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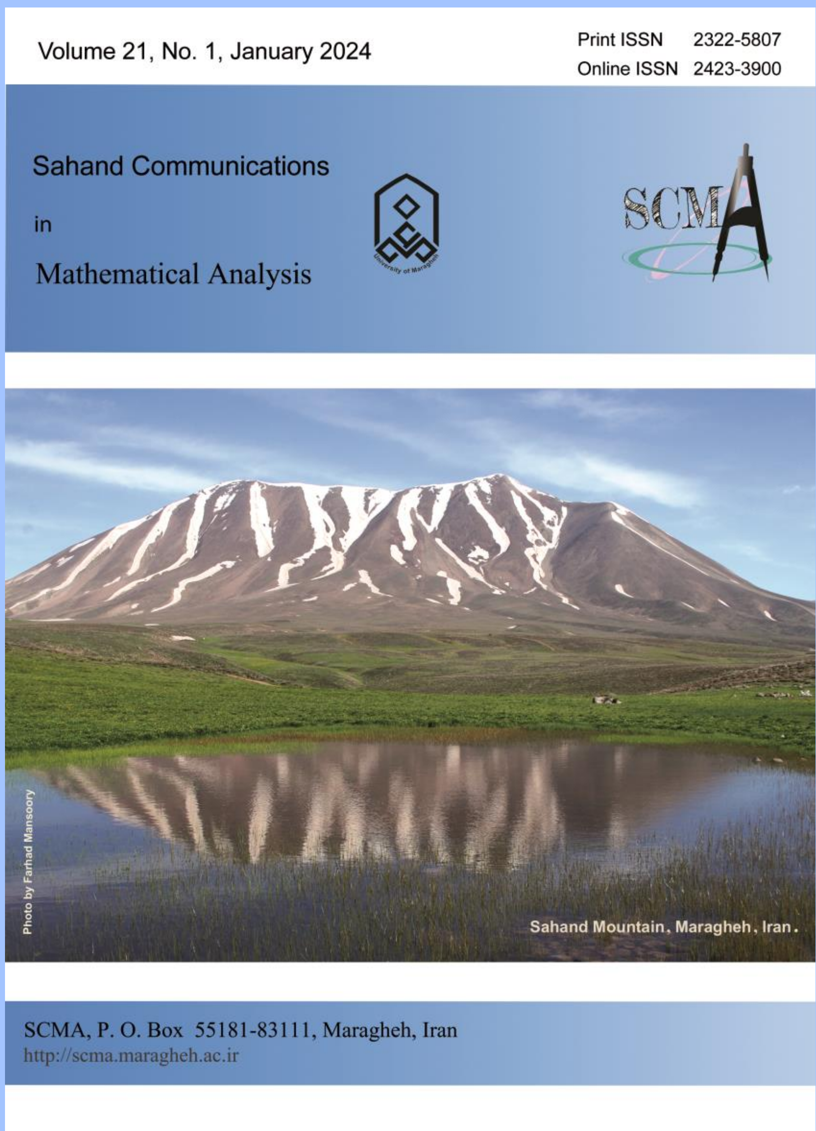
Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 1
Pages: 131-149

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.1989720.1256



Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 1 (2024), 131-149
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.1989720.1256

f_δ-Open Sets in Fine Topological Spaces

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Abstract. In this paper, the concept of δ-cluster point on a set which belongs to the collection of fine open sets generated by the topology τ on X has been introduced. Using this definition, the idea of f_δ-open sets is initiated and certain properties of these sets have also been studied. On the basis of separation axioms defined over fine topological space, certain types of f_δ-separation axioms on fine space have also been defined, along with some illustrative examples.

1. Introduction

The concept of δ-open sets in topological space has been introduced by Velicko [17] in 1968. He has also studied some of the significant characterizations of these open sets. The collection of δ-open sets is a restricted class of open sets used in defining the generalized concept of continuity by considering it in the range of continuous functions. Powar and Rajak initiated the notion of fine topological space [15], which is a special case of generalized or supra topological space [5]. The collection of fine open sets is a wider class of subsets of X containing semi-open sets, pre-open sets, α-open sets, β-open sets etc. Ameen, Asaad and Muhammed [4] considered properties of δ-preopen, δ-semiopen, a-open and e^*-open sets in topological spaces. Al-Omari and Noiri ([2], [3]) investigated some operators in minimal spaces.

In the present paper, the authors have extended the idea of δ-open sets fine topological space and defined a new class of sets called f_δ-open sets. In order to study the concepts of certain separation axioms in the

2020 *Mathematics Subject Classification.* 54A05, 54A10, 54C05, 54C10, 54D10.
Key words and phrases. Fine open sets, f_δ-open sets, f_δ-closed sets, f_δ-g-closed set, f_δ-separation axioms.
Received: 15 February 2023, Accepted: 03 May 2023.
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context of f_δ-open sets, we have also defined separation axioms in fine topological space. Some notable characterizations of these separation axioms have been presented in the form of some exciting theorems.

2. Preliminaries

In this section, some basic definitions and results have been recalled. Throughout this paper, the space (X, τ) (or simply X) represents topological space and $int(U)$ and $cl(U)$ denote the interior of a set U and closure of a set U, respectively (see [11]) for a subset U of a topological space (X, τ).

Definition 2.1 ([17])**.** A point x of topological space X is said to be the δ-cluster point of a subset V of X if $int(cl(U)) \cap V \neq \phi$ for each open set U of X containing x.

Definition 2.2 ([17])**.** The collection of all δ-cluster points of set V is called the δ-closure of V, which is denoted by $cl_\delta(V)$. A subset V of a topological space (X, τ) is said to be δ-closed if $V = cl_\delta(V)$. The family of all δ-closed subsets of a space (X, τ) is denoted by F_δ. The complement of a δ-closed set is said to be δ-open. The family of all δ-open subsets of a space (X, τ) is denoted by τ_δ. The δ-interior of a subset V of a topological space (X, τ) is the largest δ-open set contained in V and it is denoted by $\mathrm{int}_\delta(V)$.

Definition 2.3 ([6])**.** A subset V of a topological space (X, τ) is said to be an a-open set if $V \subseteq \mathrm{int}(cl(\mathrm{int}_\delta(V)))$. The complement of a-open sets is called a-closed.

Remark 2.4 ([17])**.** Let (X, τ) be the topological space, then the following hold.

(i) Every δ-open set is an open set but not conversely.
(ii) Every δ-open set is an a-open set but not conversely.

Definition 2.5 ([15])**.** Let (X, τ) be the topological space, define $\tau(V_\alpha) = \tau_\alpha$(say)$= \{G_\alpha(\neq X) : G_\alpha \cap V_\alpha \neq \phi$, for $V_\alpha \in \tau$ and $V_\alpha \neq \phi, X$ for some $\alpha \in J$, where J is the index set$\}$. Next, consider $\tau_f = \{\phi, X\} \cup_\alpha \{\tau_\alpha\}$. The collection τ_f of subsets of X is called the fine collection of subsets of X and (X, τ, τ_f) is said to be the fine topological space X or simply a fine space generated by the topology τ on X. The members of τ_f are called fine open sets of (X, τ, τ_f). The fine interior of a subset V of a fine topological space (X, τ, τ_f) is the largest fine open set contained in V denoted by $f_{\mathrm{int}}(V)$.

