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Inverse Conformable Sturm–Liouville Problems with a Transmission and Eigen–Parameter Dependent Boundary Conditions

Mohammad Shahriari[1*] and Reza Akbari[2]

Abstract. In this paper, we provide a different uniqueness results for inverse spectral problems of conformable fractional Sturm-Liouville operators of order α $(0 < \alpha \leq 1)$, with a jump and eigenparameter dependent boundary conditions. Further, we study the asymptotic form of solutions, eigenvalues and the corresponding eigenfunctions of the problem. Also, we consider three terms of the inverse problem, from the Weyl function, the spectral data and two spectra. Moreover, we can also extend Hald's theorem to the problem.

1. Introduction

In 2014, Khalil et al. in [5] defined a new well-behaved (local) simple fractional derivative called ‘‘the conformable fractional derivative(CFD) depending just on the basic limit definition of the derivative. Unlike other definitions of fractional derivative such as Riemann-Liouville, Caputo, etc., this definition enables us to prove many properties similar to derivatives of integer order. For more information about the CFD, refer to [1, 2].

However, the CFD has it's drawbacks. It's derivative has some disadvantages and some unusual properties, e.g., the zeroth derivative of a

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function does not return the function. Fractional Sturm-Liouville problems (FSLPs) have attracted much attention as an important branch of fractional derivative research, [3, 6–8, 16].

In our opinion, the most important useful property of the conformable derivative is the possibility of defining the inner product in integral form. This capability makes the Sturm-Liouville problem (SLP) well investigated in different situations. In [10] the authors investigated the existence of infinity of real eigenvalues of conformable fractional Sturm-Liouville problem (CFSLP). So that the eigenvalues of CFSLP are simple and the corresponding eigenfunctions are orthogonal.

Furthermore, in [11] the authors proved the uniqueness theorems of CFSLP for solving the inverse problem with respect to the Weyl function, two spectra and spectral data. Also, they proved the Hochstadt-Lieberman theorem.

Motivated by the above discussion, this study aims to propose how to handle a discontinuous conditions and apply the asymptotic formulas to prove several uniqueness results. In this study, we introduce a Weyl function m that uniquely determines the parameters of the problem. We also show that this Weyl function is a meromorphic Herglotz–Nevanlinna function uniquely determined by its poles and residues as well as by its poles and zeros. Moreover, we generalize the Hochstadt–Liebermann type result to the present situation. For related result in the SLP, FSLP, CFSLP, PDSLP we refer to [12–15, 17–23].

2. Preliminaries

In this section, we will present some necessary definitions and properties related to conformable fractional calculus theory which can be found in [1, 5].

Definition 2.1. For the function $h : [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$, the CFD is defined by:

(2.1)
$$D^\alpha h(\tau) = \lim_{\delta\to 0} \frac{h\left(\tau + \delta\tau^{1-\alpha}\right) - h(\tau)}{\delta}, \qquad D^\alpha h(0) = \lim_{\tau\to 0^+} D^\alpha h(\tau),$$

for all $\tau > 0$. If h is differentiable, then

$$D^\alpha f(\tau) = \tau^{1-\alpha} h'(\tau), \tag{2.2}$$

where $h'(\tau) = \lim_{\delta\to 0}[h(\tau+\delta) - h(\tau)]/\delta$. If $D^\alpha h(\tau_0)$ exists and is finite, we say that h is α-differentiable at τ_0.

Theorem 2.2. *If a function $h : [0,\infty) \to \mathbb{R}$ is conformable fractional (CF) differentiable at $\tau_0 > 0$, then h is continuous at x_0.*

Definition 2.3. For the function $h : [0, \infty) \to \mathbb{R}$ and $\alpha \in (0, 1]$, the CF integral is defined by:

$$J_\alpha h(\tau) = \int_0^\tau h(t)\mathrm{d}_\alpha t = \int_0^\tau t^{\alpha-1} h(t)\mathrm{d}t. \tag{2.3}$$

So that the above integral is the Riemann improper integral.

Theorem 2.4. *For the α-differentiable functions g and h, the following formulas hold*

(i) $D^\alpha(\eta g + \kappa h) = \eta D^\alpha g + \kappa D^\alpha h$, $\forall \eta, \kappa \in \mathbb{R}$.
(ii) $D^\alpha(\tau^p) = p\tau^{p-\alpha}$, $\forall p \in \mathbb{R}$.
(iii) $D^\alpha(k) = 0$, $\forall k$ *constant.*
(iv) $D^\alpha(gh) = hD^\alpha(g) + gD^\alpha(h)$.
(v) $D^\alpha(g/h) = \dfrac{hD^\alpha(g) - gD^\alpha(h)}{h^2}$, *for* $h \neq 0$.
(vi) *For the continuous function* $h : [0, \infty) \to \mathbb{R}$*, we get* $D^\alpha J_\alpha h(\tau) = h(\tau)$*, for all* $\tau > a$.
(vii) *For the differentiable function* $h : [0, \infty) \to \mathbb{R}$*, we have*

$$J_\alpha D^\alpha h(\tau) = h(\tau) - h(a), \quad \text{for all } \tau > a.$$

3. Main Problem and Spectral Properties

Let us consider the following boundary value problem

$$\ell_\alpha y := -D^\alpha D^\alpha y + q(x)y = \lambda y \tag{3.1}$$

with the eigen-parameter dependent conditions

$$\begin{aligned} L_1(y) &:= \lambda y(0) - hD^\alpha y(0) = 0, \\ L_2(y) &:= \lambda y(\pi) + HD^\alpha y(\pi) = 0 \end{aligned} \tag{3.2}$$

and the jump conditions

$$\begin{aligned} \mathcal{D}_1(y) &:= y(p+0) - ay(p-0) = 0, \\ \mathcal{D}_2(y) &:= D^\alpha y(p+0) - bD^\alpha y(p-0) - cy(p-0) = 0, \end{aligned} \tag{3.3}$$

where $q(x)$ is real-valued function in $L^1_\alpha[0, \pi]$. We also assume that h, H, a, b, c and p are real numbers, satisfying $ab > 0$. For the reader's convenience, we use the notation $L_\alpha = L_\alpha(q(x); h; H; p)$, for the problem (3.1)–(3.3).

