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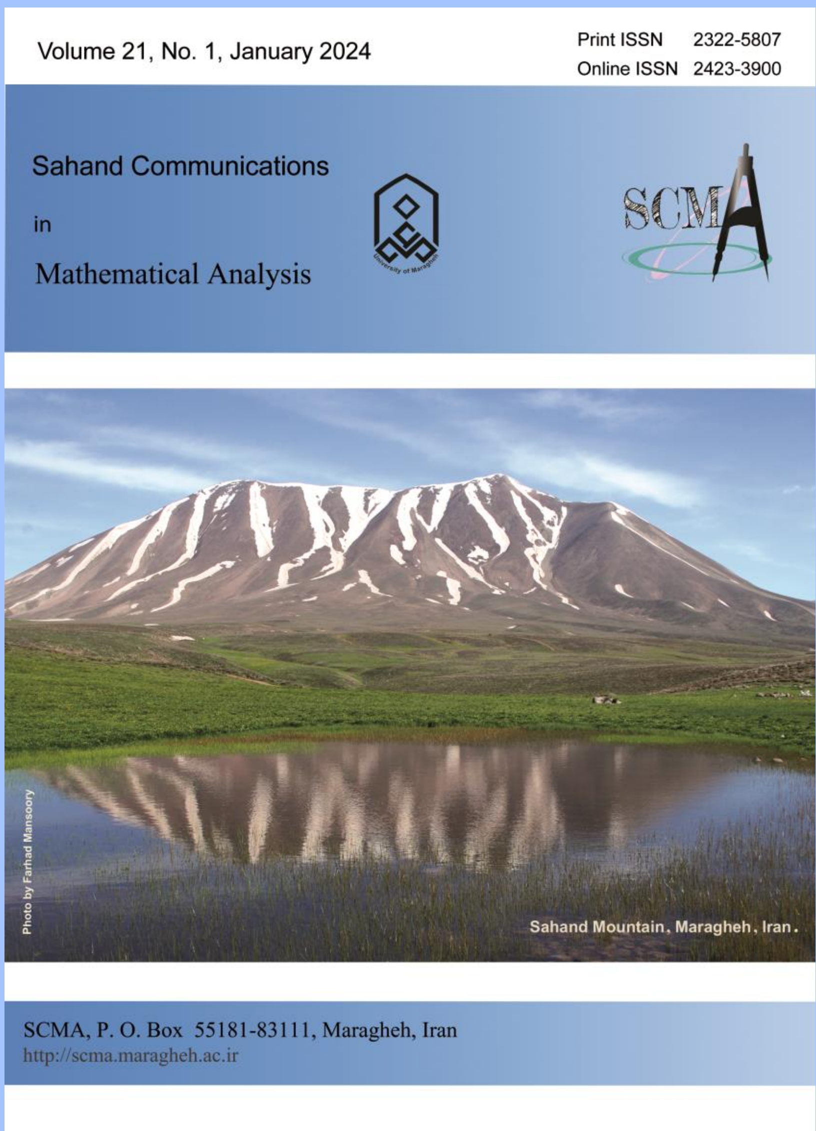

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Fractional Ostrowski Type Inequalities via $\phi - \lambda-$Convex Function

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Abstract. In this paper, we aim to state well-known Ostrowski inequality via fractional Montgomery identity for the class of $\phi-\lambda-$ convex functions. This generalized class of convex function contains other well-known convex functions from literature, allowing us to derive Ostrowski-type inequalities as specific instances. Moreover, we present Ostrowski-type inequalities for which certain powers of absolute derivatives are $\phi - \lambda-$ convex using various techniques, including Hölder's inequality and the power mean inequality. Consequently, various established results would be captured as special cases. Moreover, we provide applications in terms of special means, allowing us to derive many numerical inequalities related to special means from Ostrowski-type inequalities.

1. Introduction

In almost every field of science, inequalities play an important role. Although it is a very vast discipline, our focus is mainly on Ostrowski-type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. Additionally , one can find the numerous variants and applications in [1, 5, 9–11, 19, 23, 27, 28, 30, 33, 34, 37]. This inequality is well known in the literature as Ostrowski inequality, which is stated as:

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Theorem 1.1 ([32])**.** *Let $\varsigma : [a, b] \to \mathbb{R}$ be differentiable with $|\varsigma'(t)| \leq M$ $\forall t \in (a, b)$. Then*

$$\left|\varsigma(x) - \frac{1}{b-a}\int_a^b \varsigma(t)dt\right| \leq (b-a)M\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right], \tag{1.1}$$

$\forall x \in (a, b)$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

Nowadays, with the increasing demand of researchers to study natural phenomena, the use of fractional differential operators and fractional differential equations has become an effective means to achieve this goal. Compared with integer-order operators, Fractional operators, which can simulate natural phenomena better, are a class of operators developed in recent years. These operators have expanded and have been widely used in modeling real-world phenomena such as biomathematics, electrical circuits, medicine, disease transmission and control.

On the other hand, convexity is a simple and ordinary concept with massive applications in industry and business, greatly influencing our daily life. In solving many real-world problems, the concept of convexity plays a decisive role. In the solution of many real world problems the concept of convexity is very decisive. Problems faced in constrained control and estimation are often convex. Geometrically, a real valued function is said to be convex if the line segment segment joining any two of its points lies on or above the graph of the function in Euclidean space.