Definition 2.6 ([13])**.** The complement of a fine open set is called fine closed set in fine topological space (X, τ, τ_f). The family of all fine closed subsets of a fine space (X, τ, τ_f) is denoted by F_f. The fine closure of V is the smallest fine closed set containing the set V and it is denoted by $f_{cl}(V)$.

Remark 2.7 ([15])**.** The family of closed sets in (X, τ) is coarser than that of fine closed sets in (X, τ, τ_f). It may also be noted that the collection τ_f is a special case of generalized topology on X.

Definition 2.8 ([7])**.** A subset V of (X, τ) is said to be δ-g-closed if $cl_\delta(V) \subseteq U$ whenever $V \subseteq U$ and U is δ-open in (X, τ).

Definition 2.9 ([7])**.** A topological space (X, τ) is said to be

(i) δ-T_0 if for each $x, y \in X$ such that $x \neq y$, there exists a δ-open set U of X which either contains x or y, but not both.

(ii) δ-T_1 if for each pair of distinct points $x, y \in X$, there exist δ-open sets U and V of X such that $x \in U$ and $x \notin V$ or $y \in V$ and $y \notin U$.

(iii) δ-T_2 if for each $x, y \in X$ such that $x \neq y$ there exist δ-open sets U and V of X such that $U \cap V = \phi$, for $x \in U$ and $y \in V$.

(iv) δ-$T_{\frac{1}{2}}$ if every δ-g-closed set in X is δ-closed.

(v) δ-regular space if for each δ-closed set F and every point $p \notin F$ there exist δ-open sets G and H such that $F \subseteq G$, $p \in H$ and $G \cap H = \phi$

3. f_δ-Open Sets

Before studying f_δ-open sets, it is interesting to note that the collection of δ-closed sets defined in [17] satisfies all the properties which were fulfilled by the classical closed sets in a topological space ([[11], Theorem 17.1]).

Theorem 3.1. *Let (X, τ) be a topological space. Then the following hold:*

(i) *ϕ and X are δ-closed.*

(ii) *Arbitrary intersection of δ-closed sets is δ-closed.*

(iii) *Finite union of δ-closed sets is δ-closed.*

Proof. (i) Since, $cl_\delta(X) = X$ and $cl_\delta(\phi) = \phi$. So, ϕ and X are δ-closed.

(ii) Let $\{A_\alpha\}_{\alpha \in J}$ be an arbitrary collection of δ-closed sets then $cl_\delta(A_\alpha) = A_\alpha$. We now show that $\bigcap_{\alpha \in J} A_\alpha$ is δ-closed. So, it is sufficient to prove that $cl_\delta\left(\bigcap_{\alpha \in J} A_\alpha\right) = \bigcap_{\alpha \in J} A_\alpha$. Since, $\bigcap_{\alpha \in J} A_\alpha \subseteq A_\alpha$ for each $\alpha \in J \Rightarrow cl_\delta\left(\bigcap_{\alpha \in J} A_\alpha\right) \subseteq cl_\delta(A_\alpha)$ for all $\alpha \in J$

(see [17]). Hence, $cl_\delta\left(\bigcap_{\alpha\in J} A_\alpha\right) \subseteq \bigcap_{\alpha\in J}(cl_\delta(A_\alpha)) = \bigcap_{\alpha\in J} A_\alpha$. Let $x \in \bigcap_{\alpha\in J}(cl_\delta(A_\alpha)) \Rightarrow x \in cl_\delta(A_\alpha) = A_\alpha$ for each $\alpha \in J$. Hence, $x \in \bigcap_{\alpha\in J} A_\alpha$ and, $x \in cl_\delta\left(\bigcap_{\alpha\in J} A_\alpha\right)$. Therefore, $\bigcap_{\alpha\in J}(cl_\delta(A_\alpha)) \subseteq cl_\delta\left(\bigcap_{\alpha\in J}(A_\alpha)\right)$. Thus, $cl_\delta\left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in J} A_\alpha$.

(iii) To prove that, the finite union of δ-closed sets is δ-closed, it is sufficient to show that $A \cup B$ is δ-closed whenever A and B are δ-closed. Let $x \notin A \cup B$, which implies $x \notin A$ and $x \notin B$. Let $x \notin A$ i.e. x is not a δ-cluster point of A. Then, $int(cl(U)) \cap A = \phi$ for some open set U containing x. Similarly, let $x \notin B$, i.e. x is not a δ-cluster point of B. Then, $int(cl(V)) \cap B = \phi$ for some open set V containing x. Since $x \in U$, $x \in V$, then $x \in U \cap V = W$ (say) and W is an open set.

Consider,

$$\text{int}(cl(W)) \cap (A \cup B) = (\text{int}(cl(W)) \cap A) \cup (\text{int}(cl(W)) \cap B). \tag{3.1}$$

Since, $W \subseteq U$ and $W \subseteq V$ so,

$$\text{int}(cl(W)) \subseteq \text{int}(cl(U)), \qquad \text{int}(cl(W)) \subseteq \text{int}(cl(V)).$$

Therefore, $\text{int}(cl(W)) \cap A \subseteq \text{int}(cl(U)) \cap A = \phi \Rightarrow \text{int}(cl(W)) \cap A = \phi$. Similarly, $int(cl(W)) \cap B = \phi$.

Therefore by (3.1), $\text{int}(cl(W)) \cap (A \cup B) = \phi \cup \phi = \phi$ for some open set W containing x. So, it is clear that x is not δ-cluster point of $A \cup B$. Thus, any point out side of $A \cup B$ is not a δ-cluster point of $A \cup B$, means $A \cup B$ contains all its δ-cluster points. Hence, by Definition 2.2, $cl_\delta(A \cup B) = A \cup B$. Thus, $A \cup B$ is δ-closed. □

The following example assures that infinite union of δ-closed sets need not be δ-closed set.

Example 3.2. Let $[a, b]$ be a closed subset of R in its standard topology. Let $x \in R$ such that $x \notin [a, b]$.

Case 1: If $x > b$, then there exists an open set $(b, x + \epsilon)$ containing x such that $\text{int}(cl((b, x + \epsilon))) = (b, x + \epsilon)$ and $(b, x + \epsilon) \cap [a, b] = \phi$. Therefore, x is not δ-cluster point of $[a, b]$.

Case 2: If $x < a$, then there exists an open set $(x - \epsilon, a)$ containing x such that $\text{int}(cl((x - \epsilon, a))) = (x - \epsilon, a)$ and $(x - \epsilon, a) \cap [a, b] = \phi$. Therefore, x is not δ-cluster point of $[a, b]$.

Case 3: If $x \in [a, b]$, then for each open set U containing x there exists an open set $(x - \epsilon, x + \epsilon) \subseteq U$ i.e. $(x - \epsilon, x + \epsilon) \subseteq \text{int}(cl(U))$ and

$\text{int}(cl(x-\epsilon, x+\epsilon)) \cap [a,b] = (x-\epsilon, x+\epsilon) \cap [a,b] \neq \phi \Rightarrow \text{int}(cl(U)) \cap [a,b] \neq \phi$. Therefore, x is δ-cluster point of $[a,b]$.

