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Certain Properties Associated with Generalized M-Series using Hadamard Product

Dheerandra Shanker Sachan[1*], Dinesh Kumar[2] and Kottakkaran Sooppy Nisar[3]

Abstract. The generalized M-series is a hybrid function of generalized Mittag-Leffler function and generalized hypergeometric function. The principal aim of this paper is to investigate certain properties resembling those of the Mittag-Leffler and Hypergeometric functions including various differential and integral formulas associated with generalized M-series. Certain corollaries involving the generalized hypergeometric function are also discussed. Further, in view of Hadamard product of two analytic functions, we have represented our main findings in Hadamard product of two known functions.

1. Introduction and Preliminaries

The generalized M-series [18] which is extension of both Mittag-Leffler function and generalized hypergeometric function is defined as

$$
\begin{aligned}
(1.1) \qquad {}_uM_v^{\alpha,\beta}(z) &= {}_uM_v^{\alpha,\beta}\left(c_1,\ldots,c_u;d_1,\ldots,d_v;z\right) \\
&= \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k} \frac{z^k}{\Gamma\left(\beta+k\alpha\right)},
\end{aligned}
$$

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* Corresponding author.

where $\alpha, \beta \in \mathbb{C}, z \in \mathbb{C}, \Re(\alpha) > 0, (c_i)_k\ (i = 1, \ldots, u)$ and $(d_j)_k\ (j = 1, \ldots, v)$ are well known Pochhammer symbols defined by

$$(c)_k = \begin{cases} 1 & k = 0, c \neq 0 \\ c(c+1)\ldots(c+k-1) & k \in \mathbb{N}, c \in \mathbb{C} \end{cases}$$
$$= \frac{\Gamma(c+k)}{\Gamma(c)}, \quad k \in \mathbb{N} \cup \{0\}.$$

The series (1.1) is defined when none of the parameters $d_j\ (j = 1, \ldots, v)$ is a negative integer or zero; if any numerator parameter $c_i\ (i = 1, \ldots, u)$ is a negative integer or zero, then series terminates to a polynomial in z. The series (1.1) is convergent for all z if $u \leq v$; it is convergent for $|z| < \delta = \alpha^\alpha$ if $u = v + 1$ and divergent if $u > v + 1$. When $u = v + 1$ and $|z| = \delta$, the series is convergent under conditions that depend on the parameters. The detailed account of the M-series can be found in the paper written by Sharma and Jain[18] (see also [2, 4, 5, 17]).

The generalized M-series (1.1) has interesting relationship with elementary and various classical special functions. For example, it can be reduced to the exponential function, binomial series, cosine function, sine function, generalized Mittag-Leffler function [15](For current research of Mittag-Leffler function, see [9]), the generalized Mittag-Lefler function introduced by Prabhakar [23], generalized hypergeometric function [12] and many more by specializing the parameters as given below,

$${}_0M_0^{1,1}(-;-;z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

$${}_1M_0^{1,1}(c;-;z) = (1-z)^{-c} = \sum_{k=0}^{\infty} \frac{(c)_k z^k}{k!},$$

$${}_0M_1^{1,1}\left(-;1/2;-z^2/4\right) = \cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!},$$

$$z.{}_0M_1^{1,1}\left(-;3/2;-z^2/4\right) = \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!},$$

$$(1.2) \qquad {}_0M_0^{\alpha,\beta}(-;-;z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)},$$

$${}_1M_1^{\alpha,\beta}(\rho;1;z) = E_{\alpha,\beta}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{(1)_k \Gamma(\beta + k\alpha)},$$

$$(1.3) \qquad {}_uM_v^{1,1}(z) = {}_uF_v(c_1, \ldots, c_u; d_1, \ldots, d_v; z)$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k}{(d_1)_k \dots (d_v)_k} \frac{z^k}{k!} = {}_uF_v\left(z\right).$$

For our investigation, we consider following important formulae:
From Rainville [12, p. 87, 22, 21], we have

$$(c)_{n+k} = (c)_n \left(c+n\right)_k, \tag{1.4}$$

$$(c)_{2k} = (2)^{2k} \left(\frac{c}{2}\right)_k \left(\frac{c+1}{2}\right)_k, \tag{1.5}$$

$$\Gamma\left(z\right)\Gamma\left(1-z\right) = \frac{\pi}{\sin \pi z}, \quad \left(0 < \Re(z) < 1\right). \tag{1.6}$$

From Gasper and Rahman [14, p. 5, Eq. 1.2.27],we have

$$(c)_{-n} = \prod_{k=1}^{n} \left(\frac{1}{c-k}\right). \tag{1.7}$$

Also, for our purpose, we recall the classical Gamma and Beta function (see, e.g., [1, p. 24], [16, p. 8])

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \mathrm{d}t \quad (\Re(z) > 0). \tag{1.8}$$

$$B(\alpha, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t & (\Re(\alpha) > 0, \Re(\beta) > 0) \\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \end{cases} \tag{1.9}$$

Now, let's recall the definition of Hadamard product [24][22] which is also known as the convolution of two power series. Consider two power series with radii of convergence are denoted by R_f and R_g as:

$$f(z) = \sum_{k=0}^{\infty} u_k z^k, \quad (|z| < R_f),$$

$$g(z) = \sum_{k=0}^{\infty} v_k z^k, \quad (|z| < R_g),$$

then the Hadamard product of two power series is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} u_k v_k z^k = (g * f)(z) \quad (|z| < R), \tag{1.10}$$

where

$$R = \lim_{k \to \infty} \left| \frac{u_k v_k}{u_{k+1} v_{k+1}} \right|$$

$$= \left(\lim_{k\to\infty}\left|\frac{u_k}{u_{k+1}}\right|\right)\cdot\left(\lim_{k\to\infty}\left|\frac{v_k}{v_{k+1}}\right|\right)$$
$$= R_f.R_g.$$

where, in general we have $R \geq R_f.R_g$.