An important area in the field of applied and pure mathematics is the integral inequality. Inequalities aim to develop different mathematical methods. Nowadays, there is a need to seek accurate inequalities for proving the existence and uniqueness of the mathematical methods. The concept of convexity plays a strong role in the field of inequalities due to its definition and properties. Furthermore, there is a strong correlation between convexity and symmetry concepts.

In recent years, the generalization of classical convex function have emerged resulting in applications in the field of Mathematics. From literature, we recall some definitions for different types of convex.

Definition 1.2 ([4])**.** The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$\eta\left(tx + (1-t)y\right) \leq t\eta(x) + (1-t)\eta(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 1.3 ([4])**.** The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $MT-$convex, if $\eta(x) \geq 0$ and

$$\eta\left(tx + (1-t)y\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y), \quad \forall x, y \in I, t \in (0, 1).$$

Definition 1.4 ([20])**.** The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex, if $\eta(x) \geq 0$ and

$$\eta\left(tx+(1-t)y\right) \leq \eta(x)+\eta(y), \quad \forall x,y \in I \text{ and } t \in [0,1]$$

Definition 1.5 ([22])**.** The $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a GL convex, if $\eta(x) \geq 0$ and

$$\eta\left(tx+(1-t)y\right) \leq \frac{1}{t}\eta(x)+\frac{1}{1-t}\eta(y), \quad \forall x,y \in I \text{ and } t \in (0,1).$$

Definition 1.6 ([6])**.** Let $s \in (0,1]$. The $\eta : I \subset [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the 2^{nd} kind, if

$$\eta\left(tx+(1-t)y\right) \leq t^s\eta(x)+(1-t)^s\eta(y), \quad \forall x,y \in I, t \in [0,1].$$

Definition 1.7 ([14])**.** The $\eta : I \subset \mathbb{R} \to [0,\infty)$ is of GL $s-$convex, with $s \in [0,1]$, if

$$\eta\left(tx+(1-t)y\right) \leq \frac{1}{t^s}\eta(x)+\frac{1}{(1-t)^s}\eta(y), \quad \forall t \in (0,1) \text{ and } x,y \in I.$$

Definition 1.8 ([38])**.** Let $h : J \subseteq \mathbb{R} \to [0,\infty)$ with $h \neq 0$. The $\eta : I \subseteq \mathbb{R} \to [0,\infty)$ is an $h-$convex if $\forall x,y \in I$, we have

$$\eta\left(tx+(1-t)y\right) \leq h(t)\eta(x)+h(1-t)\eta(y), \quad \forall t \in [0,1].$$

Definition 1.9 ([15])**.** Let $\phi : (0,1) \to (0,\infty)$, the $\eta : I \subset \mathbb{R} \to [0,\infty)$ is a $\phi-$convex if $\forall x,y \in I$ we have

$$\eta\left(tx+(1-t)y\right) \leq t\phi(t)\eta(x)+(1-t)\phi(1-t)\eta(y), \quad \forall t \in (0,1).$$

Definition 1.10. The Riemann-Liouville integral operator of order $\psi > 0$ with $a \geq 0$ is defined as

$$J_a^{\psi}\varsigma(x) = \frac{1}{\Gamma(\psi)}\int_a^x (x-t)^{\psi-1}\varsigma(t)dt,$$
$$J_a^0\varsigma(x) = \varsigma(x).$$

In case of $\psi = 1$, the fractional integral reduces to the classical integral.

Definition 1.11 ([35])**.** The Riemann-Liouville integrals $I_{a^+}^{\psi}\varsigma$ and $I_{b^-}^{\psi}\varsigma$ of $\varsigma \in L_1([a,b])$ having order $\psi > 0$ with $a \geq 0, a < b$ are defined by

$$I_{a^+}^{\psi}\varsigma(x) = \frac{1}{\Gamma(\psi)}\int_a^x (x-t)^{\psi-1}\varsigma(t)dt, \quad x > a$$

and

$$I_{b^-}^{\psi}\varsigma(x) = \frac{1}{\Gamma(\psi)}\int_x^b (t-x)^{\psi-1}\varsigma(t)dt, \quad x < b,$$

respectively. Here $\Gamma(\psi) = \int_0^{\infty} e^{-u}u^{\psi-1}du$ is the GammThe and $I_{a^+}^0\varsigma(x) = I_{b^-}^0\varsigma(x) = \varsigma(x)$.

Theorem 1.12. *Let $\varsigma : I \to \mathbb{R}$ be differentiable mapping on I^0, with $a, b \in I$, with $a < b$, $\varsigma' \in L_1[a,b]$ and for $\psi > 1$, Montgomery identity for fractional integrals holds:*

$$\varsigma(x) - \frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi} J_a^{\psi}\varsigma(b) = J_a^{\psi-1}(P_1(x,b)\varsigma(b)) + J_a^{\psi}(P_1(x,b)\varsigma'(b)), \tag{1.2}$$

where $P_1(x,t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\psi}\Gamma(\psi), & \text{if} \quad t \in [a,x], \\ \\ \frac{t-b}{b-a}(b-x)^{1-\psi}\Gamma(\psi), & \text{if} \quad t \in (x,b]. \end{cases}$$

Let $[a,b] \subseteq (0, +\infty)$, we may define special means as follows:

(a) The arithmetic mean

$$A(a,b) = \frac{a+b}{2};$$

(b) The geometric mean

$$G(a,b) = \sqrt{ab};$$

(c) The harmonic mean

$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}};$$

(d) The logarithmic mean

$$L(a,b) = \begin{cases} a, & \text{if} \quad a = b \\ \dfrac{b-a}{\ln b - \ln a}, & \text{if} \quad a \neq b \end{cases};$$

(e) The identric mean

$$I(a,b) = \begin{cases} a, & \text{if} \quad a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if} \quad a \neq b. \end{cases};$$

(f) The $p-$logarithmic mean

$$L_p(a,b) = \begin{cases} a, & \text{if} \quad a = b \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if} \quad a \neq b. \end{cases};$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

We use this Lemma in [7] to prove our main results.