Hence, any point which is lying out side of $[a,b]$ is not a δ-cluster point of $[a,b]$ i.e. $[a,b]$ contains all its δ-cluster points. So, by Definition 2.2, $cl_\delta[a,b] = [a,b]$. Thus, $[a,b]$ is δ-closed.

Now, since $[a,b]$ is δ-closed set, $\left[\frac{1}{n}, 1\right]$ will be δ-closed set for all $n \in N$. But, $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0,1]$, which is not δ-closed set as it can be easily checked that $cl_\delta(0,1] = [0,1]$.

Remark 3.3. In view of the result in [11, Theorem 17.1] and Theorem 3.1 of this paper, it has been concluded that the collection of δ-open sets forms a topology on X.

We shall now define f_δ-cluster points, and generate a new class of sets called f_δ-open sets. The relationships and characterizations of f_δ-open sets are also discussed in this section.

Definition 3.4. Let (X, τ, τ_f) be a fine space and V be the subset of X, then a point x of X is called the f_δ-cluster point of V if $f_{\text{int}}(f_{cl}(U)) \cap V \neq \phi$, for every fine-open set U of a fine space X containing x. The collection of all f_δ-cluster points of set V is called the f_δ-closure of V and is denoted by $f_{cl_\delta}(V)$. A subset V of a fine space X considered f_δ-closed set if $V = f_{cl_\delta}(V)$. The family of all f_δ-closed subsets of a fine space (X, τ, τ_f) is denoted by F_{f_δ}. The complement of the f_δ-closed set is called f_δ-open set. The family of all f_δ-open subsets of a fine space (X, τ, τ_f) is denoted by τ_{f_δ}.

Definition 3.5. Let (X, τ, τ_f) be a fine space and $V \subseteq X$. A point $p \in V$ is said to be the f_δ-interior point of V if there exists a f_δ-open set U such that $p \in U \subseteq V$. The collection of all f_δ-interior points of V is called the f_δ-interior of V, which is denoted by $f_{\text{int}_\delta}(V)$. The f_δ-interior of V is the union of all f_δ-open sets of X contained in V, i.e. the most extensive f_δ-open set contained in V.

Theorem 3.6. *Let (X, τ, τ_f) be a fine topological space and $V(\neq \phi) \subseteq X$. If x is a f_δ-cluster point of V, then it is a δ-cluster point of V.*

Proof. Let (X, τ) be a topological space and $\{G_\alpha\}_{\alpha \in J}$ be a collection of members of τ and $\{U_\alpha\}_{\alpha \in J}$ be a collection of induced fine open sets. Given $V(\neq \phi) \subseteq X$, suppose $x \in X$ is a f_δ-cluster point of V i.e.,

$$f_{int}(f_{cl}(U_\alpha)) \cap V \neq \phi \tag{3.2}$$

for each fine open set U_α containing x. We have to show that x is a δ-cluster point of the set V i.e., to show that

$$int(cl(G_\alpha)) \cap V \neq \phi \tag{3.3}$$

for each open set G_α containing x. Since every open set is a fine open set, (3.2) holds for every open set G_α containing x. Hence,

$$f_{\text{int}}(f_{cl}(G_\alpha)) \cap V \neq \phi \tag{3.4}$$

In order to establish (3.3), the following cases have been considered:
Case 1: Let $cl(G_\alpha) = X$.
Then clearly $\text{int}(cl(G_\alpha)) = X$. So, (3.3) holds for all open sets G_α containing x.
Case 2: Let $cl(G_\alpha) = F(\neq X)$.
Then, $\exists$ some $\beta \in J$ such that $G_\beta \subseteq (G_\alpha)^C$ (complement of G_α) then clearly $G_\alpha \subseteq (G_\beta)^C$, where $(G_\beta)^C$ is the smallest closed set containing G_α i.e., $cl(G_\alpha) = (G_\beta)^C = F$. Then G_β is an open set such that $G_\beta \cap (G_\alpha)^C \neq \phi$. This implies that $(G_\alpha)^C$ is a fine open set $\Rightarrow G_\alpha$ is a fine closed set.

Hence, G_α is an open set containing $x \Rightarrow$ by definition of fine open set, G_α is fine open set and we have investigated that it is a fine closed set. Thus, $f_{\text{int}}(f_{cl}(G_\alpha)) = G_\alpha$ for each open set G_α containing x. $\Rightarrow$ $f_{\text{int}}(f_{cl}(G_\alpha)) = G_\alpha \subseteq \text{int}(cl(G_\alpha))$. Hence, (3.3) holds. Thus, x is a δ-cluster point of V. □

Corollary 3.7. *Let (X, τ, τ_f) be a fine topological space. Then each f_δ-cluster point of a non-empty subset V of X is a cluster point of V.*

The converse of Corollary 3.7 may not be true as shown by the following example.

Example 3.8. Let $X = \{a, b, c, d\}$, with topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. The following collections of subsets of X have been computed according to their corresponding definitions τ_f and F_f:

$$\begin{aligned}\tau_f = \{&\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\},\\ &\{b, c, d\}\{a, b, d\}, \{a, c, d\}, X\}.\\ F_f = \{&\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},\\ &\{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}.\end{aligned}$$

Let $V = \{a, b, d\} \subseteq X$, then $x = c \in X$ is a cluster point of V but it may be easily verified that c is not a f_δ-cluster point of V.

Theorem 3.9. *Let (X, τ, τ_f) be a fine topological space. Then every δ-closed set is f_δ-closed.*

Proof. Let V be a δ-closed set, then we have $V = cl_\delta(V)$. Since by Theorem 3.6, every f_δ-cluster point is δ-cluster point, then $f_{cl_\delta}(V) \subseteq cl_\delta(V) = V \Rightarrow f_{cl_\delta}(V) \subseteq V$. But $V \subseteq f_{cl_\delta}(V)$, hence $V = f_{cl_\delta}(V)$. Thus, V is f_δ-closed set. □

Example 3.10. Let $X = \{a, b, c, d\}$, with topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. The following collections of subsets of X have been computed according to their corresponding definitions:

$$\tau_\delta = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}.$$

$$F_\delta = \{\phi, \{b, c, d\}, \{a, c, d\}, \{c, d\}, X\}.$$

$$\tau_{f_\delta} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

$$F_{f_\delta} = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

It is clear from the above collection of sets that every δ-closed set is f_δ-closed set whereas the converse does not hold always e.g. $V = \{a, b\}$ is f_δ-closed set but it is not δ-closed set.

Remark 3.11. In view of Definitions 2.2, 2.5 and 3.4, the following implications present the explicit relationships amongst various open sets.