A remarkable study on various properties and applications of (1.1) has been made by many authors such as Chouhan and Saraswat [3] investigated fractional differentiation and integration of M-series using Riemann-Liouville fractional calculus operators, Suthar et al. [10] studied the properties of M-series under the new generalized fractional integral operators involving I-function as a kernel, Najafzadeh [20] introduced a new subclass of univalent functions associated with M-series based on q-derivative, Gehlot [17] obtained some integral representations and certain properties of M-series associated with fractional calculus, Sachan and Jaloree [7] investigated various integral transforms like Laplace transform, K-transform, Sumudu transform, Beta transform and many more including fractional Fourier transform. For more properties and applications of (1.1), see Chouhan and Khan [2], Kumar et al. [4], Kumar and Saxena [5], Saxena [19], Singh, [6], Ilhan[13], Suthar et al. [11], Sachan et al. [8], Najafzadeh [21] etc.

2. Main Results

In this section, we have investigated new recursive properties various differential and integral formulas of M-series.Additionally, we have explored some new or known results as corollaries in terms of hypergeometric function.

2.1. **Recurrence Relations.** We obtain certain interesting recurrence relations for the function (1.1) regarding the parameters involved in the function as follows:

Theorem 2.1. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(\beta) > 0$ *and none of the* d_j *is negative or zero, then there hold the following relation*

$$(2.1)\quad {}_uM_v^{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z\left(\frac{\prod_{i=1}^{u}(c_i)}{\prod_{j=1}^{v}(d_j)}\right) \times {}_uM_v^{\alpha,\alpha+\beta}(c_1+1,\ldots,c_u+1; d_1+1,\ldots,d_v+1; z).$$

Proof. In the definition of M-series, shifting the index k by $k+1$, we obtain

$$
\begin{aligned}
{}_uM_v^{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k}{(d_1)_k \dots (d_v)_k} \frac{z^k}{\Gamma(\beta + k\alpha)} \\
&= \sum_{k=-1}^{\infty} \frac{(c_1)_{k+1} \dots (c_u)_{k+1}}{(d_1)_{k+1} \dots (d_v)_{k+1}} \frac{z^{k+1}}{\Gamma(\beta + (k+1)\,\alpha)},
\end{aligned}
$$

using formula (1.4), we have

$$
{}_uM_v^{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{(c_1)\,(c_1+1)_k \dots (c_u)\,(c_u+1)_k}{(d_1)\,(d_1+1)_k \dots (d_v)\,(d_v+1)_k} \frac{z^{k+1}}{\Gamma(\beta + (k+1)\,\alpha)},
$$

finally, by virtue of the definition of M-series (1.1), we obtain the right-hand side of (2.1). This completes proof of Theorem 2.1. □

By setting $\alpha = \beta = 1$ in Theorem 2.1 and following the definition of generalized hypergeometric function (1.3), we arrive at the following corollary:

Corollary 2.2. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v)$ and none of the d_j is negative or zero, then there hold the following relation*

$$
\begin{aligned}
{}_uF_v(z) = 1 + z &\left(\frac{\prod_{i=1}^{u} (c_i)}{\prod_{j=1}^{v} (d_j)} \right) \\
&\times {}_{u+1}F_{v+1}\left(1, c_1+1, \dots, c_u+1; 2, d_1+1, \dots, d_v+1; z\right).
\end{aligned}
$$

Theorem 2.3. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), \Re(\alpha) > 0, r \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following formula:*

(2.2)

$$
\begin{aligned}
&{}_uM_v^{\alpha,\alpha r+\beta}\left[c_1+r, \dots, c_u+r; d_1+r, \dots, d_v+r; z\right] \\
&\quad = \frac{1}{z^r} \left(\frac{\prod_{j=1}^{v} (d_j)_r}{\prod_{i=1}^{u} (c_i)_r} \right) \times \left[{}_uM_v^{\alpha,\beta}(z) - \sum_{k=0}^{r-1} \frac{(c_1)_k \dots (c_u)_k \; z^k}{(d_1)_k \dots (d_v)_k \; \Gamma(\alpha k + \beta)} \right].
\end{aligned}
$$

Proof. To prove Theorem 2.3, we start with the expression

$$
{}_uM_v^{\alpha,\beta}(z) - \sum_{k=0}^{r-1} \frac{(c_1)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k \Gamma(\alpha k + \beta)} = \sum_{k=r}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k \Gamma(\alpha k + \beta)},
$$

by putting $k = m + r$ and using formula (1.4), we arrive at

$$
\begin{aligned}
&{}_uM_v^{\alpha,\beta}(z) - \sum_{k=0}^{r-1} \frac{(c_1)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k \Gamma(\alpha k + \beta)} \\
&= z^r \left(\frac{\prod_{i=1}^{u} (c_i)_r}{\prod_{j=1}^{v} (d_j)_r} \right) \sum_{m=0}^{\infty} \frac{(c_1 + r)_m \dots (c_u + r)_m z^m}{(d_1 + r)_m \dots (d_v + r)_m \Gamma(\alpha m + \alpha r + \beta)} \\
&= z^r \left(\frac{\prod_{i=1}^{u} (c_i)_r}{\prod_{j=1}^{v} (d_j)_r} \right) {}_uM_v^{\alpha,\alpha r+\beta} \left[c_1 + r, \dots, c_u + r; d_1 + r, \dots, d_v + r; z \right],
\end{aligned}
$$

after little simplification, we have the right-hand side of (2.2). This completes proof of Theorem 2.3. □

Corollary 2.4. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), r \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following formula*

$$
\begin{aligned}
&{}_{u+1}F_{v+1} \left[1, c_1 + r, \dots, c_u + r; d_1 + r, \dots, d_v + r, r + 1; z \right] \\
&\quad = \frac{\Gamma(r+1)}{z^r} \left(\frac{\prod_{j=1}^{v} (d_j)_r}{\prod_{i=1}^{u} (c_i)_r} \right) \left[{}_uF_v(z) - \sum_{k=0}^{r-1} \frac{(c_1)_k \dots (c_u)_k \; z^k}{(d_1)_k \dots (d_v)_k \; k!} \right].
\end{aligned}
$$

2.2. **Differentiation Formulas.** We present higher order derivative formulas for the function (1.1) in the context of the following theorems:

Theorem 2.5. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), \Re(\alpha) > 0, m \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following result*