Lemma 1.13. *Let $\varsigma : [a,b] \to \mathbb{R}$ be a differentiable with $a < b$. If $\varsigma' \in L_1([a,b])$, then $\forall x \in (a,b)$,*

$$\left(\frac{(x-a)^{\psi}+(b-x)^{\psi}}{b-a}\right)\varsigma(x) - \frac{\Gamma(\psi+1)}{b-a}\left[I^{\psi}_{x^-}\varsigma(a) + I^{\psi}_{x^+}\varsigma(b)\right]$$
$$= \frac{(x-a)^{\psi+1}}{b-a}\int_0^1 t^{\psi}\varsigma'(tx+(1-t)a)dt$$
$$- \frac{(b-x)^{\psi+1}}{b-a}\int_0^1 t^{\psi}\varsigma'(tx+(1-t)b)dt.$$

Throughout this paper, we denote

$$\sigma(\varsigma, x, a, b, \psi)$$
$$= \left(\frac{(x-a)^{\psi}+(b-x)^{\psi}}{b-a}\right)\varsigma(x) - \frac{\Gamma(\psi+1)}{b-a}\left[I^{\psi}_{x^-}\varsigma(a) + I^{\psi}_{x^+}\varsigma(b)\right].$$

We also make use of Euler's beta function, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$
$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The main aim of our study is to generalize the Ostrowski inequality (1.1) for ϕ−−convex functions, as given in Section 2. Additionally, we present Ostrowski inequalities for which certain powers of absolute derivatives are ϕ−−convex, using various techniques,including Hölder's inequality [40] and power mean inequality [39]. Furthermore, we provide the the special cases of our results and applications of midpoint inequalities in special means.

2. Fractional Ostrowski Inequality via $\phi - \lambda$−Convex

In this section, we are introducing very first time the concept of $\phi - \lambda$−convex function, which contain many classes of convex functions in literature.

Definition 2.1. Let $\lambda \in (0,1]$ and $\phi : (0,1) \to (0,\infty)$, the $\eta : I \to [0,\infty)$ is a $\phi - \lambda$−convex, if

$$\eta\left(tx+(1-t)y\right) \le t^{\lambda}\phi(t)\eta(x) + (1-t)^{\lambda}\phi(1-t)\eta(y), \tag{2.1}$$

$\forall x, y \in I$, $t \in (0,1)$.

Remark 2.2. In Definition 2.1, one can see the following.

(i) If $\lambda = 1$ in (2.1), we get ϕ−convex.

(ii) If $\lambda = 1, l(t) = t$, and by taking $h = l\phi$ in (2.1), we get $h-$convex.
(iii) If $\lambda = 1, \phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$ in (2.1), then class of GL $s-$convex.
(iv) If $\lambda = 1, \phi(t) = \frac{1}{t^2}$ in (2.1), then concept of GL convex.
(v) If $\lambda = 1, \phi(t) = t^{s-1}$ with $s \in (0,1]$ in (2.1), then concept of $s-$convex in 2^{nd} kind.
(vi) If $\lambda = 1, \phi(t) = \frac{1}{t}$ in (2.1), then concept of $P-$convex.
(vii) If $\lambda = 1, \phi(t) = 1$ in (2.1), then ordinary convex.
(viii) If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (2.1), then concept of $MT-$convex.

Theorem 2.3. *Let $\lambda \in (0,1], \varsigma : [a,b] \to \mathbb{R}$ be differentiable on (a,b), $\varsigma' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $\eta : I \subset \mathbb{R} \to \mathbb{R}$ be a $\phi-\lambda-$convex, then we have the inequalities*

$$\eta\left[\varsigma(x) - \frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi} J_a^{\psi}\varsigma(b) + J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right] \tag{2.2}$$
$$\leq \frac{(x-a)^{\lambda-1}(b-x)^{1-\psi}}{(b-a)^{\lambda}}\phi\left(\frac{x-a}{b-a}\right)\left[\int_a^x \eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right]$$
$$+ \frac{(b-x)^{\lambda-\psi}}{(b-a)^{\lambda}}\ \phi\left(\frac{b-x}{b-a}\right)\left[\int_x^b \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right],$$

$\forall x \in [a,b]$.