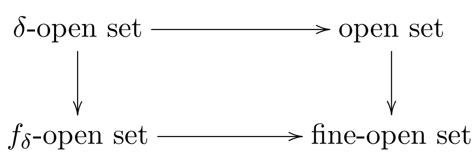

Diagram 2.1. Relationships amongst various open sets

Lemma 3.12. *Let (X, τ, τ_f) be a fine topological space and V be an f_δ-open set then $V = f_{\mathrm{int}_\delta}(V)$.*

Proof. Let V be an f_δ-open set, then by Definition 3.5, f_δ-interior of V is the largest f_δ-open set contained in V. Then, $f_{\mathrm{int}_\delta}(V) \subseteq V$. Now, let $p \in V$ and V is an f_δ-open set then p is f_δ-interior point of V, so by Definition 3.5, $p \in f_{\mathrm{int}_\delta}(V) \Rightarrow V \subseteq f_{\mathrm{int}_\delta}(V)$. Therefore, $V = f_{\mathrm{int}_\delta}(V)$. □

Lemma 3.13. *Let U and V be two subsets of a fine topological space (X, τ, τ_f). If $U \subseteq V$, then $f_{\mathrm{int}_\delta}(U) \subseteq f_{\mathrm{int}_\delta}(V)$.*

Proof. Let $p \in f_{\mathrm{int}_\delta}(U)$, then by Definition 3.5, p is an f_δ-interior point of U, so there exists an f_δ-open set W such that $p \in W \subseteq U$ but $U \subseteq V$ (given). Therefore, $p \in W \subseteq V \Rightarrow p \in f_{\mathrm{int}_\delta}(V)$. Thus, $f_{\mathrm{int}_\delta}(U) \subseteq f_{\mathrm{int}_\delta}(V)$. □

Lemma 3.14. *Let (X, τ, τ_f) be a fine topological space and let $\{V_\alpha\}_{\alpha \in J}$ be the family of subsets of X, then $\bigcup_{\alpha \in J} (f_{\mathrm{int}_\delta}(V_\alpha)) \subseteq f_{\mathrm{int}_\delta}\left(\bigcup_{\alpha \in J} (V_\alpha)\right)$.*

The next lemma is a direct consequence of Lemma 3.13.

Remark 3.15. Let (X, τ, τ_f) be a fine topological space and let $\{V_\alpha\}_{\alpha \in J}$ be the family of subsets of X, then $f_{\text{int}_\delta}\left(\bigcup_{\alpha \in J}(V_\alpha)\right) \nsubseteq \bigcup_{\alpha \in J}(f_{\text{int}_\delta}(V_\alpha))$. In order to support this assertion, please refer Example 3.16.

Example 3.16. Referring Example 3.10 of this paper, we recall the collection τ_{f_δ} and consider $E = \{a, b, c\}$ and $F = \{d\} \in \tau_{f_\delta}$, then $E \cup F = X$. It may be verified that $f_{\text{int}_\delta}(E) = E$, $f_{\text{int}_\delta}(F) = \phi$ and $f_{\text{int}_\delta}(E \cup F) = X$. Thus $f_{\text{int}_\delta}(E \cup F) \nsubseteq f_{\text{int}_\delta}(E) \cup f_{\text{int}_\delta}(F)$.

Theorem 3.17. *Let (X, τ, τ_f) be a fine topological space. Then arbitrary union of f_δ-open sets is f_δ-open set but, the finite intersections of f_δ-open sets need not be f_δ-open.*

Proof. Let $(V_\alpha)_{\alpha \in J}$ be the class of f_δ-open sets. We show that $\bigcup_{\alpha \in J}(V_\alpha)$ is f_δ-open set. By Lemma 3.12, it is sufficient to prove that $\bigcup_{\alpha \in J}(V_\alpha) = f_{\text{int}_\delta}\left(\bigcup_{\alpha \in J}(V_\alpha)\right)$. By Definition 3.5, $f_{\text{int}_\delta}(\bigcup_{\alpha \in J}(V_\alpha)) \subseteq \bigcup_{\alpha \in J}(V_\alpha)$. Now, we shall show that $\bigcup_{\alpha \in J}(V_\alpha) \subseteq f_{\text{int}_\delta}\left(\bigcup_{\alpha \in J}(V_\alpha)\right)$. Let $p \in \bigcup_{\alpha \in J}(V_\alpha)$ so, there exists some $\beta \in J$ such that $p \in V_\beta$. Since, V_β is f_δ-open set, then by Lemma 3.12, $V_\beta = f_{\text{int}_\delta}(V_\beta) \Rightarrow p \in f_{\text{int}_\delta}(V_\beta)_{\beta \in J}$. Hence, $p \in \bigcup_{\beta \in J}(f_{\text{int}_\delta}(V_\beta))$ then, by Lemma 3.14, $p \in f_{\text{int}_\delta}\left(\bigcup_{\beta \in J}(V_\beta)\right)$. Thus, $\bigcup_{\alpha \in J}(V_\alpha) \subseteq f_{\text{int}_\delta}\left(\bigcup_{\alpha \in J}(V_\alpha)\right)$ and finally $\bigcup_{\alpha \in J}(V_\alpha) = f_{\text{int}_\delta}\left(\bigcup_{\alpha \in J}(V_\alpha)\right)$. $\square$

In support of the later part of Theorem 3.17, Example 3.18 may be referred.

Example 3.18. Referring Example 3.10 of this paper, recall the collection τ_{f_δ} and consider $A = \{a, d\}$ and $B = \{b, d\} \in \tau_{f_\delta}$. It is clear that $A \cap B = \{d\} \notin \tau_{f_\delta}$.

Lemma 3.19. *Let U be a subset of a fine topological space (X, τ, τ_f) then $U \subseteq f_{cl_\delta}(U)$.*

Proof. Let $p \in U$ then there exists some fine-open set $V \subseteq X$ such that $p \in V$. We now consider the following two cases:
Case 1 If $V = X$ then the proof follows directly.
Case 2 If $V \neq X$ (i.e. V is a proper subset of X) then $p \in V \subseteq f_{cl}(V) \Rightarrow p \in f_{cl}(V)$. Since, $V \subseteq f_{cl}(V) \Rightarrow f_{\text{int}}(V) \subseteq f_{\text{int}}(f_{cl}(V))$ and since V is fine-open set, then $f_{\text{int}}(V) = V$. Hence, $V \subseteq f_{\text{int}}(f_{cl}(V)) \Rightarrow$

$f_{\text{int}}(f_{cl}(V)) \cap U \neq \phi$. Thus, by Definition 3.4, p is f_δ-cluster point of $U \Rightarrow p \in f_{cl_\delta}(U)$. Hence, $U \subseteq f_{cl_\delta}(U)$. □

Lemma 3.20. *Let U be a subset of fine topological space (X, τ, τ_f). Then $f_{cl_\delta}(U)$ is the smallest f_δ-closed set containing U.*

Proof. Suppose $f_{cl_\delta}(U)$ is not the most miniature f_δ-closed set containing U. Then, there exists some f_δ-closed set V such that $U \subseteq V \subseteq f_{cl_\delta}(U)$. Implies, V consists of all the elements of U and some f_δ-cluster points of U (if it contains all f_δ-cluster points, then $f_{cl_\delta}(U) \subseteq V$, which is a contradiction to our assumption). So, there exists an f_δ-cluster point p of U not in V then, by Definition 3.4, $f_{\text{int}}(f_{cl}(W)) \cap U \neq \phi$, for every fine-open set W of fine space X containing p.