(2.3)

$$
\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^m {}_uM_v^{\alpha,\beta}(z) \\
&\quad = m! \left(\frac{\prod_{i=1}^{u} (c_i)_m}{\prod_{j=1}^{v} (d_j)_m} \right) \\
&\qquad \times {}_{u+1}M_{v+1}^{\alpha,\beta+m\alpha} \left(1 + m, c_1 + m, \dots, c_u + m; 1, d_1 + m, \dots, d_v + m; z \right).
\end{aligned}
$$

Proof. Differentiating the series in (1.1) term by term m times in succession, we arrive at

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m {}_uM_v^{\alpha,\beta}(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k}{(d_1)_k \dots (d_v)_k} \frac{z^k}{\Gamma(\beta + k\alpha)}$$
$$= \sum_{k=m}^{\infty} \frac{(c_1)_k \dots (c_u)_k \; k!}{(d_1)_k \dots (d_v)_k (k-m)!} \frac{z^{k-m}}{\Gamma(\beta + k\alpha)},$$

by putting $k = m + r$, we obtain

$$= \sum_{r=0}^{\infty} \frac{(c_1)_{m+r} \dots (c_u)_{m+r} (m+r)! z^r}{(d_1)_{m+r} \dots (d_v)_{m+r} r! \Gamma\left(\beta + (m+r)\alpha\right)},$$

by using formula (1.4), we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m {}_uM_v^{\alpha,\beta}(z) = \left(\frac{\prod_{i=1}^{u}(c_i)_m}{\prod_{j=1}^{v}(d_j)_m}\right)$$
$$\times \sum_{r=0}^{\infty} \frac{(c_1+m)_r \dots (c_u+m)_r (m+r)! z^r}{(d_1+m)_r \dots (d_v+m)_r r! \Gamma\left(\beta + m\alpha + r\alpha\right)}$$
$$= m! \left(\frac{\prod_{i=1}^{u}(c_i)_m}{\prod_{j=1}^{v}(d_j)_m}\right)$$
$$\times \sum_{r=0}^{\infty} \frac{(c_1+m)_r \dots (c_u+m)_r (1+m)_r z^r}{(1)_r (d_1+m)_r \dots (d_v+m)_r \Gamma\left(\beta + m\alpha + r\alpha\right)},$$

finally, by virtue of the definition of M-series (1.1), we obtain right-hand side of (2.3). This completes proof of Theorem 2.5. □

Corollary 2.6. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), m \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following known formula*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m {}_uF_v(z) = \left(\frac{\prod_{i=1}^{u}(c_i)_m}{\prod_{j=1}^{v}(d_j)_m}\right)$$
$$\times {}_uF_v\left(c_1+m, \dots, c_u+m; d_1+m, \dots, d_v+m; z\right).$$

Theorem 2.7. *If $\beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\beta) > 0, m, n \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following formula*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left[z^{\beta-1}{}_uM_v^{\frac{m}{n},\beta}\left(z^{\frac{m}{n}}\right)\right]$$
$$= z^{\beta-1}\left(\frac{\prod_{i=1}^{u}(c_i)_n}{\prod_{j=1}^{v}(d_j)_n}\right)$$
$$\times\left[\sum_{k=1}^{n}\frac{(c_1+n)_{-k}\ldots(c_u+n)_{-k}z^{-\frac{mk}{n}}}{(d_1+n)_{-k}\ldots(d_v+n)_{-k}\Gamma\left(\beta-\frac{mk}{n}\right)}\right.$$
$$\left.+\,{}_uM_v^{\frac{m}{n},\beta}\left(c_1+n,\ldots,c_u+n;d_1+n,\ldots,d_v+n;z^{\frac{m}{n}}\right)\right].$$

Proof. Differentiating the series in (1.1) term by term m times in succession, we arrive at

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left[z^{\beta-1}{}_uM_v^{\frac{m}{n},\beta}(z^{\frac{m}{n}})\right] = \left(\frac{d}{dz}\right)^m\sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k}{(d_1)_k\ldots(d_v)_k}\frac{z^{\frac{mk}{n}+\beta-1}}{\Gamma\left(\beta+\frac{mk}{n}\right)}$$
$$= z^{\beta-m-1}\sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k}{(d_1)_k\ldots(d_v)_k}\frac{z^{\frac{mk}{n}}}{\Gamma\left(\beta-m+\frac{mk}{n}\right)},$$

for $\Re(\beta-m) > 0$, replacing k by $k+n$ and using (1.4), we obtain

$$= z^{\beta-m-1}\sum_{k=-n}^{\infty}\frac{(c_1)_n(c_1+n)_k\ldots(c_u)_n(c_u+n)_k}{(d_1)_n(d_1+n)_k\ldots(d_v)_n(d_v+n)_k}\frac{z^{\frac{mk}{n}+m}}{\Gamma\left(\beta+\frac{mk}{n}\right)}$$
$$= z^{\beta-1}\left(\frac{\prod_{i=1}^{u}(c_i)_n}{\prod_{j=1}^{v}(d_j)_n}\right)\sum_{k=-n}^{\infty}\frac{(c_1+n)_k\ldots(c_u+n)_kz^{\frac{mk}{n}}}{(d_1+n)_k\ldots(d_v+n)_k\Gamma\left(\beta+\frac{mk}{n}\right)}$$
$$= z^{\beta-1}\left(\frac{\prod_{i=1}^{u}(c_i)_n}{\prod_{j=1}^{v}(d_j)_n}\right)\left[\sum_{k=1}^{n}\frac{(c_1+n)_{-k}\ldots(c_u+n)_{-k}z^{-\frac{mk}{n}}}{(d_1+n)_{-k}\ldots(d_v+n)_{-k}\Gamma\left(\beta-\frac{mk}{n}\right)}\right.$$
$$\left.+\sum_{k=0}^{\infty}\frac{(c_1+n)_k\ldots(c_u+n)_kz^{\frac{mk}{n}}}{(d_1+n)_k\ldots(d_v+n)_k\Gamma\left(\beta+\frac{mk}{n}\right)}\right]$$

$$= z^{\beta-1}\left(\frac{\prod_{i=1}^{u}(c_i)_n}{\prod_{j=1}^{v}(d_j)_n}\right)\left[\sum_{k=1}^{n}\frac{(c_1+n)_{-k}\ldots(c_u+n)_{-k}z^{-\frac{mk}{n}}}{(d_1+n)_{-k}\ldots(d_v+n)_{-k}\Gamma\left(\beta-\frac{mk}{n}\right)}\right.$$
$$\left.+\, {}_uM_v^{\frac{m}{n},\beta}\left(c_1+n,\ldots,c_u+n;d_1+n,\ldots,d_v+n;z^{\frac{m}{n}}\right)\right],$$