Proof. Utilizing Theorem 1.12, we get

$$\begin{aligned}
&\varsigma(x) - \frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi} J_a^{\psi}\varsigma(b) + J_a^{\psi-1}(P_1(x,b)\varsigma(b)) \\
&= J_a^{\psi}(P_1(x,b)\varsigma'(b)) \\
&= \frac{1}{\Gamma(\psi)}\int_a^b P_1(x,t)\frac{\varsigma'(t)}{(b-t)^{1-\psi}}dt \\
&= \left(\frac{x-a}{b-a}\right)\left[\frac{(b-x)^{1-\psi}}{x-a}\int_a^x \frac{\{t-a\}\varsigma'(t)}{(b-t)^{1-\psi}}dt\right] \\
&\quad + \left(\frac{b-x}{b-a}\right)\left[\frac{(b-x)^{1-\psi}}{b-x}\int_x^b \frac{\{t-b\}\varsigma'(t)}{(b-t)^{1-\psi}}dt\right],
\end{aligned}$$

$\forall x \in [a,b]$. Next by using the $\eta : I \subset [0,\infty) \to \mathbb{R}$, is $\phi - \lambda-$convex, we get

$$\eta\left[\varsigma(x) - \frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi} J_a^{\psi}\varsigma(b) + J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq \left(\frac{x-a}{b-a}\right)^{\lambda}\phi\left(\frac{x-a}{b-a}\right)\eta\left[\frac{(b-x)^{1-\psi}}{x-a}\int_a^x \frac{\{t-a\}\varsigma'(t)}{(b-t)^{1-\psi}}dt\right]$$

$$+\left(\frac{b-x}{b-a}\right)^{\lambda}\phi\left(\frac{b-x}{b-a}\right)\eta\left[\frac{(b-x)^{1-\psi}}{b-x}\int_x^b\frac{\{t-b\}\varsigma'(t)}{(b-t)^{1-\psi}}dt\right],$$

$\forall x\in[a,b]$. Applying Jensen's integral inequality [12], we get the desired result. □

Corollary 2.4. *In Theorem* 2.3, *one can see the following.*

(i) If $\lambda=1$ *in* (2.2), *then Fractional Ostrowski type inequality for* $\phi-$*convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(b-x)^{1-\psi}}{(b-a)}\left[\phi\left(\frac{x-a}{b-a}\right)\int_a^x\eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right.$$
$$\left.+\phi\left(\frac{b-x}{b-a}\right)\int_x^b\eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right].$$

(ii) If $\lambda=1,l(t)=t$ *and* $h=l\phi$ *in* (2.2), *then Fractional Ostrowski type inequality for* $h-$*convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq h\left(\frac{x-a}{b-a}\right)\left[\frac{(b-x)^{1-\psi}}{x-a}\int_a^x\eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right]$$
$$+h\left(\frac{b-x}{b-a}\right)\left[\frac{1}{(b-x)^{\psi}}\int_x^b\eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right].$$

(iii) If $\lambda=1,\phi(t)=\frac{1}{t^{s+1}}$ *with* $s\in[0,1]$ *in* (2.2), *then Ostrowski inequality for Godunova-Levin* $s-$*convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(b-a)^s(b-x)^{1-\psi}}{(x-a)^{1+s}}\int_a^x\eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt$$
$$+\frac{(b-a)^s}{(b-x)^{\psi+s}}\int_x^b\eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt.$$

(iv) If $\lambda=1,\phi(t)=\frac{1}{t^2}$ *in* (2.2), *then Fractional Ostrowski type inequality for Godunova-Levin convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(b-a)(b-x)^{1-\psi}}{(x-a)^2}\int_a^x\eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt$$

$$+\frac{(b-a)}{(b-x)^{\psi+1}}\int_x^b \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt.$$

(v) *If* $\lambda=1,\phi(t)=t^{s-1}$ *with* $s\in(0,1]$ *in* (2.2), *then Fractional Ostrowski type inequality for* $s-$*convex in* 2^{nd} *kind:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(x-a)^{s-1}(b-x)^{1-\psi}}{(b-a)^s}\int_a^x \eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt$$
$$+\frac{(b-x)^{s-\psi}}{(b-a)^s}\int_x^b \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt.$$

(vi) *If* $\lambda=1,\phi(t)=\frac{1}{t}$ *in* (2.2), *then Fractional Ostrowski type inequality for* $P-$*convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(b-x)^{1-\psi}}{(x-a)}\int_a^x \eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt+\frac{1}{(b-x)^{\psi}}\int_x^b \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt.$$

(vii) *If* $\lambda=\phi(t)=1$ *in* (2.2), *then Fractional Ostrowski type inequality for convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(b-x)^{1-\psi}}{b-a}\left[\int_a^x \eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt+\int_x^b \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right].$$

(viii) *If* $\psi=\lambda=\phi(t)=1$ *in* (2.2), *then we get inequality* (2.1) *of Theorem* 2.1 *in [12].*

(ix) *If* $\lambda=1,\phi(t)=\frac{1}{2\sqrt{t(1-t)}}$ *in* (2.2), *then Fractional Ostrowski type inequality for* $MT-$*convex:*

$$\eta\left[\varsigma(x)-\frac{\Gamma(\psi)}{b-a}(b-x)^{1-\psi}J_a^{\psi}\varsigma(b)+J_a^{\psi-1}(P_1(x,b)\varsigma(b))\right]$$
$$\leq\frac{(b-x)^{\frac{1}{2}-\psi}}{2\sqrt{(x-a)}}\left[\int_a^x \eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt+\int_x^b \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right]dt\right].$$

Theorem 2.5. *Suppose all the assumptions of Lemma* 1.13 *hold. Additionally, assume that* $\lambda\in(0,1],|\varsigma'|$ *is* $\phi-\lambda-$*convex function on* $[a,b]$ *with* $\phi(t)\neq\frac{1}{t^2}$ *and* $|\varsigma'(x)|\leq M$. *Then*

(2.3) $$|\sigma(\varsigma,x,a,b,\psi)|$$

$$\leq M\left(\int_0^1\left[t^{\psi+\lambda}\phi(t)+t^{\psi}(1-t)^{\lambda}\phi(1-t)\right]dt\right)\ {}^{\psi}\kappa_a^b(x),$$

$\forall x\in(a,b),$ *where* ${}^{\psi}\kappa_a^b(x)=\frac{(x-a)^{\psi+1}+(b-x)^{\psi+1}}{b-a}$.