But, $U \subseteq V$ then, $f_{\text{int}}(f_{cl}(W)) \cap V \neq \phi$, for every fine-open set W of fine space X containing p. This implies that, p is also f_δ-cluster point of $V \Rightarrow p \in f_{cl_\delta}(V)$. Since, V is f_δ-closed set then, $V = f_{cl_\delta}(V) \Rightarrow p \in V$, which is a contradiction. Therefore, our assumption that there exists f_δ-closed set V satisfying $U \subseteq V \subseteq f_{cl_\delta}(U)$ is not true and hence $f_{cl_\delta}(U)$ is the smallest f_δ-closed set containing U. □

Theorem 3.21. *Let U and V be two subsets of a fine topological space (X, τ, τ_f). Then following hold:*

(i) $f_{cl_\delta}(X) = X$ *and* $f_{cl_\delta}(\phi) = \phi$.
(ii) *If* $U \subseteq V$, *then* $f_{cl_\delta}(U) \subseteq f_{cl_\delta}(V)$.
(iii) $f_{cl_\delta}(U \cap V) \subseteq f_{cl_\delta}(U) \cap f_{cl_\delta}(V)$.
(iv) $f_{cl_\delta}(U) \cup f_{cl_\delta}(V) \subseteq f_{cl_\delta}(U \cup V)$.
(v) $f_{cl_\delta}(f_{cl_\delta}(U)) = f_{cl_\delta}(U)$.

Proof. (i) Since, X and ϕ both are f_δ-closed sets, then (1) holds by Definition 3.4.

(ii) $V \subseteq f_{cl_\delta}(V)$ and given that $U \subseteq V$, we have $U \subseteq f_{cl_\delta}(V)$. But, $f_{cl_\delta}(V)$ is f_δ-closed set. Therefor, $f_{cl_\delta}(V)$ is a f_δ-closed set containing U. Since, by Lemma 3.20, $f_{cl_\delta}(U)$ is the smallest f_δ-closed set containing U, hence $f_{cl_\delta}(U) \subseteq f_{cl_\delta}(V)$.

(iii) Applying part 2 of this theorem consider the following:

$$U \cap V \subseteq U \quad \Rightarrow \quad f_{cl_\delta}(U \cap V) \subseteq f_{cl_\delta}(U) \tag{3.5}$$

$$U \cap V \subseteq V \quad \Rightarrow \quad f_{cl_\delta}(U \cap V) \subseteq f_{cl_\delta}(V) \tag{3.6}$$

then $f_{cl_\delta}(U \cap V) \subseteq f_{cl_\delta}(U) \cap f_{cl_\delta}(V)$. (Using (3.5) and (3.6))

(iv) Again applying part 2 of this theorem, we get

$$U \subseteq U \cup V \quad \Rightarrow \quad f_{cl_\delta}(U) \subseteq f_{cl_\delta}(U \cup V). \tag{3.7}$$

$$V \subseteq U \cup V \quad \Rightarrow \quad f_{cl_\delta}(V) \subseteq f_{cl_\delta}(U \cup V). \tag{3.8}$$

Then $f_{cl_\delta}(U) \cup f_{cl_\delta}(V) \subseteq f_{cl_\delta}(U \cup V)$. (Referring (3.7) and (3.8))

(v) Since, $U \subseteq f_{cl_\delta}(U) \Rightarrow f_{cl_\delta}(U) \subseteq f_{cl_\delta}(f_{cl_\delta}(U))$. Now, let $x \in f_{cl_\delta}(f_{cl_\delta}(U))$. Then x is a f_δ-cluster point of $f_{cl_\delta}(U)$. But $f_{cl_\delta}(U)$ is a set containing all its f_δ-cluster points. Implies $x \in f_{cl_\delta}(U)$. Therefore, $f_{cl_\delta}(f_{cl_\delta}(U)) \subseteq f_{cl_\delta}(U)$. Hence,

$$f_{cl_\delta}(f_{cl_\delta}(U)) = f_{cl_\delta}(U).$$ □

The following example assures that the equality in parts 3 and 4 of Theorem 3.21, may not hold in general.

Example 3.22. Referring Example 3.10 of this paper, we consider the collection τ_{f_δ}.

(i) Consider $U = \{a, b, c\}$ and $V = \{d\}$ be the subsets of X, then $f_{cl_\delta}(U) = X$ and $f_{cl_\delta}(V) = V$, so, $f_{cl_\delta}(U \cap V) = \phi$, $f_{cl_\delta}(U) \cap f_{cl_\delta}(V) = V$. Thus, $f_{cl_\delta}(U) \cap f_{cl_\delta}(V) \not\subseteq f_{cl_\delta}(U \cap V)$.

(ii) Again consider $U = \{a, b\}$, $V = \{c\}$, then $f_{cl_\delta}(U) = U$, $f_{cl_\delta}(V) = V$ and $f_{cl_\delta}(U \cup V) = X$. Thus, $f_{cl_\delta}(U \cup V) \not\subset f_{cl_\delta}(U) \cup f_{cl_\delta}(V)$.

Theorem 3.23. *Let U and V be two subsets of a fine topological space (X, τ, τ_f). Then following hold:*

(i) $f_{\text{int}_\delta}(X) = X$ *and* $f_{\text{int}_\delta}(\phi) = \phi$.

(ii) $f_{\text{int}_\delta}(U \cap V) \subseteq f_{\text{int}_\delta}(U) \cap f_{\text{int}_\delta}(V)$.

(iii) $f_{\text{int}_\delta}(U) \cup f_{\text{int}_\delta}(V) \subseteq f_{\text{int}_\delta}(U \cup V)$. *(It is a special case of Lemma 3.14)*

(iv) $f_{\text{int}_\delta}(f_{\text{int}_\delta}(U)) = f_{\text{int}_\delta}(U)$.

Proof. (i) Since, X and ϕ both are f_δ-open sets then, by Lemma 3.12, (1) holds.

(ii) In view of Lemma 3.13, we get the following

$$U \cap V \subseteq U \quad \Rightarrow \quad f_{\text{int}_\delta}(U \cap V) \subseteq f_{\text{int}_\delta}(U) \tag{3.9}$$

$$U \cap V \subseteq V \quad \Rightarrow \quad f_{\text{int}_\delta}(U \cap V) \subseteq f_{\text{int}_\delta}(V) \tag{3.10}$$

then $f_{\text{int}_\delta}(U \cap V) \subseteq f_{\text{int}_\delta}(U) \cap f_{\text{int}_\delta}(V)$. (by (3.9) and (3.10))

(iii) Following is again a consequence of Lemma 3.13:

$$U \subseteq U \cup V \quad \Rightarrow \quad f_{\text{int}_\delta}(U) \subseteq f_{\text{int}_\delta}(U \cup V). \tag{3.11}$$

$$V \subseteq U \cup V \quad \Rightarrow \quad f_{\text{int}_\delta}(V) \subseteq f_{\text{int}_\delta}(U \cup V). \tag{3.12}$$

Then $f_{\text{int}_\delta}(U) \cup f_{\text{int}_\delta}(V) \subseteq f_{\text{int}_\delta}(U \cup V)$. (by (3.11) and (3.12))

(iv) Let $W = f_{\text{int}_\delta}(U)$ hence, W is f_δ-open set, therefore by Lemma 3.12, $W = f_{\text{int}_\delta}(W)$. Thus, $f_{\text{int}_\delta}(f_{\text{int}_\delta}(U)) = f_{\text{int}_\delta}(U)$. □

The following example illustrates that the equality in parts 2 and 3 of Theorem 3.23 may not necessarily hold in general:

Example 3.24. Referring Example 3.10 of this paper, we consider the collection τ_{f_δ}.