This completes proof of Theorem 2.7. □

If $n=1$ and using formula (1.7), then we arrive at following result:

Corollary 2.8. *If $\beta, c_i, d_j, z \in \mathbb{C}, (i=1,\ldots,u; j=1,\ldots,v), \Re(\beta) > 0, m \in \mathbb{N}, \Re(\beta-m) > 0$ and none of the d_j is negative or zero, then there hold the following formula*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m\left[z^{\beta-1}{}_uM_v^{m,\beta}(z^m)\right]$$
$$= z^{\beta-1}\frac{z^{-m}}{\Gamma(\beta-m)}$$
$$+\, z^{\beta-1}\left(\frac{\prod_{i=1}^{u}(c_i)}{\prod_{j=1}^{v}(d_j)}\right){}_uM_v^{m,\beta}\left[c_1+1,\ldots,c_u+1;d_1+1,\ldots,d_v+1;z^m\right].$$

Theorem 2.9. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i=1,\ldots,u; j=1,\ldots,v), \Re(\alpha) > 0, n \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following formula:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left[z^{c_1+n-1}{}_uM_v^{\alpha,\beta}(z)\right] \tag{2.4}$$
$$= (c_1)_n z^{c_1-1}{}_uM_v^{\alpha,\beta}\left[c_1+n,c_2,\ldots,c_u;d_1,d_2,\ldots,d_v;z\right].$$

Proof. Differentiating the series in (1.1) term by term m times in succession, we arrive at

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left[z^{c_1+n-1}{}_uM_v^{\alpha,\beta}(z)\right] = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left[\sum_{k=0}^{\infty}\frac{(c_1)_k(c_2)_k\ldots(c_u)_kz^{k+c_1+n-1}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\right]$$
$$= \sum_{k=0}^{\infty}\frac{(c_1)_k(k+c_1)_n(c_2)_k\ldots(c_u)_kz^{k+c_1-1}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)},$$

using formula (1.4), then R.H.S. of above expression

$$= \sum_{k=0}^{\infty}\frac{(c_1)_{k+n}(c_2)_k\ldots(c_u)_kz^{k+c_1-1}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_n (c_1+n)_k (c_2)_k \dots (c_u)_k z^{k+c_1-1}}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)}$$

$$= (c_1)_n z^{c_1-1} \sum_{k=0}^{\infty} \frac{(c_1+n)_k (c_2)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)},$$

finally, by virtue of the definition of M-series (1.1), we obtain right-hand side of (2.4). This completes proof of Theorem 2.9. □

Corollary 2.10. *If* $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), n \in \mathbb{N}$ *and none of the* d_j *is negative or zero, then we have following known formula*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[z^{c_1+n-1} {}_uF_v(z)\right]$$
$$= (c_1)_n z^{c_1-1} {}_uF_v \left[c_1+n, c_2, \dots, c_u; d_1, d_2, \dots, d_v; z\right].$$

2.3. **Integral Representations.** Now we investigate certain integrals involving the generalized M-series applying well known Beta and Gamma functions.

Theorem 2.11. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), \Re(\alpha) > 0, \Re(\lambda) > 0, \Re(d_1 - \lambda) > 0$ *and none of the* d_j *is negative or zero,then there hold the following formula*

$$\int_0^1 t^{\lambda-1}(1-t)^{d_1-\lambda-1} {}_uM_v^{\alpha,\beta}\left(c_1, \dots, c_u; \lambda, d_2, \dots, d_v; zt\right) \mathrm{d}t \tag{2.5}$$
$$= B\left(\lambda, d_1 - \lambda\right) {}_uM_v^{\alpha,\beta}(z).$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.5). Then we have

$$\mathcal{L} = \int_0^1 t^{\lambda-1}(1-t)^{d_1-\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k t^k}{(\lambda)_k (d_2)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \mathrm{d}t$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k}{(\lambda)_k (d_2)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^1 t^{\lambda+k-1}(1-t)^{d_1-\lambda-1} \mathrm{d}t$$

with the use of (1.9), we obtain

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k}{(\lambda)_k (d_2)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} B\left(\lambda + k, d_1 - \lambda\right)$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k}{(\lambda)_k (d_2)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \frac{\Gamma(\lambda + k)\Gamma(d_1 - \lambda)}{\Gamma(d_1 + k)}$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k}{(d_1)_k (d_2)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \frac{\Gamma(\lambda)\Gamma(d_1 - \lambda)}{\Gamma(d_1)}$$

$$= B(\lambda, d_1 - \lambda)\, {}_uM_v^{\alpha,\beta}(z).$$

This completes proof of Theorem 2.11. □

Corollary 2.12. *If* $c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\lambda) > 0, \Re(d_1 - \lambda) > 0$ *and none of the* d_j *is negative or zero, then we have following result*

$$\int_0^1 t^{\lambda-1}(1-t)^{d_1-\lambda-1}\, {}_uF_v(c_1, \ldots, c_u; \lambda, d_2, \ldots, d_v; zt)\,\mathrm{d}t$$
$$= B(\lambda, d_1 - \lambda)\, {}_uF_v(z).$$

Theorem 2.13. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(c_1) > 0$ *and none of the* d_j *is negative or zero, then there hold the following formula*

(2.6)
$$\frac{1}{\Gamma(c_1)} \int_0^\infty e^{-t} t^{c_1-1}\, {}_{u-1}M_v^{\alpha,\beta}(-, c_2, \ldots, c_u; d_1, \ldots, d_v; zt)\,\mathrm{d}t = {}_uM_v^{\alpha,\beta}(z).$$