To obtain a self-adjoint operators, we define the weighted inner products as follows

$$\begin{aligned} \langle F, G\rangle_{\mathcal{H}} &:= \int_0^\pi f\overline{g}w\mathrm{d}_\alpha x + \frac{w(0)}{h} f_1\overline{g_1} + \frac{w(\pi)}{H} f_2\overline{g_2}, \\ F &= \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}, \qquad G = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix}, \end{aligned} \tag{3.4}$$

where the weight function is

$$w(x) = \begin{cases} 1, & 0 \le x < p, \\ \frac{1}{ab}, & p < x < \pi. \end{cases} \tag{3.5}$$

In order to obtain a new self-adjoint eigenvalue problem using Hilbert space $\mathcal{H} := L^2_\alpha((0,\pi);w) \oplus \mathbb{C}^2$ with the following operator

$$A_\alpha : \mathcal{H} \to \mathcal{H}.$$

Next we introduce

$$R_1(y) := y(0), \qquad R'_1(y) := hD^\alpha y(0),$$

$$R_2(y) := y(\pi), \qquad R'_2(y) := HD^\alpha y(\pi).$$

In this space we construct with domain

$$\operatorname{dom}(A_\alpha) = \left\{ F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \middle| \begin{array}{c} f, D^\alpha f \in AC\left([0,p) \cup (p,\pi]\right),\ \ell f \in L(0,\pi) \\ f_1 = R_1(f),\ f_2 = R_2(f),\ \mathcal{D}_1(f) = \mathcal{D}_2(f) = 0, \end{array} \right\}$$

by

$$A_\alpha F = \begin{pmatrix} \ell f \\ R'_1(f) \\ -R'_2(f) \end{pmatrix} \quad \text{with} \quad F = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} \in \operatorname{dom}(A_\alpha).$$

By construction the eigenvalue problem of the form

$$A_\alpha Y = \lambda Y, \qquad Y := \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in \operatorname{dom}(A_\alpha),$$

it is easy to see that, this problem is equivalent to the eigenvalue problem (3.1)–(3.3).

Throughout the paper, using $AC\left([0,p) \cup (p,\pi]\right)$ represents the set of all functions whose restriction to $[0,p)$ or $(p,\pi]$ is absolutely continuous. From the linear differential equations, we obtain that the modified fractional Wronskian

$$W_\alpha(f,g) = w(x)\left(f(x)D^\alpha g(x) - D^\alpha f(x) g(x)\right) \tag{3.6}$$

is constant on $x \in [0,p) \cup (p,\pi]$ for two solutions $\ell_\alpha f = \lambda f$, $\ell_\alpha g = \lambda g$ satisfying the transmission conditions (3.3).

Lemma 3.1. *If $0 < \alpha \le 1$, then the PDCFSL operator A_α is self-adjoint on $L^2_\alpha((0,\pi);w) \oplus \mathbb{C}^2$.*

Proof. After using α-integration by parts twice, we arrive at the following expression:

$$\langle A_\alpha F, G\rangle = W_\alpha (f,g)\big|_\pi - W_\alpha (f,g)\big|_0 + \langle F, A_\alpha G\rangle . \tag{3.7}$$

So, from Eqs (3.2) and (3.3) we have:

$$W_\alpha (f,g)\big|_\pi - W_\alpha (f,g)\big|_0 = 0$$

Then A_α is self-adjoint operator on $L^2_\alpha((0,\pi);w)\oplus\mathbb{C}^2$. □

Especially, the eigenvalues of A_α and consequently L_α are real and simple. Further, for each function $f\in \operatorname{dom}(A_\alpha)$ we will denote by f_j, $(j=1,2)$, the restriction of f to the subinterval (p_{j-1},p_j), $(p_0=0\ p_2=\pi)$. Moreover, we will set $f_2(p)=f(p+0)$ and $f_1(p)=f(p-0)$. Assume that $u(x,\lambda)$ and $v(x,\lambda)$ are solutions of (3.1) with

$$u(0,\lambda)=h, \qquad D^\alpha u(0,\lambda)=\lambda, \tag{3.8}$$

and

$$v(\pi,\lambda)=-H, \qquad D^\alpha v(\pi,\lambda)=\lambda, \tag{3.9}$$

and the transmission conditions (3.3), respectively. Note that the equation (3.1) with initial conditions (3.8) or (3.9) has a unique solution. These solutions are the entire function in $\lambda\in\mathbb{C}$ for $x\in[0,d)\cup(d,\pi]$. Similarly, we can write

$$\begin{aligned}\Delta(\lambda) &:= W_\alpha(u(\lambda),v(\lambda))\\ &= L_1(v(\lambda))\\ &= -w(\pi)L_2(u(\lambda)).\end{aligned} \tag{3.10}$$

The function $\Delta(\lambda)$ is called the characteristic function. All roots λ_n of $\Delta(\lambda)$ are the eigenvalues of L_α. Furthermore, $u(x,\lambda_n)$ and $v(x,\lambda_n)$ are the eigenfunctions of λ_n, satisfy the relation $v(x,\lambda_n)=\beta_n u(x,\lambda_n)$, from (3.8),

$$\beta_n = \frac{v(0,\lambda_n)}{h}. \tag{3.11}$$

Also, define

$$\gamma_n := \|u(x,\lambda_n)\|^2_{\mathcal{H}}.$$

Thus it can be verified that:

Lemma 3.2. *The eigenvalues of L_α, λ_n, are real and simple. The derivative of $\Delta(\lambda)$ in λ_n has the following form*

$$\dot\Delta(\lambda_n) = -\gamma_n\beta_n, \tag{3.12}$$

where $\dot\Delta(\lambda)=\frac{d}{d\lambda}\Delta(\lambda)$.