Proof. From the Lemma 1.13 we have

$$\begin{aligned}|\sigma(\varsigma,x,a,b,\psi)| &\leq \frac{(x-a)^{\psi+1}}{b-a}\int_0^1 t^{\psi}\left|\varsigma'(tx+(1-t)a)\right|dt\\ &+\frac{(b-x)^{\psi+1}}{b-a}\int_0^1 t^{\psi}\left|\varsigma'(tx+(1-t)b)\right|dt.\end{aligned}\tag{2.4}$$

Since $|\varsigma'|$ is $\phi-\lambda-$convex on $[a,b]$ and $|\varsigma'(x)|\leq M$, we have

$$\int_0^1 t^{\psi}\left|\varsigma'(tx+(1-t)a)\right|dt\leq M\int_0^1 t^{\psi}\left[t^{\lambda}\phi(t)+(1-t)^{\lambda}\phi(1-t)\right]dt\tag{2.5}$$

and similarly

$$\int_0^1 t^{\psi}\left|\varsigma'(tx+(1-t)b)\right|dt\leq M\int_0^1 t^{\psi}\left[t^{\lambda}\phi(t)+(1-t)^{\lambda}\phi(1-t)\right]dt.\tag{2.6}$$

We get the desired result. □

Corollary 2.6. *In Theorem* 2.5, *one can see the following.*

(i) If $\lambda=1$, *then Fractional Ostrowski type inequality for* $\phi-$*convex:*

$$|\sigma(\varsigma,x,a,b,\psi)|\leq M\left(\int_0^1\left[t^{\psi+1}\phi(t)+t^{\psi}(1-t)\phi(1-t)\right]dt\right)\ {}^{\psi}\kappa_a^b(x).$$

(ii) If $\lambda=1,l(t)=t$ *and* $h=l\phi$, *then Fractional Ostrowski type inequality for* $h-$*convex:*

$$|\sigma(\varsigma,x,a,b,\psi)|\leq M\left(\int_0^1 t^{\psi}\left[h(t)+h(1-t)\right]dt\right)\ {}^{\psi}\kappa_a^b(x).$$

(iii) If $\lambda=1,\phi(t)=\frac{1}{t^{s+1}}$ *with* $s\in[0,1)$, *then Ostrowski inequality for GL* $s-$*convex:*

$$|\sigma(\varsigma,x,a,b,\psi)|\leq M\left(\frac{1}{1+\psi-s}+\frac{\Gamma(1+\psi)\Gamma(1-s)}{\Gamma(2+\psi-s)}\right)\ {}^{\psi}\kappa_a^b(x).$$

(iv) If If $\lambda=1,\phi(t)=t^{s-1}$ *with* $s\in(0,1]$, *then inequality* (2.6) *of Theorem* 7 *in [36].*

(v) If $\lambda=\psi=1,\phi(t)=t^{s-1}$ *with* $s\in(0,1]$, *then inequality* (2.1) *of Theorem* 2 *in [2].*

(vi) *If* $\lambda = 1, \phi(t) = \frac{1}{t}$, *then Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{2M}{1+\psi} \ {}^{\psi}\kappa_a^b(x).$$

(vii) *If* $\lambda = 1, \phi(t) = 1$, *then inequality of Corollary* 1 *in [36].*

(viii) *If* $\lambda = \psi = \phi(t) = 1$, *then one has inequality* (1.3) *of Theorem* 3 *in [36].*

(ix) *If* $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$, *then Fractional Ostrowski type inequality for* $MT-$*convex:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq M \left(\frac{\sqrt{\pi}\ \Gamma\left[\frac{1}{2}+\psi\right]}{2\ \Gamma[1+\psi]} \right) \ {}^{\psi}\kappa_a^b(x).$$

Theorem 2.7. *Suppose all the assumptions of Lemma* 1.13 *hold. Additionally, assume that* $\lambda \in (0,1], |\varsigma'|^q$ *is* $\phi - \lambda-$*convex function on* $[a,b], q \geq 1$ *with* $\phi(t) \neq \frac{1}{t^2}$ *and* $|\varsigma'(x)| \leq M$. *Then*

(2.7)

$$\begin{aligned} &|\sigma(\varsigma, x, a, b, \psi)| \\ &\leq \frac{M}{(1+\psi)^{1-\frac{1}{q}}} \left(\int_0^1 \left[t^{\psi+\lambda}\phi(t) + t^{\psi}(1-t)^{\lambda}\phi(1-t) \right] dt \right)^{\frac{1}{q}} \ {}^{\psi}\kappa_a^b(x), \end{aligned}$$

$\forall x \in (a,b)$, *where* ${}^{\psi}\kappa_a^b(x) = \frac{(x-a)^{\psi+1}+(b-x)^{\psi+1}}{b-a}$.