Proof. Let $\mathcal{L}$ be the right-handed member of (2.6). Then we have

$$\mathcal{L} = \frac{1}{\Gamma(c_1)} \int_0^\infty e^{-t} t^{c_1-1} \sum_{k=0}^\infty \frac{(c_2)_k \ldots (c_u)_k z^k t^k}{(d_1)_k \ldots (d_v)_k \Gamma(\beta + k\alpha)} \mathrm{d}t$$
$$= \frac{1}{\Gamma(c_1)} \sum_{k=0}^\infty \frac{(c_2)_k \ldots (c_u)_k z^k}{(d_1)_k \ldots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^\infty e^{-t} t^{c_1+k-1} \mathrm{d}t$$

using (1.8), we have

$$\mathcal{L} = \frac{1}{\Gamma(c_1)} \sum_{k=0}^\infty \frac{(c_2)_k \ldots (c_u)_k z^k}{(d_1)_k \ldots (d_v)_k \Gamma(\beta + k\alpha)} \Gamma(c_1 + k)$$
$$= \sum_{k=0}^\infty \frac{(c_1)_k \ldots (c_u)_k z^k}{(d_1)_k \ldots (d_v)_k \Gamma(\beta + k\alpha)}$$
$$= {}_uM_v^{\alpha,\beta}(z).$$

This completes proof of Theorem 2.13. □

Corollary 2.14. *If* $c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v)$, $\Re(c_1) > 0$ *and none of the* d_j *is negative or zero,then we have*

$$\frac{1}{\Gamma(c_1)} \int_0^\infty e^{-t} t^{c_1-1}\, {}_{u-1}F_v(-, c_2, \ldots, c_u; d_1, \ldots, d_v; zt)\,\mathrm{d}t = {}_uF_v(z).$$

Theorem 2.15. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0$ *and none of the* d_j *is negative or zero, then there hold the following formula*

(2.7)
$$\int_0^\infty e^{-t}\, {}_uM_v^{\alpha,\beta}(zt)\,\mathrm{d}t = {}_{u+1}M_v^{\alpha,\beta}(1, c_1, \ldots, c_u; d_1, \ldots, d_v; z).$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.7). Then we have

$$\mathcal{L} = \int_0^\infty e^{-t} \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k t^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \mathrm{d}t$$
$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k \; z^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^\infty e^{-t} t^k \mathrm{d}t$$

on applying (1.8), we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k \; z^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \Gamma(k+1)$$
$$= \sum_{k=0}^{\infty} \frac{(1)_k (c_1)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)}$$
$$= {}_{u+1}M_v^{\alpha,\beta}(1, c_1, \dots, c_u; d_1, \dots, d_v; z).$$

This completes proof of Theorem 2.15. □

Corollary 2.16. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v)$ and none of the d_j is negative or zero, then we have the following formula*

$$\int_0^\infty e^{-t} {}_uF_v(zt)\,\mathrm{d}t = {}_{u+1}F_v(1, c_1, \dots, c_u; d_1, \dots, d_v; z).$$

Theorem 2.17. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), \Re(\alpha) > 0, \Re(c_1) > 0$ and none of the d_j is negative or zero,then the following formula holds true*

$$\int_0^\infty e^{-t^2} t^{2c_1 - 1} {}_{u-1}M_v^{\alpha,\beta}\left(c_2, \dots, c_u; d_1, \dots, d_v; z^2 t^2\right) \mathrm{d}t = \frac{1}{2}\Gamma(c_1)\, {}_uM_v^{\alpha,\beta}(z^2). \tag{2.8}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.8). Then we have

$$\mathcal{L} = \int_0^\infty e^{-t^2} t^{2c_1 - 1} \sum_{k=0}^{\infty} \frac{(c_2)_k \dots (c_u)_k z^{2k} t^{2k}}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \mathrm{d}t$$
$$= \sum_{k=0}^{\infty} \frac{(c_2)_k \dots (c_u)_k z^{2k}}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^\infty e^{-t^2} t^{2c_1 + 2k - 1} \mathrm{d}t,$$

by substituting $t^2 = x$, we arrive at

$$\mathcal{L} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(c_2)_k \dots (c_u)_k z^{2k}}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^\infty e^{-x} x^{c_1 + k - 1} \mathrm{d}x$$

with the use of (1.8), we have

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(c_2)_k\ldots(c_u)_k z^{2k}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\Gamma(c_1+k)\\ &= \frac{1}{2}\Gamma(c_1)\sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k z^{2k}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\\ &= \frac{1}{2}\Gamma(c_1)_u M_v^{\alpha,\beta}(z^2).\end{aligned}$$

This completes proof of Theorem 2.17. □

Corollary 2.18. *If $c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v)$, $\Re(c_1) > 0$ and none of the d_j is negative or zero, then we have*

$$\int_0^\infty e^{-t^2} t^{2c_1-1}{}_{u-1}F_v\left(c_2,\ldots,c_u; d_1,\ldots,d_v; z^2t^2\right)\mathrm{d}t = \frac{1}{2}\Gamma(c_1)\,{}_uF_v\left(z^2\right).$$

Theorem 2.19. *If $\alpha,\beta,c_i,d_j,z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v), \Re(\alpha) > 0$ and none of the d_j is negative or zero, then there hold the following formula*

$$\begin{aligned}&\int_0^t\left[\frac{1}{\sqrt{z(t-z)}}\right]{}_uM_v^{\alpha,\beta}\left[4z(t-z)\right]\mathrm{d}z\\ &\quad= \pi_{u+1}M_{v+1}^{\alpha,\beta}\left(\frac{1}{2},c_1,\ldots,c_u;1,d_1,\ldots,d_v;t^2\right).\end{aligned}\tag{2.9}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.9). Then we have

$$\begin{aligned}\mathcal{L} &= \int_0^t[z(t-z)]^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k\left[4z(t-z)\right]^k}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\mathrm{d}z\\ &= \sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k 2^{2k}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\int_0^t z^{k-\frac{1}{2}}(t-z)^{k-\frac{1}{2}}\,\mathrm{d}z,\end{aligned}$$

on substituting $z = \omega t$, we have

$$\mathcal{L} = \sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k 2^{2k}t^{2k}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\int_0^1 \omega^{k-\frac{1}{2}}(1-\omega)^{k-\frac{1}{2}}\mathrm{d}\omega$$

using (1.9), we have

$$\begin{aligned}\mathcal{L} &= \sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k 2^{2k}t^{2k}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+1)}\\ &= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(c_1)_k\ldots(c_u)_k 2^{2k}t^{2k}}{(d_1)_k\ldots(d_v)_k\Gamma(\beta+k\alpha)}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{(1)_{2k}},\end{aligned}$$

by using formula (1.5) and finally, by virtue of the definition of M-series (1.1), we obtain right-hand side of (2.9). This completes proof of Theorem 2.19. □