In the sequel, we consider a simple unitary transformation for our eigenvalue problem, then it can be seen that:

Remark 3.3. Without losing of generality of the problem (3.1)–(3.3), by [21, Lemma 2.3], we can get $ab = 1$.

4. Asymptotic Formulas for Eigendata

In this section, we study the asymptotic forms of solutions and eigenvalues. For these aims, we prove some lemmas and theorems as follows.

Theorem 4.1. *Let $\lambda = \rho^2$ and $\tau := |\mathrm{Im}\rho|$. The asymptotic forms of solutions and the characteristic function for PDCFSLP (3.1)– (3.3) as $|\lambda| \to \infty$, are established as follows:*

(4.1)

$$u(x,\lambda) = \begin{cases} \rho^2 \cos\left(\frac{\rho}{\alpha}x^\alpha\right) + \rho\left[q_1(x)\sin\left(\frac{\rho}{\alpha}x^\alpha\right) \right. \\ \quad \left. +\frac{1}{2}\int_0^x q(t)\sin\left(\frac{\rho}{\alpha}\left(x^\alpha - 2t^\alpha\right)\right)\mathrm{d}_\alpha t\right] + O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0 \le x < p, \\ \rho^2\left[b_1\cos\left(\frac{\rho}{\alpha}x^\alpha\right) + b_2\cos\left(\frac{\rho}{\alpha}(2p^\alpha - x^\alpha)\right)\right] + \rho\left[f_1(x)\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right. \\ \quad \left. +f_2(x)\sin\left(\frac{\rho}{\alpha}(2p^\alpha - x^\alpha)\right)\right] + O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & p \le x < \pi, \end{cases}$$

(4.2)

$$D_x^\alpha u(x,\lambda) = \begin{cases} -\rho^3 \sin\left(\frac{\rho}{\alpha}x^\alpha\right) + \rho^2\left[q_1(x)\cos\left(\frac{\rho}{\alpha}x^\alpha\right) \right. \\ \quad \left. +\frac{1}{2}\int_0^x q(t)\cos\left(\frac{\rho}{\alpha}\left(x^\alpha - 2t^\alpha\right)\right)\mathrm{d}_\alpha t\right] + O\left(\rho\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0 \le x < p, \\ \rho^3\left[-b_1\sin\left(\frac{\rho}{\alpha}x^\alpha\right) + b_2\sin\left(\frac{\rho}{\alpha}(2d^\alpha - x^\alpha)\right)\right] + \rho^2\left[f_1(x)\cos\left(\frac{\rho}{\alpha}x^\alpha\right)\right. \\ \quad \left. -f_2(x)\cos\left(\frac{\rho}{\alpha}(2p^\alpha - x^\alpha)\right)\right] + O\left(\rho\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d \le x < \pi, \end{cases}$$

where

$$b_1 = \frac{a+b}{2},\ b_2 = \frac{a-b}{2},\ q_1(x) = \frac{1}{2}\int_0^x q(t)\mathrm{d}_\alpha t - h,$$
$$f_1(x) = b_1\left(\frac{1}{2}\int_0^x q(t)\mathrm{d}_\alpha t - h\right) + \frac{c}{2},$$
$$f_2(x) = b_2\left(-\frac{1}{2}\int_0^x q(t)\mathrm{d}_\alpha t + \int_0^p q(t)\mathrm{d}_\alpha t - h\right) - \frac{c}{2}.$$

The characteristic function satisfies

(4.3)

$$\Delta(\lambda) = w(\pi)\Bigg[\rho^4\left(b_1\cos\left(\frac{\rho}{\alpha}\pi^\alpha\right)+b_2\cos\left(\frac{\rho}{\alpha}(2p^\alpha-\pi^\alpha)\right)\right) + \rho^3\left[-w_1\sin\left(\frac{\rho}{\alpha}\pi^\alpha\right)+w_2\sin\left(\frac{\rho}{\alpha}(2d^\alpha-\pi^\alpha)\right)\right]+o\left(\exp\left(\frac{\tau}{\alpha}\pi^\alpha\right)\right)\Bigg].$$

where

$$w_1 = b_1\left(H-h+\frac{1}{2}\int_0^\pi q(t)\mathrm{d}_\alpha t\right)+\frac{c}{2}, \tag{4.4}$$
$$w_2 = b_2\left(H-h-\frac{1}{2}\int_0^\pi q(t)\mathrm{d}_\alpha t+\int_0^p q(t)\mathrm{d}_\alpha t\right)-\frac{c}{2}.$$

Proof. Suppose that $C_1(x,\lambda)$ and $S_1(x,\lambda)$ are solutions (3.1) and jump conditions (3.3) with initial conditions:

$$S_1(0,\lambda)=0, \qquad D^\alpha S_1(0,\lambda)=1,$$

and

$$C_1(0,\lambda)=1, \qquad D^\alpha C_1(0,\lambda)=0.$$

Using the jump conditions (3.3), we get

$$S_2(x,\lambda) = A_1C_1(x,\lambda)+B_1S_1(x,\lambda), \tag{4.5}$$
$$C_2(x,\lambda) = A_2C_1(x,\lambda)+B_2S_1(x,\lambda),$$

where

(4.6)

$$A_1 = (a-b)S_1(p,\lambda)D^\alpha S_1(p,\lambda)-cS_1^2(p,\lambda),$$
$$B_1 = bC_1(p,\lambda)D^\alpha S_1(p,\lambda)-aS_1(p,\lambda)D^\alpha C_1(p,\lambda)+cS_1(p,\lambda)C_1(p,\lambda),$$
$$A_2 = aC_1(p,\lambda)D^\alpha S_1(p,\lambda)-bS_1(p,\lambda)D^\alpha C_1(p,\lambda)-cS_1(p,\lambda)C_1(p,\lambda),$$
$$B_2 = (a-b)C_1(p,\lambda)D^\alpha C_1(p,\lambda)+cC_1^2(p,\lambda).$$