Proof. From the Lemma 1.13 and using power mean inequality [39], we have

(2.8)

$$\begin{aligned} &|\sigma(\varsigma, x, a, b, \psi)| \\ &\leq \frac{(x-a)^{\psi+1}}{b-a} \left(\int_0^1 t^{\psi} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\psi} \left|\varsigma'(tx+(1-t)a)\right|^q dt \right)^{\frac{1}{q}} \\ &\quad + \frac{(b-x)^{\psi+1}}{b-a} \left(\int_0^1 t^{\psi} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\psi} \left|\varsigma'(tx+(1-t)b)\right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|\varsigma'|^q$ is $\phi - \lambda-$convex on $[a,b]$. and $|\varsigma'(x)| \leq M$, we get

$$\begin{aligned} &\int_0^1 t^{\psi} \left|\varsigma'(tx+(1-t)a)\right|^q dt \\ &\leq M^q \int_0^1 t^{\psi} \left[t^{\lambda}\phi(t) + (1-t)^{\lambda}\phi(1-t) \right] dt, \end{aligned} \tag{2.9}$$

and

$$\int_0^1 t^{\psi} \left|\varsigma'(tx+(1-t)b)\right|^q dt \leq M^q \int_0^1 t^{\psi} \left[t^{\lambda}\phi(t)+(1-t)^{\lambda}\phi(1-t)\right] dt. \tag{2.10}$$

We get the desired result. ☐

Corollary 2.8. *In Theorem* 2.7, *one can see the following.*

(i) If $q = 1$, *Theorem* 2.5.

(ii) If $\lambda = 1$, *then Fractional Ostrowski type inequality for* $\phi-$*convex:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(1+\psi)^{1-\frac{1}{q}}} \left(\int_0^1 t^{\psi} \left[t\phi(t)+(1-t)\phi(1-t)\right] dt\right)^{\frac{1}{q}} {}^{\psi}\kappa_a^b(x).$$

(iii) If $\lambda = 1, l(t) = t$ *and* $h = l\phi$, *then Fractional Ostrowski type inequality for* $h-$*convex:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(1+\psi)^{1-\frac{1}{q}}} \left(\int_0^1 t^{\psi} \left[h(t)+h(1-t)\right] dt\right)^{\frac{1}{q}} {}^{\psi}\kappa_a^b(x).$$

(iv) If $\lambda = 1, \phi(t) = \frac{1}{t^{s+1}}$ *with* $s \in [0, 1)$, *then Ostrowski inequality for GL* $s-$*convex:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(1+\psi)^{1-\frac{1}{q}}} \left(\frac{1}{1+\psi-s} + \frac{\Gamma(1+\psi)\Gamma(1-s)}{\Gamma(2+\psi-s)}\right)^{\frac{1}{q}} {}^{\psi}\kappa_a^b(x).$$

(v) If $\lambda = 1, \phi(t) = t^{s-1}$ *with* $s \in (0, 1]$, *then inequality* (2.8) *of Theorem* 9 *in [36].*

(vi) If $\lambda = \psi = 1, \phi(t) = t^{s-1}$ *with* $s \in [0, 1]$, *then inequality* (2.3) *of Theorem* 4 *in [2].*

(vii) If $\lambda = 1, \phi(t) = \frac{1}{t}$, *then Ostrowski inequality for* $P-$*convex via fractional integrals:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{2^{\frac{1}{q}} M}{1+\psi} {}^{\psi}\kappa_a^b(x).$$

(viii) If $\lambda = \phi(t) = 1$, *then one has the inequality of Corollary* 3 *in [36].*

(ix) If $\lambda = \psi = \phi(t) = 1$, *then one has inequality* (1.5) *of Theorem* 5 *in [36].*

(x) If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$*, then Fractional Ostrowski type inequality for* $MT-$*convex:*

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(1+\psi)^{1-\frac{1}{q}}} \left(\frac{\sqrt{\pi}\ \Gamma\left[\frac{1}{2}+\psi\right]}{2\ \Gamma[1+\psi]} \right)^{\frac{1}{q}}\ {}^{\psi}\kappa_a^b(x).$$

Theorem 2.9. *Suppose all the assumptions of Lemma* 1.13 *hold. Additionally, assume that* $\lambda \in (0,1], |\varsigma'|^q$ *is* $\phi-\lambda-$*convex function on* $[a,b], q>1$ *with* $p^{-1}+q^{-1}=1$, $\phi(t) \neq \frac{1}{t^2}$ *and* $|\varsigma'(x)| \leq M$. *Then*

$$\begin{aligned} (2.11) \quad & |\sigma(\varsigma, x, a, b, \psi)| \\ & \leq \frac{M}{(\psi p+1)^{\frac{1}{p}}} \left(\int_0^1 \left[t^{\lambda}\phi(t) + (1-t)^{\lambda}\phi(1-t) \right] dt \right)^{\frac{1}{q}}\ {}^{\psi}\kappa_a^b(x), \end{aligned}$$

$\forall x \in (a,b),$ *and* ${}^{\psi}\kappa_a^b(x) = \frac{(x-a)^{\psi+1}+(b-x)^{\psi+1}}{b-a}$.