Corollary 2.20. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v)$ and none of the d_j is negative or zero,then we have*

$$\int_0^t \left[\frac{1}{\sqrt{z(t-z)}}\right] {}_uF_v\left[4z\left(t-z\right)\right] \mathrm{d}z$$
$$= \pi_{u+1}F_{v+1}\left(\frac{1}{2}, c_1, \ldots, c_u; 1, d_1, \ldots, d_v; t^2\right).$$

Theorem 2.21. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and none of the d_j is negative or zero,then there hold the following formula*

(2.10)
$$\int_{-1}^1 (1+z)^{\mu-1}\left(1-z\right)^{\lambda-1} {}_uM_v^{\alpha,\beta}\left(\frac{1-z}{2}\right) \mathrm{d}z$$
$$= 2^{\mu+\lambda-1}B(\mu,\lambda)_{u+1}M_{v+1}^{\alpha,\beta}\left(\lambda, c_1, \ldots, c_u; \mu+\lambda, d_1, \ldots, d_v; 1\right).$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.10). Then we have

$$\mathcal{L} = \int_{-1}^1 (1+z)^{\mu-1}(1-z)^{\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k \Gamma\left(\beta+k\alpha\right)} \left(\frac{1-z}{2}\right)^k \mathrm{d}z$$
$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k 2^k \Gamma\left(\beta+k\alpha\right)} \int_{-1}^1 (1+z)^{\mu-1}(1-z)^{\lambda+k-1}\mathrm{d}z,$$

by substituting $1+z = 2t$ and using the definition of beta function, we get

$$\mathcal{L} = 2^{\mu+\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k \Gamma\left(\beta+k\alpha\right)} \int_0^1 t^{\mu-1}(1-t)^{\lambda+k-1}\mathrm{d}t$$

on applying (1.9), we have

$$\mathcal{L} = 2^{\mu+\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k \Gamma\left(\beta+k\alpha\right)} \frac{\Gamma\left(\lambda+k\right)\Gamma\left(\mu\right)}{\Gamma\left(\mu+\lambda+k\right)}$$
$$= 2^{\mu+\lambda-1} B\left(\mu,\lambda\right) \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k \Gamma\left(\beta+k\alpha\right)} \frac{(\lambda)_k}{(\mu+\lambda)_k}$$

finally, by virtue of the definition of M-series (1.1), we obtain right-hand side of (2.10). This completes proof of Theorem 2.21. □

Special Cases of (2.10)
Case-I. If $\mu = 1$ then there hold the formula

$$\int_{-1}^{1} (1-z)^{\lambda-1} {}_uM_v^{\alpha,\beta}\left(\frac{1-z}{2}\right) \mathrm{d}z$$
$$= \frac{2^\lambda}{\lambda} {}_{u+1}M_{v+1}^{\alpha,\beta}\left(\lambda, c_1, \ldots, c_u; 1+\lambda, d_1, \ldots, d_v; 1\right).$$

Case-II. If $\lambda = 1$, then the following formula holds true

$$\int_{-1}^{1} (1+z)^{\mu-1} {}_uM_v^{\alpha,\beta}\left(\frac{1-z}{2}\right) \mathrm{d}z$$
$$= \frac{2^\mu}{\mu} {}_{u+1}M_{v+1}^{\alpha,\beta}\left(1, c_1, \ldots, c_u; 1+\mu, d_1, \ldots, d_v; 1\right).$$

Case-III. If $\mu + \lambda = 1$ and using (1.6), then there hold the following formula

$$\int_{-1}^{1} (1+z)^{\mu-1} (1-z)^{\lambda-1} {}_uM_v^{\alpha,\beta}\left(\frac{1-z}{2}\right) \mathrm{d}z$$
$$= \frac{\pi}{\sin \pi\mu} {}_{u+1}M_{v+1}^{\alpha,\beta}\left(1-\mu, c_1, \ldots, c_u; 1, d_1, \ldots, d_v; 1\right).$$

Corollary 2.22. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\mu) > 0, \Re(\lambda) > 0$ and none of the d_j is negative or zero, then there hold the following formula*

$$\int_{-1}^{1} (1+z)^{\mu-1} (1-z)^{\lambda-1} {}_uF_v\left(\frac{1-z}{2}\right) \mathrm{d}z$$
$$= 2^{\mu+\lambda-1} B\left(\mu, \lambda\right) {}_{u+1}F_{v+1}\left(\lambda, c_1, \ldots, c_u; \mu+\lambda, d_1, \ldots, d_v; 1\right).$$