It was shown in [10] that the function $C_1(x,\lambda)$ is the unique solution of the integral equation

$$C_1(x,\lambda)=\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+\int_0^x\frac{\sin\left(\frac{\rho}{\alpha}(x^\alpha-t^\alpha)\right)}{\rho}q(t)C_1(t,\lambda)\mathrm{d}_\alpha t \tag{4.7}$$

and for $|\rho|\to\infty$, we have

$$C_1(x,\lambda)=\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}x^\alpha\right).$$

From (4.7) we calculate

(4.8)
$$C_1(x,\lambda) = \cos\left(\frac{\rho}{\alpha}x^\alpha\right) + \frac{\sin\left(\frac{\rho}{\alpha}x^\alpha\right)}{2\rho}\int_0^x q(t)\mathrm{d}_\alpha t + \frac{1}{2\rho}\int_0^x \sin\left(\frac{\rho}{\alpha}(x^\alpha - 2t^\alpha)\right) q(t)\mathrm{d}_\alpha t + O\left(\frac{1}{\rho^2}\exp\frac{\tau}{\alpha}x^\alpha\right),$$

that $0 \le x < p$ and

(4.9)
$$D^\alpha C_1(x,\lambda) = -\rho\sin\left(\frac{\rho}{\alpha}x^\alpha\right) + \frac{\cos\left(\frac{\rho}{\alpha}x^\alpha\right)}{2}\int_0^x q(t)\mathrm{d}_\alpha t + \frac{1}{2}\int_0^x \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2t^\alpha)\right) q(t)\mathrm{d}_\alpha t + O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}x^\alpha\right),$$

that $0 \le x < p$. Analogously,

(4.10)
$$S_1(x,\lambda) = \frac{1}{\rho}\sin\left(\frac{\rho}{\alpha}x^\alpha\right) - \frac{\cos\left(\frac{\rho}{\alpha}x^\alpha\right)}{2\rho^2}\int_0^x q(t)\mathrm{d}_\alpha t + \frac{1}{2\rho^2}\int_0^x \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2t^\alpha)\right) q(t)\mathrm{d}_\alpha t + O\left(\frac{1}{\rho^3}\exp\frac{\tau}{\alpha}x^\alpha\right),$$

that $0 \le x < p$ and

(4.11)
$$D^\alpha S_1(x,\lambda) = \cos\left(\frac{\rho}{\alpha}x^\alpha\right) + \frac{\sin\left(\frac{\rho}{\alpha}x^\alpha\right)}{2\rho}\int_0^x q(t)\mathrm{d}_\alpha t - \frac{1}{2\rho}\int_0^x \sin\left(\frac{\rho}{\alpha}(x^\alpha - 2t^\alpha)\right) q(t)\mathrm{d}_\alpha t + O\left(\frac{1}{\rho^2}\exp\frac{\tau}{\alpha}x^\alpha\right),$$

that $0 \le x < p$.

By virtue of (4.6) and (4.7)–(4.11)

$$\begin{aligned}
A_1 &= \frac{b_2}{\rho}\sin\left(\frac{2\rho}{\alpha}p^\alpha\right) + O\left(\frac{1}{\rho^2}\exp\frac{\tau}{\alpha}p^\alpha\right),\\
B_1 &= b_1 + b_2\cos\left(\frac{2\rho}{\alpha}p^\alpha\right) + O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}p^\alpha\right),\\
A_2 &= b_1 + b_2\cos\left(\frac{2\rho}{\alpha}p^\alpha\right) + \frac{\sin\left(\frac{2\rho}{\alpha}p^\alpha\right)}{\rho}\left(b_2\int_0^p q(t)\mathrm{d}_\alpha t - \frac{c}{2}\right)\\
&\quad + O\left(\frac{1}{\rho^2}\exp\frac{\tau}{\alpha}p^\alpha\right),\\
B_2 &= b_2\left(\rho\sin\left(\frac{2\rho}{\alpha}p^\alpha\right) - \cos\left(\frac{2\rho}{\alpha}p^\alpha\right)\int_0^p q(t)\mathrm{d}_\alpha t\right.
\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&-\int_0^p \cos\left(\frac{2\rho}{\alpha}(p^\alpha - t^\alpha)\right) q(t)\mathrm{d}_\alpha t\Bigg) \\
&+\frac{c}{2}\left(1+\cos\left(\frac{2\rho}{\alpha}p^\alpha\right)\right) + O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}p^\alpha\right).
\end{aligned}
$$

Using (4.7)–(4.12), we get the asymptotic forms of $C(x,\lambda)$, $S(x,\lambda)$, $D^\alpha C(x,\lambda)$ and $D^\alpha S(x,\lambda)$. Clearly, $u(x;\lambda) = \lambda C(x;\lambda) - hD^\alpha S(x;\lambda)$. So, using (4.7)–(4.12), we calculate (4.1)-(4.2). Using (3.10), we obtain the characteristic function (4.3). □

As a direct consequence of Theorem 4.1, we get

$$
\begin{aligned}
|u(x,\lambda)| &= O\left(|\rho|^2 \exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), \\
|D^\alpha u(x,\lambda)| &= O\left(|\rho|^3 \exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), \quad 0 \le x \le \pi
\end{aligned}
$$

By changing x to $\pi - x$ in Eqs. (3.1) and using the jump conditions (3.3), asymptotic formulas of $v(x,\lambda)$ and $D^\alpha v(x,\lambda)$ can be obtained. Specially,

$$
\begin{aligned}
|v(x,\lambda)| &= O\left(|\rho|^2 \exp\left(\frac{\tau}{\alpha}(\pi - x)^\alpha\right)\right), \\
|D^\alpha v(x,\lambda)| &= O\left(|\rho|^3 \exp\left(\frac{\tau}{\alpha}(\pi - x)^\alpha\right)\right), \quad 0 \le x \le \pi.
\end{aligned} \tag{4.13}
$$

One can see that from Valiron's theorem ([9, Thm. 13.4]), [10] and the above calculations, we obtain:

Theorem 4.2. *The eigenvalues $\lambda_n = \rho_n^2$ of the PDCFSLP* (3.1)–(3.3) *satisfy*

$$
\rho_n = \alpha\pi^{1-\alpha} n + O(1) \tag{4.14}
$$

as $n \to \infty$.