(i) If $U = \{a, d\}$, $V = \{b, d\}$, then $f_{\text{int}_\delta}(U) = U$, $f_{\text{int}_\delta}(V) = V$ and $f_{\text{int}_\delta}(U) \cap f_{\text{int}_\delta}(V) = \{d\}$, $U \cap V = \{d\}$. But it may be easily verified that $f_{\text{int}_\delta}(U \cap V) = \phi$. Thus, $f_{\text{int}_\delta}(U) \cap f_{\text{int}_\delta}(V) \not\subset f_{\text{int}_\delta}(U \cap V)$.

(ii) Similarly, if we choose $E = \{a, b, c\}$, $F = \{d\}$ then it may be checked easily that $f_{\text{int}_\delta}(E \cup F) \not\subset f_{\text{int}_\delta}(E) \cup f_{\text{int}_\delta}(F)$.

Theorem 3.25. *Let (X, τ, τ_f) be a fine topological space and V be a subset of X. Then $x \in f_{cl_\delta}(V)$, if and only if $V \cap U \neq \phi$ for every f_δ-open set U of X containing x.*

Proof. Given $x \in f_{cl_\delta}(V)$ and we wish to show that $V \cap U \neq \phi$. Let if possible $V \cap U = \phi$ for some f_δ-open set U of X containing x then, $V \subseteq X \backslash U$ and $X \backslash U$ is f_δ-closed set in X. Then $f_{cl_\delta}(V) \subseteq X \backslash U$. So, $x \in X \backslash U$, which is a contradiction to the hypothesis. Hence, $V \cap U \neq \phi$ for every f_δ-open set U of X containing x.

Conversely, given $V \cap U \neq \phi$ for every f_δ-open set U of X containing x and our aim is to show that $x \in f_{cl_\delta}(V)$. Let if possible, $x \notin f_{cl_\delta}(V)$ and $f_{cl_\delta}(V)$ be f_δ-closed such that $V \subseteq f_{cl_\delta}(V)$. It is clear that, $X \backslash f_{cl_\delta}(V)$ is f_δ-open set containing x and, $X \backslash f_{cl_\delta}(V) \subseteq X \backslash V$. Therefore, $V \cap (X \backslash f_{cl_\delta}(V)) = \phi$ which is contradiction to the hypothesis. Thus, $x \in f_{cl_\delta}(V)$. □

Definition 3.26. A subset V of a fine topological space (X, τ, τ_f) is said to be f_δ-g-closed (f_δ-generalized closed) if $f_{cl_\delta}(V) \subseteq U$ whenever $V \subseteq U$ for f_δ-open set U of X.

In view of [13, Theorem 4.3], the following result is established.

Theorem 3.27. *Let (X, τ, τ_f) be a fine topological space and U be the f_δ-g-closed subset of X, then $f_{cl_\delta}(U) \backslash U$ does not contain any non-empty f_δ-closed set.*

Proof. Let V be a non-empty f_δ-closed subset of X such that $V \subseteq f_{cl_\delta}(U) \backslash U$, then $V \subseteq X \backslash U \Rightarrow U \subseteq X \backslash V$. Since $X \backslash V$ is f_δ-open set and U is f_δ-g-closed set then by Definition 3.26, $f_{cl_\delta}(U) \subseteq X \backslash V \Rightarrow V \subseteq X \backslash f_{cl_\delta}(U)$. We get $V \subseteq X \backslash f_{cl_\delta}(U) \cap f_{cl_\delta}(U) \backslash U \subseteq X \backslash f_{cl_\delta}(U) \cap f_{cl_\delta}(U) = \phi \Rightarrow V = \phi$ which is a contradiction. Thus, $V \nsubseteq f_{cl_\delta}(U) \backslash U$. □

4. Fine-Separation Axioms

In this section, certain separation axioms have been defined with respect to fine open sets.

Definition 4.1. A subset V of fine topological space (X, τ, τ_f) is said to be f-g-closed (fine generalized closed) if $f_{cl}(V) \subseteq U$ whenever $V \subseteq U$ for a fine open set U of X.

Definition 4.2. A fine topological space (X, τ, τ_f) is said to be

(i) T_0^f if for each pair of distinct points $x, y \in X$, there exists a fine open set U of X which either contains x or y, but not both.
(ii) T_1^f if for given a pair of distinct points $x, y \in X$, there exist two fine open sets U and V such that $x \in U$, $x \notin V$ and $y \in V$, $y \notin U$.
(iii) T_2^f (or f-Hausdorff) if for each pair of distinct points $x, y \in X$ there exist two fine open sets U and V of X containing x and y respectively such that $U \cap V = \phi$.
(iv) $T_{\frac{1}{2}}^f$ if every f-g-closed set in X is fine closed.

Theorem 4.3. *Let (X, τ, τ_f) be a fine topological space with τ non-trivial consisting of at least one subset of X having minimum two distinct elements of X. Then, (X, τ, τ_f) is T_2^f space.*

Proof. Let X be any non-empty set with the topology τ given by $\tau = \{\phi, X, \{x_\alpha, x_\beta\} : (x_\alpha \neq x_\beta) \text{ for } x_\alpha, x_\beta \in X\}$. The fine topology, τ_f consists of the sets of following types:

$$\tau_f = \begin{Bmatrix} \phi, X, & & \\ U_f & : & x_\alpha \in U_f \\ V_f & : & x_\beta \in V_f \\ W_f & : & x_\alpha, x_\beta \in W_f \end{Bmatrix} \tag{4.1}$$

It may be easily verified that τ_f is a fine topology on (X, τ). We now show that (X, τ, τ_f) is T_2^f space.

Case 1 If x_α, x_β are two distinct points of X, then there exist two fine open sets $\{x_\alpha\}$, $\{x_\beta\}$ such that $\{x_\alpha\} \in U_f$ and $\{x_\beta\} \in V_f$ of X and $\{x_\alpha\} \cap \{x_\beta\} = \phi$.

Case 2 If x, x_α, $x_\beta \in X$ such that $x_\alpha \neq x$, $x \neq x_\beta$ then there exist two fine open sets $\{x_\alpha\} \in U_f$ and $\{x_\beta, x\} \in V_f$ of X such that $\{x_\alpha\} \cap \{x_\beta, x\} = \phi$.

Case 3 If x_1, x_2 are two distinct points of X such that $x_1, x_2 \neq x_\alpha, x_\beta$ then there exist two fine open sets $\{x_\alpha, x_1\} \in U_f$ and $\{x_\beta, x_2\} \in V_f$ of X such that $\{x_\alpha, x_1\} \cap \{x_\beta, x_2\} = \phi$. Thus, (X, τ, τ_f) is T_2^f. □

Example 4.4. Let $X = \{a, b, c\}$, with topology $\tau = \{\phi, \{a\}, X\}$ and the fine topology τ_f generated by τ is given by $\tau_f = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$.

It may be easily checked that this space is T_0^f but not T_1^f. But as soon as one proper subset of X is added (viz. $\{a, b\}$ or $\{a, c\}$) to the topology it will turn out to be T_2^f spaces.