Proof. From the Lemma 1.13 and using Hölder's inequality [40], we have

$$\begin{aligned} (2.12) \quad & |\sigma(\varsigma, x, a, b, \psi)| \\ & \leq \frac{(x-a)^{\psi+1}}{b-a} \left(\int_0^1 t^{\psi p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| \varsigma'\left(tx+(1-t)a\right) \right|^q dt \right)^{\frac{1}{q}} \\ & + \frac{(b-x)^{\psi+1}}{b-a} \left(\int_0^1 t^{\psi p} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| \varsigma'\left(tx+(1-t)b\right) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|\varsigma'|^q$ is $\phi-\lambda-$convex and $|\varsigma'(x)| \leq M$, we have
(2.13)

$$\int_0^1 \left| \varsigma'\left(tx+(1-t)a\right) \right|^q dt \leq M^q \int_0^1 \left[t^{\lambda}\phi(t) + (1-t)^{\lambda}\phi(1-t) \right] dt$$

and
(2.14)

$$\int_0^1 \left| \varsigma'\left(tx+(1-t)b\right) \right|^q dt \leq M^q \int_0^1 \left[t^{\lambda}\phi(t) + (1-t)^{\lambda}\phi(1-t) \right] dt$$

We get the desired result. □

Corollary 2.10. *In Theorem* 2.9, *one can see the following.*

(i) If $\lambda = 1$, *then Fractional Ostrowski type inequality for* $\phi-$*convex:*

$$\begin{aligned} & |\sigma(\varsigma, x, a, b, \psi)| \\ & \leq \frac{M}{(\psi p+1)^{\frac{1}{p}}} \left(\int_0^1 \left[t\phi(t) + (1-t)\phi(1-t) \right] dt \right)^{\frac{1}{q}}\ {}^{\psi}\kappa_a^b(x). \end{aligned}$$

(ii) If $\lambda = 1, l(t) = t$ and $h = l\phi$, then Fractional Ostrowski type inequality for $h-$convex:

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(\psi p+1)^{\frac{1}{p}}} \left(\int_0^1 [h(t) + h(1-t)]\, dt \right)^{\frac{1}{q}} \ {}^{\psi}\kappa_a^b(x).$$

(iii) If $\lambda = 1, \phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1)$, then Ostrowski inequality for GL $s-$convex:

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(\psi p+1)^{\frac{1}{p}}} \left(\frac{2}{1-s} \right)^{\frac{1}{q}} \ {}^{\psi}\kappa_a^b(x).$$

(iv) If $\lambda = 1, \phi(t) = t^{s-1}$ with $s \in (0,1]$, then inequality (2.7) of Theorem 8 in [36].

(v) If $\lambda = \psi = 1, \phi(t) = t^{s-1}$ with $s \in (0,1]$, then inequality (2.2) of Theorem 3 in [2].

(vi) If $\lambda = 1, \phi(t) = \frac{1}{t}$, then Ostrowski inequality for $P-$convex via fractional integrals:

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{2^{\frac{1}{q}} M}{(\psi p+1)^{\frac{1}{p}}} \ {}^{\psi}\kappa_a^b(x).$$

(vii) If $\lambda = \phi(t) = 1$, then one has Corollary 2 in [36].

(viii) If $\lambda = \psi = \phi(t) = 1$, then one has inequality (1.4) of Theorem 4 in [36].

(ix) If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$, then Fractional Ostrowski type inequality for $MT-$convex:

$$|\sigma(\varsigma, x, a, b, \psi)| \leq \frac{M}{(\psi p+1)^{\frac{1}{p}}} \left(\frac{\pi}{2} \right)^{\frac{1}{q}} \ {}^{\psi}\kappa_a^b(x).$$

3. Applications of Midpoint Inequalities

If we replace ς by $-\varsigma$ and $x = \frac{a+b}{2}$ in Theorem 2.3, we get

Theorem 3.1. *Let $\varsigma : [a,b] \to \mathbb{R}$ be differentiable on (a,b), $\varsigma' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $\eta : I \subset \mathbb{R} \to \mathbb{R}$, be a $\phi - \lambda-$convex, then*

$$\eta \left[\frac{\Gamma(\psi)\left(\frac{b-a}{2}\right)^{1-\psi}}{b-a} J_a^{\psi}\varsigma(b) - \varsigma\left(\frac{a+b}{2}\right) - J_a^{\psi-1}\left(P_1\left(\frac{a+b}{2}, b\right)\varsigma(b)\right) \right]$$
$$\leq \frac{2^{\psi-\lambda}\ \phi\left(\frac{1}{2}\right)}{(b-a)^{\psi}} \left[\int_{\frac{a+b}{2}}^{a} \eta\left[\frac{(t-a)\varsigma'(t)}{(b-t)^{1-\psi}}\right] dt + \int_b^{\frac{a+b}{2}} \eta\left[\frac{(t-b)\varsigma'(t)}{(b-t)^{1-\psi}}\right] dt \right].$$

Remark 3.2. In Theorem 3.1, if $\psi=1$, we get

$$\eta\left(\frac{1}{b-a}\int_a^b \varsigma(t)dt-\varsigma\left(\frac{a+b}{2}\right)\right)$$
$$\leq \frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta[(a-t)\varsigma'(t)]dt+\int_{\frac{a+b}{2}}^b \eta[(b-t)\varsigma'(t)]dt\right].$$

Remark 3.3. In Theorem 3.1, assume that $\eta: I\subset[0,\infty)\to\mathbb{R}$ be an $\phi-\lambda-$convex function:

(i) If $\psi=1,\varsigma(t)=\frac{1}{t}$, where $t\in[a,b]\subset(0,\infty)$, then we have

$$\eta\left[\frac{A(a,b)-L(a,b)}{A(a,b)L(a,b)}\right]$$
$$\leq \frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[\frac{t-a}{t^2}\right]dt+\int_{\frac{a+b}{2}}^b \eta\left[\frac{t-b}{t^2}\right]dt\right].$$