Lemma 4.3. *The specification of the spectrum $\{\lambda_n\}$, $n \ge 0$, are uniquely determined by the characteristic function $\Delta(\lambda)$ by the formulaes*

$$
\Delta(\lambda) = C \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n^\circ}, \tag{4.15}
$$

where $C = -\lambda_0 \Omega \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^\circ}$.

Proof. By Hadamard's factorization theorem [4, P. 289], $\Delta(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$
\Delta(\lambda) = \frac{C(\lambda - \lambda_0)^2}{\lambda_0^2} \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right). \tag{4.16}
$$

(when $\Delta(0) = 0$ needs minor modifications). Define

$$\Delta_\circ(\lambda) = \rho^4 w(\pi)\left(b_1 \cos\left(\frac{\rho}{\alpha}\pi^\alpha\right) + b_2 \cos\left(\frac{\rho}{\alpha}(2d^\alpha - \pi^\alpha)\right)\right) \tag{4.17}$$

Let $\lambda_n = \rho_n^2$ and $\lambda_n^\circ = (\rho_n^\circ)^2$ be zeros of the functions (4.3) and (4.17) respectively, then

$$\rho_n = \rho_n^\circ + o(1), \quad n \to \infty.$$

Using the Hadamard's factorization [9, Sec. 4.2] for the function $\Delta_\circ(\lambda)$ defined in (4.17), we obtain the infinite product

$$\Delta_\circ(\lambda) = \Omega\lambda^2 \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n^\circ}\right), \qquad \Omega = aw(\pi).$$

Then

$$\frac{\Delta(\lambda)}{\Delta_\circ(\lambda)} = \frac{C(\lambda_0 - \lambda)^2}{\Omega\lambda_0^2\lambda^2} \prod_{n=1}^{\infty}\frac{\lambda_n^\circ}{\lambda_n} \prod_{n=1}^{\infty}\left(1 + \frac{\lambda_n - \lambda_n^\circ}{\lambda_n^\circ - \lambda}\right).$$

Taking (4.3) into account, we calculate

$$\lim_{\lambda\to-\infty}\frac{\Delta(\lambda)}{\Delta_\circ(\lambda)} = 1, \qquad \lim_{\lambda\to-\infty}\prod_{n=1}^{\infty}\left(1 + \frac{\lambda_n - \lambda_n^\circ}{\lambda_n^\circ - \lambda}\right) = 1$$

and hence

$$C = \lambda_0^2\Omega\prod_{n=1}^{\infty}\frac{\lambda_n}{\lambda_n^\circ}.$$

Substituting this into (4.16), we arrive at (4.15). □

Example 4.4. Consider the following PDCSLP with $q(x) = 0$ and $h = H = 1$ with one jump point $p = \frac{\pi}{4}$

$$\begin{aligned}
&-D^\alpha D^\alpha y = \lambda y \\
&\lambda y(0) - D^\alpha y(0) = 0, \\
&\lambda y(\pi) + D^\alpha y(\pi) = 0, \\
&y\left(\frac{\pi}{4} + 0\right) - 2y\left(\frac{\pi}{4} - 0\right) = 0, \\
&D^\alpha y\left(\frac{\pi}{4} + 0\right) - \frac{1}{2}D^\alpha y\left(\frac{\pi}{4} - 0\right) = 0.
\end{aligned} \tag{4.18}$$

The characteristic function and eigenfunctions are

$$\begin{aligned}
\Delta(\lambda) =& \rho^4\left[\frac{5}{4}\cos\left(\frac{\rho}{\alpha}\pi^\alpha\right) + \frac{3}{4}\cos\left(\frac{\rho}{\alpha}\left(\pi^\alpha - 2\left(\frac{\pi}{4}\right)^\alpha\right)\right)\right] \\
&+ \rho^3\left[-\frac{5}{4}\sin\left(\frac{\rho}{\alpha}\pi^\alpha\right) - \frac{3}{4}\sin\left(\frac{\rho}{\alpha}\left(\pi^\alpha - 2\left(\frac{\pi}{4}\right)^\alpha\right)\right)\right] \\
&+ \rho^2\left[\frac{5}{4}\cos\left(\frac{\rho}{\alpha}\pi^\alpha\right) - \frac{3}{4}\cos\left(\frac{\rho}{\alpha}\left(\pi^\alpha - 2\left(\frac{\pi}{4}\right)^\alpha\right)\right)\right]
\end{aligned}$$

$$+\rho\left[-\frac{5}{4}\sin\left(\frac{\rho}{\alpha}\pi^{\alpha}\right)+\frac{3}{4}\sin\left(\frac{\rho}{\alpha}\left(\pi^{\alpha}-2\left(\frac{\pi}{4}\right)^{\alpha}\right)\right)\right]$$

$$u_{n,\alpha}(x)=\begin{cases}\rho_n^2\cos\left(\frac{\rho_n}{\alpha}x^{\alpha}\right)-\rho_n\sin\left(\frac{\rho_n}{\alpha}x^{\alpha}\right), & 0\le x<\frac{\pi}{4},\\ \quad\rho_n^2\left[\frac{5}{4}\cos\left(\frac{\rho_n}{\alpha}x^{\alpha}\right)+\frac{3}{4}\cos\left(\frac{\rho_n}{\alpha}\left(x^{\alpha}-2\left(\frac{\pi}{4}\right)^{\alpha}\right)\right)\right] & \\ \quad-\rho_n\left[\frac{5}{4}\sin\left(\frac{\rho_n}{\alpha}x^{\alpha}\right)-\frac{3}{4}\sin\left(\frac{\rho_n}{\alpha}\left(x^{\alpha}-2\left(\frac{\pi}{4}\right)^{\alpha}\right)\right)\right], & \frac{\pi}{4}\le x\le\pi.\end{cases}$$

The eigenvalues and eigenfunctions are presented in Table 1 and Figure 1. We use the **fzero** function in **MATLAB R2015a** to compute the zeros $\rho_{n,\alpha}$ of the function $\Delta(\lambda)$.