Theorem 2.23. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(\mu) > -1, \Re(\lambda) > 0$ and none of the d_j is negative or zero, then the following formula holds true*

$$\int_0^t z^\mu (t-z)^{\lambda-1} {}_uM_v^{\alpha,\beta}(z) \mathrm{d}z \tag{2.11}$$
$$= t^{\mu+\lambda} B\left(1+\mu, \lambda\right)$$
$$\times {}_{u+1}M_{v+1}^{\alpha,\beta}\left(1+\mu, c_1, \ldots, c_u; 1+\mu+\lambda, d_1, \ldots, d_v; t\right).$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.11). Then we have

$$\mathcal{L} = \int_0^t z^\mu (t-z)^{\lambda-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_u)_k}{(d_1)_k \ldots (d_v)_k} \frac{z^k}{\Gamma\left(\beta + k\alpha\right)} \mathrm{d}z$$

$$= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^t z^{\mu+k} (t-z)^{\lambda-1} \, \mathrm{d}z,$$

on substituting $z = \omega t$, we obtain

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k t^{\mu+\lambda+k}}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \int_0^1 \omega^{\mu+k} (1-\omega)^{\lambda-1} \mathrm{d}\omega$$

with the use of (1.9), we have

$$\begin{aligned}
\mathcal{L} &= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k t^{\mu+\lambda+k}}{(d_1)_k \dots (d_v)_k \Gamma(\beta + k\alpha)} \frac{\Gamma(1+\mu+k)\Gamma(\lambda)}{\Gamma(1+\mu+\lambda+k)} \\
&= t^{\mu+\lambda} \frac{\Gamma(1+\mu)\Gamma(\lambda)}{\Gamma(1+\mu+\lambda)} \sum_{k=0}^{\infty} \frac{(1+\mu)_k (c_1)_k \dots (c_u)_k}{(1+\mu+\lambda)_k (d_1)_k \dots (d_v)_k} \frac{t^k}{\Gamma(\beta+k\alpha)} \\
&= t^{\mu+\lambda} B(1+\mu,\lambda) \, {}_{u+1}M_{v+1}^{\alpha,\beta} (1+\mu, c_1, \dots, c_u; 1+\mu+\lambda, d_1, \dots, d_v; t).
\end{aligned}$$

This completes proof of Theorem 2.23. □

Corollary 2.24. *If $c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), \Re(\mu) > -1, \Re(\lambda) > 0$ and none of the d_j is negative or zero, then we have following formula*

$$\begin{aligned}
&\int_0^t z^{\mu} (t-z)^{\lambda-1} \, {}_uF_v(z) \mathrm{d}z \\
&\quad = t^{\mu+\lambda} B(1+\mu,\lambda) \\
&\qquad \times {}_{u+1}F_{v+1} (1+\mu, c_1, \dots, c_u; 1+\mu+\lambda, d_1, \dots, d_v; t).
\end{aligned}$$

Theorem 2.25. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \dots, u; j = 1, \dots, v), \Re(\alpha) > 0, \Re(\beta) > 0$ and none of the d_j is negative or zero, then there hold the following formula*

$$\int_0^{\infty} e^{-t} t^{\beta-1} \, {}_uM_v^{\alpha,\beta} (t^{\alpha} z) \, \mathrm{d}t = {}_{u+1}F_v (1, c_1, \dots, c_u; d_1, \dots, d_v; z). \tag{2.12}$$

Proof. Let $\mathcal{L}$ be the left-handed member of (2.12). Then we have

$$\begin{aligned}
\mathcal{L} &= \int_0^{\infty} e^{-t} t^{\beta-1} \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k t^{\alpha k} z^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta+k\alpha)} \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \frac{(c_1)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k \Gamma(\beta+k\alpha)} \int_0^{\infty} e^{-t} t^{\alpha k+\beta-1} \mathrm{d}t
\end{aligned}$$

using (1.8), we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(1)_k (c_1)_k \dots (c_u)_k z^k}{(d_1)_k \dots (d_v)_k k!}$$

$$= {}_{u+1}F_v\left(1, c_1, \ldots, c_u; d_1, \ldots, d_v; z\right).$$

This completes proof of Theorem 2.25. □

3. Hadamard Product Representation

In this section, we present our main findings in terms of the product of two known functions applying Hadamard product concept of power series defined in (1.10).

Theorem 3.1. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, m \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following result*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m {}_uM_v^{\alpha,\beta}(z) = m!\left(\frac{\prod_{i=1}^{u}(c_i)_m}{\prod_{j=1}^{v}(d_j)_m}\right) \times {}_{u+1}F_v\left(1+m, c_1+m, \ldots, c_u+m; d_1+m, \ldots, d_v+m; z\right) * E_{\alpha,m\alpha+\beta}(z).$$

Proof. Applying (1.10) and in view of (1.3) and (1.2), we have

$$\begin{aligned}&{}_{u+1}M_{v+1}^{\alpha,\beta+m\alpha}\left(1+m, c_1+m, \ldots, c_u+m; 1, d_1+m, \ldots, d_v+m; z\right)\\ &\quad= {}_{u+1}F_v\left(1+m, c_1+m, \ldots, c_u+m; d_1+m, \ldots, d_v+m; z\right) * E_{\alpha,m\alpha+\beta}(z)\end{aligned} \tag{3.1}$$

use of (3.1) in Theorem 2.5, leads to proof of the Theorem 3.1. □

Following similar lines as of Theorem 3.1, Theorems 2.7 to 2.25 can easily be represented in Hadamard product of two power series as:

Theorem 3.2. *If $\beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\beta) > 0, m, n \in \mathbb{N}$ and none of the d_j is negative or zero, then there hold the following formula*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left[z^{\beta-1}{}_uM_v^{\frac{m}{n},\beta}\left(z^{\frac{m}{n}}\right)\right] = z^{\beta-1}\left(\frac{\prod_{i=1}^{u}(c_i)_n}{\prod_{j=1}^{v}(d_j)_n}\right) \times \left[\sum_{k=1}^{n}\frac{(c_1+n)_{-k}\ldots(c_u+n)_{-k}z^{-\frac{mk}{n}}}{(d_1+n)_{-k}\ldots(d_v+n)_{-k}\Gamma\left(\beta-\frac{mk}{n}\right)}\right.$$

$$+\,{}_{u+1}F_v\left(1, c_1+n,\ldots,c_u+n; d_1+n,\ldots,d_v+n; z^{\frac{m}{n}}\right) * E_{\frac{m}{n},\beta}(z^{\frac{m}{n}})\Bigg].$$

Theorem 3.3. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v), \Re(\alpha) > 0, n \in \mathbb{N}$ *and none of the* d_j *is negative or zero, then there hold the following formula:*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left[z^{c_1+n-1}{}_uM_v^{\alpha,\beta}(z)\right] = (c_1)_n z^{c_1-1}{}_{u+1}F_v\left(1, c_1+n, c_2,\ldots,c_u; d_1,\ldots,d_v; z\right) * E_{\alpha,\beta}(z).$$