5. f_δ-Separation Axioms

In this section, certain separation axioms have been defined with respect to f_δ-open sets.

Definition 5.1. A fine topological space (X, τ, τ_f) is said to be

(i) f_δ-T_0 if for each pair of distinct points $x, y \in X$, there exists an f_δ-open set U of X which either contains x or y, but not both.
(ii) f_δ-T_1 if for given a pairs of distinct points $x, y \in X$, there exist two f_δ-open sets U and V such that $x \in U$, $x \notin V$ and $y \in V$, $y \notin U$.
(iii) f_δ-T_2 (or f_δ-Hausdorff) if for each pair of distinct points $x, y \in X$ there exist two f_δ-open sets U and V of X containing x and y respectively such that $U \cap V = \phi$.
(iv) f_δ-$T_{\frac{1}{2}}$ if every f_δ-g-closed set in X is f_δ-closed.

Theorem 5.2. *A fine topological space (X, τ, τ_f) is f_δ-T_0 if and only if $f_{cl_\delta}(\{p\}) \neq f_{cl_\delta}(\{q\})$, for every pair of distinct points $p, q \in X$.*

Proof. Let X be an f_δ-T_0 space and p, q be two distinct points of X. Then, there exists an f_δ-open set H containing p or q (say p, but not q). So $X \backslash H$ is a f_δ-closed set such that $p \notin X \backslash H$ but $q \in X \backslash H$. But $f_{cl_\delta}(\{q\})$ is the smallest f_δ-closed set containing q and $f_{cl_\delta}(\{q\}) \subseteq X \backslash H$, so $p \notin f_{cl_\delta}(\{q\})$. Therefore, $f_{cl_\delta}(\{p\}) \neq f_{cl_\delta}(\{q\})$.

Conversely, Let p and q be two distinct points of fine space X such that $f_{cl_\delta}(\{p\}) \neq f_{cl_\delta}(\{q\})$. We show that X is f_δ-T_0 space. Since, $f_{cl_\delta}(\{p\}) \neq f_{cl_\delta}(\{q\})$, implies $f_{cl_\delta}(\{p\}) \nsubseteq f_{cl_\delta}(\{q\})$ and $f_{cl_\delta}(\{q\}) \nsubseteq f_{cl_\delta}(\{p\})$. We now show that $p \notin f_{cl_\delta}(\{q\})$. Let if possible $p \in f_{cl_\delta}(\{q\}) \Rightarrow \{p\} \subseteq f_{cl_\delta}(\{q\})$ therefor, $f_{cl_\delta}(\{p\}) \subseteq f_{cl_\delta}(\{q\})$ (by Theorem 3.21 (3)), this is a contradiction to our hypothesis. So, $p \in X \backslash f_{cl_\delta}(\{q\})$ i.e. there exists f_δ-open set $X \backslash f_{cl_\delta}(\{q\})$ such that $p \in X \backslash f_{cl_\delta}(\{q\})$ but $q \notin X \backslash f_{cl_\delta}(\{q\})$. Thus, X is f_δ-T_0. □

Example 5.3. Let $X = \{a, b, c, d\}$, with topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. The following collections of subsets of X have been computed as per their corresponding definitions:

$$\tau_\delta = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$$
$$\tau_{f_\delta} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}.$$

Thus, the given fine topological space is f_δ-T_i and T_i^f for $i = 0, \frac{1}{2}, 1, 2$, but it is not δ-T_i, for $i = 0, \frac{1}{2}, 1, 2$.

Theorem 5.4. *Let X be a non-empty set with topology $\tau = \{\phi, X, \{x_0\} : x_0 \in X\}$. Then, the only sets which are f_δ-closed are ϕ and X.*

Proof. Let X be a non-empty set with topology $\tau = \{\phi, X, \{x_0\} : x_0 \in X\}$. Now applying Definitions 2.5 and 2.6, $\tau_f = \{\phi, X, \{V_0 : x_0 \in V_0\}\}$ and $F_f = \{\phi, X, \{U_0 : x_0 \notin U_0\}$.

We now show that x_0 is an f_δ-cluster point of U_0. Since, $f_{\text{int}}(f_{cl}(U)) = X$ for each fine open set U of X containing x_0, then $f_{\text{int}}(f_{cl}(U)) \cap U_0 \neq \phi$. It is clear that x_0 is f_δ-cluster point of U_0, then $x_0 \in f_{cl_\delta}(U_0)$. Therefore, $f_{cl_\delta}(U_0) \neq U_0$ and hence U_0 is not f_δ-closed set for all fine closed set U_0 of X. Thus, ϕ and X are the only f_δ-closed sets of (X, τ). □

Lemma 5.5. *Let (X, τ, τ_f) be the fine topological space with the topology τ consisting of at least one subset of X having minimum two distinct elements of X. Then singleton set $\{x\}$, $x \in X$ is fine closed.*

Proof. Let X be any non-empty set with the topology τ given by $\tau = \{\phi, X, \{x_\alpha, x_\beta\} : (x_\alpha \neq x_\beta)$ for $x_\alpha, x_\beta \in X\}$. The fine topology, τ_f consists of the sets of following types:

$$\tau_f = \left\{ \phi, X, \quad \begin{array}{rcl} U_f & : & x_\alpha \in U_f \\ V_f & : & x_\beta \in V_f \\ W_f & : & x_\alpha, x_\beta \in W_f \end{array} \right\} \tag{5.1}$$

It may be easily verified that τ_f is a fine topology on (X, τ). The corresponding collection of fine closed sets is given by

$$F_f = \left\{ \phi, X, \quad \begin{array}{rcl} U_F & : & x_\alpha \notin U_F \\ V_F & : & x_\beta \notin V_F \\ W_F & : & x_\alpha, x_\beta \notin W_F \end{array} \right\} \tag{5.2}$$

We show that $\{x\}$ is fine closed.
Case 1 If $x = x_\alpha$ then there exists $v_F = \{x_\alpha\} \in V_F$ which is fine closed.
Case 2 If $x = x_\beta$ then there exists $u_F = \{x_\beta\} \in U_F$ which is fine closed.
Case 3 If $x \neq x_\alpha, x_\beta$ then there exists $w_F = \{x\} \in W_F$ which is fine closed.
Hence, for $x \in X$, $\{x\}$ is fine closed. □

Theorem 5.6. *Let (X, τ, τ_f) be a fine topological space with τ non-trivial consisting of at least one subset of X having minimum two distinct elements of X and F_f be the collection of fine closed sets. If F_{f_δ} is the collection of f_δ-closed sets in (X, τ, τ_f), then $F_f \cong F_{f_\delta}$.*

Proof. Let X be any non-empty set with the topology τ given by $\tau = \{\phi, X, \{x_\alpha, x_\beta\} : (x_\alpha \neq x_\beta) \text{ for } x_\alpha, x_\beta \in X\}$. The fine topology, τ_f consists of the sets of following types:

$$\tau_f = \Bigg\{\phi, X, \quad U_f : x_\alpha \in U_f \quad V_f : x_\beta \in V_f \quad W_f : x_\alpha, x_\beta \in W_f\Bigg\} \tag{5.3}$$