TABLE 1. Eigenvalues and asymptotic results for Example 4.4.

	$\rho_{n,\alpha}$				$\zeta_{n,\alpha}$			
n	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=0.99$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=0.99$
2	1.2747	1.3266	1.3520	1.3501	0.646	0.657	0.670	0.674
3	2.2651	2.2556	2.2608	2.2801	0.765	0.745	0.747	0.759
4	3.5339	3.5366	3.4232	3.3073	0.895	0.876	0.848	0.826
5	4.2004	4.5073	4.6585	4.5392	0.851	0.893	0.923	0.907
10	9.2301	9.6384	9.3520	9.6268	0.935	0.955	0.926	0.961
15	14.3062	14.3736	14.753	14.2997	0.966	0.965	0.974	0.952
20	19.3728	19.7019	19.5224	19.4549	0.973	0.976	0.967	0.971
25	24.3658	24.7062	24.7421	24.6936	0.981	0.979	0.981	0.986
30	29.2630	29.4561	29.8097	29.5722	0.985	0.983	0.984	0.988
35	34.1072	34.8532	34.7168	34.3422	0.987	0.986	0.982	0.985
40	38.9354	39.7493	40.0483	39.5571	0.986	0.984	0.988	0.988

We compared the eigenvalues with first term of asymptotic form (4.14) as $\zeta_{n,\alpha}=\frac{\rho_n}{n\alpha\pi^{1-\alpha}}$. The eigenvalue and ratio ζ_n are presented in Tables 1. According to asymptotic form (4.14), the values of $\zeta_{n,\alpha}$ must be tend to one, that hold for results of $\zeta_{n,\alpha}$ in Tables 1. The first four eigenfunctions for different values of α are plotted in Figures 1. It is well known that, the nth eigenfunction of eigen-parameter SLP defined on $[0,\pi]$, has n zero in interval $(0,\pi)$. The graphs in Figures 1 indicate that this result hold also for PDCSLP with jump conditions.

5. Uniqueness Results

The main goal of this section is to study the inverse problem of the reconstruction of a boundary value problem L_{α} from its spectral characteristics. Moreover, we have used the three statements of the inverse problem of the reconstruction of the boundary value problem L_{α} : from the Weyl function, from the spectral data $\{\lambda_n,\gamma_n\}_{n\geq 0}$ and from two spectra $\{\lambda_n,\mu_n\}_{n\geq 0}$.

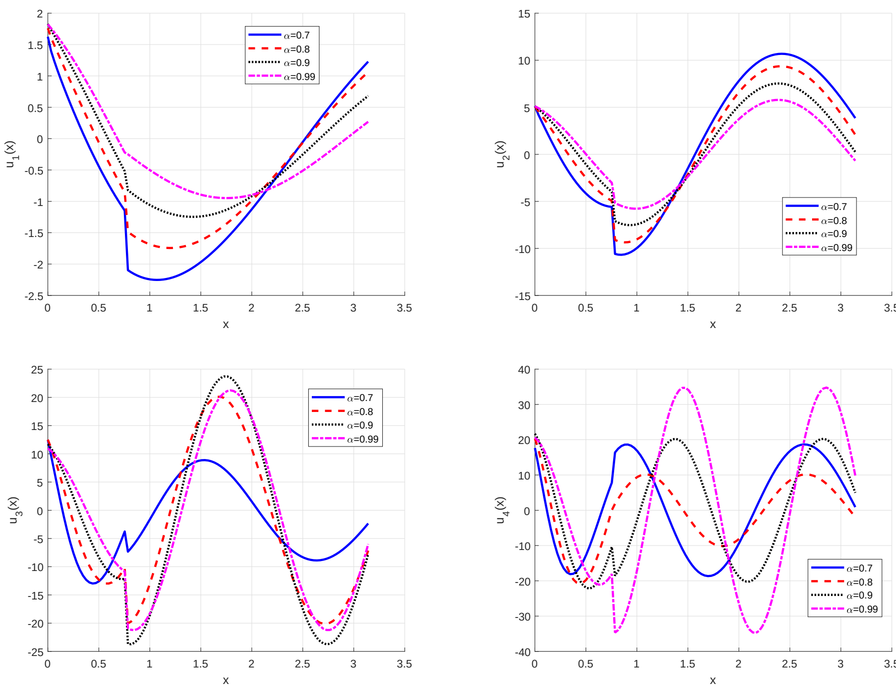

FIGURE 1. Eigenfunctions of Example 4.4 for different values of n and α.

Define the Weyl m-function by

$$m(\lambda) = -\frac{R_1(v)}{h\Delta(\lambda)} = -\frac{v(0,\lambda)}{\Delta(\lambda)}. \tag{5.1}$$

From (3.8) and (4.13), we can get the asymptotic expansion

$$m(\lambda) = \frac{1}{\sqrt{-\lambda}} + O(\lambda^{-1}) \tag{5.2}$$

along any ray except the positive real axis.

