2$$
$$\leq 100r^{-2}\left(\|u\|^*_{2,B(x,r)}\right)^2+\left(\|u\|^*_{2,B(x,r)}\right)^4.$$

Subsequently, we have

$$\|v\|^*_{1,B\left(x,\frac{3}{4}r\right)}\leq\frac{Vol(B(x,r))}{Vol\left(B\left(x,\frac{3}{4}r\right)\right)}\oint_{B(x,r)}\eta^2|\nabla u|^2 \tag{3.12}$$
$$\leq C(n,N,K,\alpha,\rho)\left[r^{-2}\left(\|u\|^*_{2,B(x,r)}\right)^2+\left(\|u\|^*_{2,B(x,r)}\right)^4\right].$$

Combining (3.12) and (3.11), we arrive at

$$\sup_{B\left(x,\frac{1}{2}r\right)}|\nabla u|^2\leq\|v\|_{\infty,B\left(x,\frac{1}{2}r\right)}$$

$$\leq C(n, N, K, \alpha, \rho) \left[r^{-2} \left(\|u\|^*_{2,B(x,r)} \right)^2 + \left(\|u\|^*_{2,B(x,r)} \right)^4 \right]. \square$$

In particular, if $\alpha = 0$, then the conclusions hold without the non-collapsing condition (2.3).

The classical Liouville theorem states that any bounded harmonic function is constant. In fact, the operator Δ enjoys the Liouville property if the following holds:

$$u \in L^\infty(\mathbb{R}^n) \quad \text{and} \quad \forall \phi \in C_c^\infty(\mathbb{R}^n) : \langle \Delta u, \phi \rangle := \langle u, \Delta \phi \rangle = 0,$$

then u is a constant. Here $\langle ., . \rangle$ denotes the real dual pairing used in the theory of distributions. If the Liouville property holds for $u \geq 0$, we speak of the strong Liouville property. For more information and the importance of Liouville property, see [2].

For an application of Theorem 2.1, we obtain the following Liouville type theorem for almost Ricci solitons:

Theorem 3.3. *Let M be a complete compact n-dimensional almost Ricci soliton satisfies in (2.1) and (2.2). If $|u| \leq 1$ is a positive solution of (1.1), then u is constant on M.*

Proof. The proof is directly based on the statement of theorem the function u is bounded, fixing a point $x \in M$ and using Theorem 2.1 for u on $B\left(x, \frac{1}{2} r\right)$, we have

$$|\nabla u|^2 \leq \frac{C(n, N, K, \alpha, \rho)}{r^2}.$$

Letting $r \to \infty$, it follows that $|\nabla u| = 0$. Since x is arbitrary, one gets u is a constant. $\square$

4. CONCLUSION

In view of this paper, we observe that our results developed in the previous sections, are also applicable to expanding and steady Ricci solitons. It should emphasize that the volume non-collapsing condition is necessary to guarantee that the solutions of (1.1) decay at infinity.

Our achievement here may generalize to the corresponding heat equation

$$\Delta u(x, t) - \partial_t u(x, t) + (1 - u^2) u = 0.$$

However, it seems more difficult, one may wonder whether these results can be extended with other solitons such as hyperbolic Ricci solitons.

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