It may be easily verified that τ_f is a fine topology on (X, τ). The corresponding collection of fine closed sets is given by

$$F_f = \Bigg\{\phi, X, \quad U_F : x_\alpha \notin U_F \quad V_F : x_\beta \notin V_F \quad W_F : x_\alpha, x_\beta \notin W_F\Bigg\} \tag{5.4}$$

We show that $G \in F_f$ implies G is f_δ-closed set. It is enough if we show that $x \notin G$ implies x is not a δ-cluster point of G.
Case 1 Let $x = x_\alpha \Rightarrow x_\alpha \notin G$. It may be seen that $\{x_\alpha\} \in U_f$ (by relation 5.3) and $\{x_\alpha\} \in V_F$ (by relation 5.4), then $f_{\text{int}}(f_{cl}\{x_\alpha\}) \cap G = \{x_\alpha\} \cap G = \phi$. Hence, x_α is not a f_δ-cluster point of G. Similarly, it may be shown that $x_\beta \notin G$ is also not a f_δ-cluster point of G.
Case 2 Let $x \neq x_\alpha, x_\beta$ and $x \notin G$. Consider a neighborhood of x, $\{x, x_\alpha\} \in U_f$, it is also clear that $\{x, x_\alpha\} \in V_F$. Hence, $f_{\text{int}}(f_{cl}\{x, x_\alpha\}) \cap G = \{x, x_\alpha\} \cap G = \phi$ i.e. x is not a f_δ-cluster point of G.
We have shown that any point which is out side of the set G is not a f_δ-cluster point of G, means G contains all its f_δ-cluster points. Therefore, G is f_δ-closed set.

Next, we show that if $G \notin F_f$, then G is not f_δ-closed set. If $G \notin F_f$ implies $x_\alpha, x_\beta \in G$. Consider $x \notin G \Rightarrow x \neq x_\alpha, x_\beta$ there exist open sets of the type U_f, V_f and W_f containing x.
Case 1 If $x \in u_f \in U_f$ such that $x_\alpha \in u_f$, then there exists some $v_F \in V_F$ such that $u_f \cong v_F$. Hence, $f_{\text{int}}(f_{cl}(u_f)) \cap G = u_f \cap G = A \neq \phi$, $(x_\alpha \in A)$ where A is a subset of X.

Case 2 If $x \in v_f \in V_f$ then by the same reasoning $f_{\text{int}}(f_{cl}(v_f)) \cap G = v_f \cap G = B \neq \phi$, $(x_\beta \in B)$ where B is a subset of X.
Case 3 If $x \in W_f$ and since $x_\alpha, x_\beta \notin W_F$ then $f_{cl}(W_f) = X$ and $f_{\text{int}}(f_{cl}(W_f)) \cap G = X \cap G = G \neq \phi$.
Implies any point lying out side of G is a f_δ-cluster point of G. Hence, $G(\notin F_f)$ is not f_δ-closed set. Thus, $F_f \cong F_{f_\delta}$ □

Moreover, if there exists any arbitrary topology τ_α finer than τ, then it is obvious that the same conclusion holds for τ_α as well.

Theorem 5.7. *Let (X, τ, τ_f) be a fine topological space, then for an element $p \in X$, the set $X\backslash\{p\}$ is f_δ-g-closed or f_δ-open.*

Proof. Suppose that $X\backslash\{p\}$ is not f_δ-open. Then X is the only f_δ-open set containing $X\backslash\{p\} \Rightarrow f_{cl_\delta}(X\backslash\{p\}) \subseteq X$. Hence $X\backslash\{p\}$ is f_δ-g-closed. Second part of this Theorem is a direct consequence of Lemma 5.5 and Theorem 5.6. □

Theorem 5.8. *A fine topological space (X, τ, τ_f) is f_δ-$T_{\frac{1}{2}}$ space if and only if for each point $p \in X$, the set $\{p\}$ is f_δ-closed or f_δ-open.*

Proof. Let X be a f_δ-$T_{\frac{1}{2}}$ space and let if possible for $p \in X$, $\{p\}$ be not f_δ-closed. Then, by Theorem 5.7, $X\backslash\{p\}$ is f_δ-g-closed. Since (X, τ, τ_f) is f_δ-$T_{\frac{1}{2}}$, then $X\backslash\{p\}$ is f_δ-closed set (by Definition 5.1(4)) that implies that $\{p\}$ is f_δ-open set.

Conversely, let V be any f_δ-g-closed set. It is enough if we show that V is f_δ-closed (i.e. $f_{cl_\delta}(V) = V$). Let $p \in f_{cl_\delta}(V)$. By assumption $\{p\}$ is f_δ-closed or f_δ-open for each $p \in X$. Consider following two cases:
Case 1. Let $\{p\}$ be f_δ-closed and $p \notin V$, then $p \in f_{cl_\delta}(V)\backslash V \Rightarrow \{p\} \subseteq f_{cl_\delta}(V)\backslash V$. By Theorem 3.27, this is contradiction. So, $p \in V \Rightarrow f_{cl_\delta}(V) \subseteq V$, but $V \subseteq f_{cl_\delta}(V)$. Hence, $f_{cl_\delta}(V) = V$. Hence, V is f_δ-closed set.
Case 2. Let $\{p\}$ be f_δ-open set. Then by Theorem 3.25, $V \cap \{p\} \neq \phi \Rightarrow p \in V$, hence $f_{cl_\delta}(V) \subseteq V$ but $V \subseteq f_{cl_\delta}(V)$. Therefore, $f_{cl_\delta}(V) = V$, it implies that V is f_δ-closed set. Thus, in both the cases (X, τ, τ_f) is f_δ-$T_{\frac{1}{2}}$ space. □

Remark 5.9. Following implications hold for some separation axioms with concerning the collection of f_δ-open sets when τ consists of at least one subset of X containing a minimum of two distinct elements of X. (by Theorem 5.4)

$$\begin{array}{ccccccc} T_2^f & \longleftrightarrow & T_1^f & \longleftrightarrow & T_{\frac{1}{2}}^f & \longleftrightarrow & T_0^f \\ \updownarrow & & & & & & \\ f_\delta\text{-}T_2 & \longleftrightarrow & f_\delta\text{-}T_1 & \longleftrightarrow & f_\delta\text{-}T_{\frac{1}{2}} & \longleftrightarrow & f_\delta\text{-}T_0 \end{array}$$

Diagram 5.1. Relation amongst different separation axioms with respect to fine collection and f_δ-collection of open sets

6. Conclusion

In the present paper, the collection of f_δ-open sets is defined, which is a special case of generalized topology. Some interesting properties of these f_δ-open sets have also been studied. The significant contribution to this paper is the definitions and particular examples of f_δ-separation axioms. It was interesting to note that the collection τ_{f_δ} satisfies the f_δ-T_2 or f_δ-Hausdorff axiom when the collection τ_{f_δ} is not the power set of the space X whereas, in the finite point set topology, the topological space is Hausdorff if it is a discrete space.

Moreover, the concept of the f_δ-g-closed set has also been initiated. These ideas may be applied to define some new class of continuous functions and by considering the ideal role, the concept may be extended over the Ideal Topological space.

Acknowledgment. The authors are thankful to the editors and anonymous reviewers for their valuable comments and suggestions that helped us improve the paper significantly.

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