For this purpose, first we consider $\chi(x,\lambda)$ be a solution of (3.1) from the initial conditions

$$\chi(0,\lambda) = 0, \qquad D^{\alpha}\chi(0,\lambda) = \frac{1}{h}$$

and the jump conditions (3.3). It's obvious that $W_\alpha(u,\chi) = 1 \neq 0$ and the function $v(x,\lambda)$ we obtain

$$\begin{aligned}\theta(x,\lambda) &:= \frac{v(x,\lambda)}{\Delta(\lambda)} \\ &= \chi(x,\lambda) - m(\lambda)u(x,\lambda).\end{aligned} \tag{5.3}$$

The functions $\theta(x,\lambda)$ and $m(\lambda)$ are called the Weyl solution and the Weyl function, respectively for the boundary value problem L_α. Clearly

$$W_\alpha(u(x,\lambda),\theta(x,\lambda)) = 1. \tag{5.4}$$

Lemma 5.1. *The Weyl function $m(\lambda)$ is a meromorphic, Herglotz–Nevanlinna function,*

$$\operatorname{Im}(m(\lambda)) = \operatorname{Im}(\lambda)\|\Theta(\lambda)\|^2_{\mathcal{H}}, \qquad \Theta(x,\lambda) = \begin{pmatrix} \theta(x) \\ -R_1(\theta) \\ R_2(\theta) \end{pmatrix} \tag{5.5}$$

and can be represented as

$$m(\lambda) = \sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda_n - \lambda}, \tag{5.6}$$

where

$$\sum_{n=0}^{\infty} \gamma_n = \frac{1}{h}. \tag{5.7}$$

Proof. The first relation follows after a straightforward calculation using

$$\begin{aligned}&\operatorname{Im}\left(\theta(\pi,\lambda)\overline{D^\alpha\theta(\pi,\lambda)}\right) - \operatorname{Im}\left(\theta(0,\lambda)\overline{D^\alpha\theta(0,\lambda)}\right) \\ &= \operatorname{Im}(\lambda)\int_0^\pi |\theta(x,\lambda)|^2 w(x)d_\alpha x.\end{aligned}$$

Hence $m(z)$ is a Herglotz–Nevanlinna function (i.e. it maps the upper half plane to the upper half plane) and by the asymptotic (5.2) it has a representation of the form ([24, Lem. 9.20])

$$m(\lambda) = \int_{\mathbb{R}} \frac{d\rho(t)}{\lambda_n - t},$$

where ρ is a Borel measure satisfying

$$\int_{\mathbb{R}} \frac{d\rho(t)}{1+|\lambda|^\gamma}, \quad \forall\gamma > \frac{1}{2}.$$

Since by (5.1) the Weyl function is meromorphic it follows that ρ is a pure point measure supported at the poles with masses given by the negative residues. Hence the result follows from Lemma 3.2. $\square$

Now, we are ready to prove our main uniqueness theorem for the solutions of the problems (3.1)–(3.3). For this purpose, we agree that together with L_α we consider a boundary value problem $\tilde{L}_\alpha$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$. If a certain symbol η denotes an object related to L_α, then $\tilde{\eta}$ will denote the analogous object related to $\tilde{L}_\alpha$.

Theorem 5.2. *If $m(\lambda) = \tilde{m}(\lambda)$ and $w(x) = \tilde{w}(x)$ then $L_\alpha = \tilde{L}_\alpha$. Thus the specification of the Weyl function and the weight function $w(x)$ uniquely determines the operator.*

Proof. It follows from (4.13) and (5.3) that

(5.8)
$$|\theta(x,\lambda)| \le C|\rho|^{-1}\exp\left(\frac{-\tau}{\alpha}x^\alpha\right), \qquad |D^\alpha\theta(x,\lambda)| \le C\exp\left(\frac{-\tau}{\alpha}x^\alpha\right),$$

as $\lambda \to \infty$ along any ray except the positive real axis. Define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix} \tilde{u}(x,\lambda) & \tilde{\theta}(x,\lambda) \\ D^\alpha\tilde{u}(x,\lambda) & D^\alpha\tilde{\theta}(x,\lambda)\end{pmatrix} = \begin{pmatrix} u(x,\lambda) & \theta(x,\lambda) \\ D^\alpha u(x,\lambda) & D^\alpha\theta(x,\lambda)\end{pmatrix}.$$

Taking (5.4) into account, we calculate

(5.9)
$$\begin{pmatrix} P_{11}(x,\lambda) & P_{12}(x,\lambda) \\ P_{21}(x,\lambda) & P_{22}(x,\lambda)\end{pmatrix} = \begin{pmatrix} uD^\alpha\tilde{\theta} - D^\alpha\tilde{u}\theta & \tilde{u}\theta - u\tilde{\theta} \\ D^\alpha u D^\alpha\tilde{\theta} - D^\alpha\tilde{u}D^\alpha\theta & \tilde{u}D^\alpha\theta - D^\alpha u\tilde{\theta}\end{pmatrix}$$

and

$$\begin{pmatrix} u(x,\lambda) \\ \theta(x,\lambda)\end{pmatrix} = \begin{pmatrix} P_{11}(x,\lambda)\tilde{u}(x,\lambda) + P_{12}(x,\lambda)D^\alpha\tilde{u}(x,\lambda) \\ P_{11}(x,\lambda)\tilde{\theta}(x,\lambda) + P_{12}(x,\lambda)D^\alpha\tilde{\theta}(x,\lambda)\end{pmatrix}. \tag{5.10}$$

It is easy to see that the functions $P_{jk}(x,\lambda)$, $j,k = 1,2$ are meromorphic in λ with simple poles in the points λ_n and $\tilde{\lambda}_n$. Moreover, if $m(\lambda) = \tilde{m}(\lambda)$ then from (5.3) and (5.9), $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire functions of growth order $1/2$ in λ. From (5.8)

$$|P_{11}(x,\lambda)| \le C, \qquad |P_{12}(x,\lambda)| \le \frac{C}{|\rho|} \tag{5.11}$$

along any ray except the positive real axis. Moreover, by our hypothesis this function has an order of growth s and thus we can apply the Phragmén–Lindelöf theorem (e.g., [9, Sect. 6.1]) the two half-planes bounded by the imaginary axis. This shows that the functions P_{11} and P_{12} are bounded on all of $\mathbb{C}$ and thus constant by Liouville's theorem. Since P_{12} vanishes along a ray it must be zero and we obtain $P_{11}(x,\lambda) = A(x)$ and $P_{12}(x,\lambda) = 0$. Using (5.10), we get

$$u(x,\lambda) = A(x)\tilde{u}(x,\lambda),\ \theta(x,\lambda) = A(x)\tilde{\theta}(x,\lambda). \tag{5.12}$$