Theorem 3.4. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v)$, $\Re(\alpha) > 0, \Re(\lambda) > 0, \Re(d_1-\lambda) > 0$ *and none of the* d_j *is negative or zero,then there hold the following formula*

$$\int_0^1 t^{\lambda-1}(1-t)^{d_1-\lambda-1}{}_uM_v^{\alpha,\beta}\left(c_1,\ldots,c_u; \lambda, d_2,\ldots,d_v; zt\right)\mathrm{d}t = B\left(\lambda, d_1-\lambda\right){}_{u+1}F_v\left(1, c_1,\ldots,c_u; d_1,\ldots,d_v; z\right) * E_{\alpha,\beta}(z).$$

Theorem 3.5. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v), \Re(\alpha) > 0, \Re(c_1) > 0$ *and none of the* d_j *is negative or zero, then there hold the following formula*

$$\frac{1}{\Gamma\left(c_1\right)}\int_0^\infty e^{-t}t^{c_1-1}{}_{u-1}M_v^{\alpha,\beta}\left(-, c_2,\ldots,c_u; d_1,\ldots,d_v; zt\right)\mathrm{d}t = {}_{u+1}F_v\left(1, c_1,\ldots,c_u; d_1,\ldots,d_v; z\right) * E_{\alpha,\beta}(z).$$

Theorem 3.6. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v), \Re(\alpha) > 0$ *and none of the* d_j *is negative or zero, then there hold the following formula*

$$\int_0^\infty e^{-t}{}_uM_v^{\alpha,\beta}\left(zt\right)\mathrm{d}t = {}_{u+2}F_v\left(1, 1, c_1,\ldots,c_u; d_1,\ldots,d_v; z\right) * E_{\alpha,\beta}(z).$$

Theorem 3.7. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v), \Re(\alpha) > 0, \Re(c_1) > 0$ *and none of the* d_j *is negative or zero, then the following formula holds true*

$$\int_0^\infty e^{-t^2}t^{2c_1-1}{}_{u-1}M_v^{\alpha,\beta}\left(c_2,\ldots,c_u; d_1,\ldots,d_v; z^2t^2\right)\mathrm{d}t = \frac{1}{2}\Gamma\left(c_1\right){}_{u+1}F_v\left(1, c_1,\ldots,c_u; d_1,\ldots,d_v; z^2\right) * E_{\alpha,\beta}(z^2).$$

Theorem 3.8. *If* $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1,\ldots,u; j = 1,\ldots,v), \Re(\alpha) > 0$ *and none of the* d_j *is negative or zero, then there hold the following formula*

$$\int_0^t \left[\frac{1}{\sqrt{z\left(t-z\right)}}\right]{}_uM_v^{\alpha,\beta}\left[4z\left(t-z\right)\right]\mathrm{d}z$$

$$= \pi_{u+1}F_v\left(\frac{1}{2}, c_1, \ldots, c_u; d_1, \ldots, d_v; t^2\right) * E_{\alpha,\beta}(t^2).$$

Theorem 3.9. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(\mu) > 0, \Re(\lambda) > 0$ and none of the d_j is negative or zero, then there hold the following formula*

$$\begin{aligned}\int_{-1}^{1} &(1+z)^{\mu-1}(1-z)^{\lambda-1}\,{}_uM_v^{\alpha,\beta}\left(\frac{1-z}{2}\right)\mathrm{d}z\\ &= 2^{\mu+\lambda-1}B(\mu,\lambda)\\ &\quad\times {}_{u+2}F_{v+1}\left(1,\lambda,c_1,\ldots,c_u;\mu+\lambda,d_1,\ldots,d_v;1\right) * E_{\alpha,\beta}(1).\end{aligned}$$

Theorem 3.10. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(\mu) > -1, \Re(\lambda) > 0$ and none of the d_j is negative or zero, then the following formula holds true*

$$\begin{aligned}\int_0^t &z^{\mu}(t-z)^{\lambda-1}\,{}_uM_v^{\alpha,\beta}(z)\mathrm{d}z\\ &= t^{\mu+\lambda}B(1+\mu,\lambda)\\ &\quad\times {}_{u+2}F_{v+1}\left(1,1+\mu,c_1,\ldots,c_u;1+\mu+\lambda,d_1,\ldots,d_v;t\right) * E_{\alpha,\beta}(t).\end{aligned}$$

Theorem 3.11. *If $\alpha, \beta, c_i, d_j, z \in \mathbb{C}, (i = 1, \ldots, u; j = 1, \ldots, v), \Re(\alpha) > 0, \Re(\beta) > 0$ and none of the d_j is negative or zero, then there hold the following formula*

$$\int_0^{\infty} e^{-t}t^{\beta-1}\,{}_uM_v^{\alpha,\beta}\left(t^{\alpha}z\right)\mathrm{d}t = {}_{u+2}F_v\left(1,1,c_1,\ldots,c_u;d_1,\ldots,d_v;t\right) * e^z.$$

4. Concluding Remarks

In this article, we studied various important properties of generalized M-series (1.1) such as recurrence relations, higher order differential as well as integral formulas. We also represented our main results in Hadamard product of two analytic functions. The results presented here, being general, can be reduced to certain other type of special functions. Most of the results presented here, have been expressed in a compact form to avoid the occurrence of infinite series so that they can be useful from the point of view of applications.

The generalized hypergeometric and Mittag-Leffler functions have a variety of applications in physics, mathematics and in many branches of engineering. Since, generalized M-series is a hybridization of above mentioned functions. Hence, the results derived in this paper, may lead to significant applications in physics, mathematics and in engineering sciences.

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[1] Department of Mathematics, St.Mary's PG College, 464001, Vidisha (M.P.), India.

Email address: sachan.dheerandra17@gmail.com

[2] Department of Applied Sciences, College of Agriculture-Jodhpur, Agriculture university Jodhpur, 342304, Jodhpur, (Raj.), India.

Email address: dinesh.dino03@yahoo.com

[3] Department of Mathematics, College of d Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Alkharj, 11942, Saudi Arabia.

Email address: ksnisar1@gmail.com