(ii) If $\lambda=\psi=1,\varsigma(t)=-\ln t$, where $t\in[a,b]\subset(0,\infty)$, then we have

$$\eta\left[\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right]$$
$$\leq \frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[\frac{t-a}{t}\right]dt+\int_{\frac{a+b}{2}}^b \eta\left[\frac{t-b}{t}\right]dt\right].$$

(iii) If $\psi=1,\varsigma(t)=t^p, p\in\mathbb{R}\setminus\{0,-1\}$, where $t\in[a,b]\subset(0,\infty)$, then we have

$$\eta\left[L_p^p(a,b)+A^p(a,b)\right]$$
$$\leq \frac{2^{1-\lambda}\phi\left(\frac{1}{2}\right)}{b-a}\left[\int_a^{\frac{a+b}{2}}\eta\left[\frac{p\,(a-t)}{t^{1-p}}\right]dt+\int_{\frac{a+b}{2}}^b \eta\left[\frac{p\,(b-t)}{t^{1-p}}\right]dt\right].$$

Remark 3.4. In Theorem 2.7, one can see the following.

(i) Let $x=\frac{a+b}{2},\psi=1, 0<a<b$, $q\geq 1$ and $\varsigma:\mathbb{R}\to\mathbb{R}^+$, $\varsigma(x)=x^n$, then

$$\left|A^n\left(a,b\right)-L_n^n\left(a,b\right)\right|$$
$$\leq \frac{M\left(b-a\right)}{\left(2\right)^{2-\frac{1}{q}}}\left(\int_0^1\left[t^{\lambda+1}\phi(t)+t(1-t)^{\lambda}\phi(1-t)\right]dt\right)^{\frac{1}{q}}.$$

(ii) Let $x=\frac{a+b}{2},\psi=1, 0<a<b$, $q\geq 1$ and $\varsigma:(0,1]\to\mathbb{R}$, $\varsigma(x)=-\ln x$, then

$$\left|\ln\left(\frac{A\left(a,b\right)}{I\left(a,b\right)}\right)\right|\leq \frac{M\left(b-a\right)}{\left(2\right)^{2-\frac{1}{q}}}\left(\int_0^1\left[t^{\lambda+1}\phi(t)+t(1-t)^{\lambda}\phi(1-t)\right]dt\right)^{\frac{1}{q}}.$$

Remark 3.5. In Theorem 2.9, one can see the following.

(i) Let $x = \frac{a+b}{2}, \psi = 1, 0 < a < b$, $p^{-1} + q^{-1} = 1$ and $\varsigma : \mathbb{R} \to \mathbb{R}^{+}$, $\varsigma(x) = x^{n}$, then

$$\begin{aligned} &\left| A^{n}\left(a,b\right) - L_{n}^{n}\left(a,b\right) \right| \\ &\leq \frac{M\left(b-a\right)}{2\left(p+1\right)^{\frac{1}{p}}} \left(\int_{0}^{1} \left[t^{\lambda}\phi(t) + (1-t)^{\lambda}\phi(1-t) \right] dt \right)^{\frac{1}{q}}. \end{aligned}$$

(ii) Let $x = \frac{a+b}{2}, \psi = 1, 0 < a < b$, $p^{-1} + q^{-1} = 1$ and $\varsigma : (0,1] \to \mathbb{R}$, $\varsigma(x) = -\ln x$, then

$$\left| \ln\left(\frac{A\left(a,b\right)}{I\left(a,b\right)} \right) \right| \leq \frac{M\left(b-a\right)}{2\left(p+1\right)^{\frac{1}{p}}} \left(\int_{0}^{1} \left[t^{\lambda}\phi(t) + (1-t)^{\lambda}\phi(1-t) \right] dt \right)^{\frac{1}{q}}.$$

4. Conclusion and Remarks

4.1. **Conclusion.** In this paper, we present the generalization of Ostrowski inequality via fractional Montgomery identity with $\phi-\lambda-$convex. This class of functions include of $\phi-$convex [15], $h-$convex [38], GL $s-$convex [14], $s-$convex in the 2^{nd} kind [6] and hence the class convex and $MT-$convex [4]. It also includes the class of $P-$convex [20] and class of GL functions [22]. In Section 2, we present the generalization of the Ostrowski inequality via the generalized Montgomery identity using fractional integrals for $-$ $-$convex functions. Furthermore, we used different techniques including Hölder's inequality [40] and power mean inequality [39] for generalization of Ostrowski inequality. In the second-to-last section, we provide applications of the obtained results in terms of special means, including arithmetic, geometric, harmonic, logarithmic, identric and $p-$logarithmic means, using the midpoint inequalities.

4.2. **Remarks and Future Ideas.**

(i) One may also do similar work by using various different classes of convex functions.
(ii) One may do similar work to generalize all results stated in this research work by applying weights.
(iii) One may also state all results stated in this research work by higher order derivatives.
(iv) One may also state all results stated in this research work by multivariable functions and gernalized fractional integral operators.

(v) One may also do the similar work by using various different generalized forms for the Korkine's and Montgomery identities, improved power mean inequality, Hölder's Iscan inequality, Jensen's integral inequality with weights, generalized fuzzy metric spaces on set of all fuzzy numbers.

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