It follows from (3.10), $W_\alpha(u(x,\lambda),\theta(x,\lambda)) = W_\alpha(\tilde{u}(x,\lambda),\tilde{\theta}(x,\lambda)) = 1$ and so we deduce $A(x) = \frac{\tilde{w}(x)}{w(x)} = 1$, that is, $u(x,\lambda) = \tilde{u}(x,\lambda)$, $\theta(x,\lambda) = \tilde{\theta}(x,\lambda)$ and $v(x,\lambda) = \tilde{v}(x,\lambda)$. Therefore from (3.1), (3.3), (3.10) and (3.9) we get $q(x) = \tilde{q}(x)$, a.e. on $[0,\pi]$ and $a = \tilde{a}$, $b = \tilde{b}$, $c = \tilde{c}$, $d = \tilde{d}$, $h = \tilde{h}$ and $H = \tilde{H}$. Consequently $L_\alpha = \tilde{L}\alpha$. □

Note that this theorem is optimal in the sense that the weight function cannot be determined from $m(\lambda)$ since a unitary transformation as in Lemma 3.3 can be used to change the weight without changing $m(\lambda)$. Note that the condition $w(x) = \tilde{w}(x)$ will be hold if we have for example $ab = \tilde{a}\tilde{b} = 1$ and $d = \tilde{d}$.

By virtue of Lemma 5.1 we also get:

Corollary 5.3. *If $\lambda_n = \tilde{\lambda}_n$ and $\gamma_n = \tilde{\gamma}_n$, for $n = 0,1,2,...$ and $w(x) = \tilde{w}(x)$ then $L_\alpha = \tilde{L}_\alpha$.*

Finally, let us consider the boundary value problem L_α^k which is the problem where the boundary condition $L_1(y)$ is replaced by

$$L_1'(y) = \begin{cases} \lambda y(0) - kD^\alpha y(0) = 0, & k \in \mathbb{R}, \\ D^\alpha y(0) = 0, & k = \infty. \end{cases}$$

Let $\{\mu_n\}_{n\geq 0}$ be the eigenvalues of the problem L_α^k.

Corollary 5.4. *Suppose $k \neq h$. If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for $n = 0,1,2,\ldots$ and $w(x) = \tilde{w}(x)$, then $L_\alpha = \tilde{L}_\alpha$.*

Proof. We begin with the case $k = \infty$. The numbers λ_n, μ_n are the poles and zeros of $m(\lambda)$ and hence determine it uniquely up to a constant by Krein's theorem [9, Thm. 27.2.1]. This unknown constant can be determined from (5.2). The case $k \neq h$ follows in the same manner using $m(\lambda) + (k-h)^{-1}$. □

Finally, we are also able to extend Hald's theorem to the case of CF Sturm-Liouville problems with transmission conditions.

Theorem 5.5. *If $\lambda_n = \tilde{\lambda}_n$, $w(x) = \tilde{w}(x)$, $L_1 = \tilde{L}_1$, $q(x) = \tilde{q}(x)$ for a.e. $x < \frac{\pi}{2}$ and $U = \tilde{U}$, $V = \tilde{V}$ for the case $d < \frac{\pi}{2}$, then $L_\alpha = \tilde{L}_\alpha$.*

Proof. It follows from Hadamard's factorization theorem $W_\alpha(\tilde{u},\tilde{v}) = K\,W_\alpha(u,v)$ for some constant K which can be determined from Lemma 4.3 and the asymptotic as $\lambda \to \infty$:

$$K = \begin{cases} 1, & d < \pi/2; \\ \frac{\tilde{b}_1}{b_1}, & d \geq \pi/2. \end{cases}$$

Furthermore, our assumptions imply

$$\tilde{v}(x,\lambda) = K\,v(x,\lambda) + F(\lambda)u(x,\lambda), \quad x < \frac{\pi}{2},$$

for some entire function $F(\lambda)$ of growth order at most $\frac{1}{2}$. Solving for F and taking the limit $x \uparrow \frac{\pi}{2}$ we obtain

$$\begin{aligned} F(\lambda) &= \frac{\tilde{v}(\frac{\pi}{2}-,\lambda) - Kv(\frac{\pi}{2}-,\lambda)}{u(\frac{\pi}{2}-,\lambda)} \\ &= K\frac{v(\frac{\pi}{2}-,\lambda)}{u(\frac{\pi}{2}-,\lambda)}\left(\frac{\tilde{v}(\frac{\pi}{2}-,\lambda)}{Kv(\frac{\pi}{2}-,\lambda)} - 1\right). \end{aligned}$$

Now, using the asymptotic form of the expression in parenthesis, we see that this vanishes for every ray different from the positive real axis. Furthermore, applying the asymptotic (4.1) for u and the analogous result for v, $\tilde{v}$ in the first part of $F(\lambda)$ this is bounded in every ray different from the positive real axis. Using Phragmén–Lindelöf theorem we conclude that this function must be identical to zero. From $\tilde{S}(x,\lambda) = S(x,\lambda)$, $\tilde{u}(x,\lambda) = u(x,\lambda)$ and $\tilde{v}(x,\lambda) = Kv(x,\lambda)$ for $x < \frac{\pi}{2}$, applying Eq. (5.3), we conclude that $M(\lambda) = \tilde{M}(\lambda)$ and the result obtain using Theorem 5.2. ☐

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