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Meryem Slimani and Fethia Ouikene

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Asymptotically Almost Periodic Generalized Ultradistributions and Application

Meryem Slimani[1*] and Fethia Ouikene[2,3]

Abstract. The paper aims to introduce and study an algebra of asymptotically almost periodic generalized ultradistributions. These generalized ultradistributions contain asymptotically almost periodic ultradistributions and asymptotically almost periodic generalized functions. The definition and main properties of these generalized ultradistributions are studied. An application to difference differential systems is given.

1. Introduction

Almost periodic functions were introduced by H. Bohr; see [4]. M. Fréchet introduced and studied asymptotically almost periodic functions in [18] as a perturbation of almost periodic functions by functions vanishing at infinity. The concept of almost periodicity in the distributions setting is due to L. Schwartz, see [23]. Asymptotically almost periodic distributions were introduced and studied in [14]. The papers [15] and [19] deal with almost periodic ultradistributions, while asymptotically almost periodic ultradistributions are considered in [21]. It is well known that the space of ultradistributions is strictly more significant than that of distributions.

An algebra of generalized functions has been introduced in [16] in connection with the problem of multiplication of distributions. The concept of almost periodicity and asymptotic almost periodicity in the setting of algebras of generalized functions were introduced and studied in the

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* Corresponding author.

works of C. Bouzar and col., see [5], [6], [7] and [8]. These algebras of almost periodic and asymptotically almost periodic generalized functions contain respectively their classical analogue of almost periodic functions and distributions and asymptotically almost periodic functions and distributions. In the same way, an algebra of generalized ultradistributions containing classical ultradistributions has been introduced in [1] and [2]. Almost periodicity in the framework of the algebra of generalized ultradistributions is tackled in the paper [9], while an application to linear ordinary differential equations is given in [10].

This work aims is to introduce and study an algebra of asymptotically almost periodic generalized ultradistributions containing asymptotically almost periodic ultradistributions and asymptotically almost periodic generalized functions. First, we introduce the algebra of asymptotically almost periodic generalized ultradistributions, denoted by $\mathcal{G}^M_{aap}$, then, we study their main properties. We prove a fundamental result on the uniqueness of the decomposition of an asymptotically almost periodic generalized ultradistribution as in the classical case of functions, distributions and ultradistributions. To do this, generalise of the Seeley theorem of [24] in the context of generalized ultradistributions. An application to linear difference differential equations in the framework of the algebra $\mathcal{G}^M_{aap}$ is given. Our results generalize the result given in [8].

The paper is organized as follows: Section two recalls some preliminary results needed in the sequel. We introduce in section three an algebra of asymptotically almost periodic generalized ultradistributions and also we investigate some of their main properties. Section four is devoted to an extension result in the context of bounded generalized ultradistributions needed to prove of the uniqueness of the decomposition of an asymptotically almost periodic generalized ultradistribution. In section five, we show that an asymptotically almost periodic generalized ultradistribution is uniquely decomposed as in the classical case of functions, distributions and ultradistributions. Section six deals with a nonlinear operation on asymptotically almost periodic generalized ultradistribution. The last section studies of asymptotically almost periodic generalized ultradistributional solutions of linear difference differential systems.

2. Preliminaries

This section recalls some preliminary results needed in the sequel. The space of continuous and bounded Complex-valued functions defined and continuously on $\mathbb{R}$, are denoted by $\mathcal{C}_b$. It is well known that $\left(\mathcal{C}_b, \|\cdot\|_{L^\infty(\mathbb{R})}\right)$ is a Banach algebra. For the definition of almost periodic functions and their properties, see [4] and [17] for more details.

Definition 2.1. A complex valued function g defined and continuous on $\mathbb{R}$ is said almost periodic if it satisfies one of the following equivalent assertions

(i) $\forall \varepsilon > 0$, the set $E\{\varepsilon, g\} := \left\{\tau \in \mathbb{R} : \|g(\cdot + \tau) - g(\cdot)\|_{L^\infty(\mathbb{R})} < \varepsilon\right\}$ is relatively dense in $\mathbb{R}$, i.e. $\exists l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains a $\tau \in E\{\varepsilon, g\}$.

(ii) $\forall \varepsilon > 0$, there exists a trigonometric polynomial P such that $\|g - P\|_{L^\infty(\mathbb{R})} < \varepsilon$.

(iii) Any sequence $(s_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ admits a subsequence $(s_{m_k})_k$ such that the sequence of functions $(g(\cdot + s_{m_k}))_k$ is uniformly convergent on $\mathbb{R}$.

We denote by $\mathcal{C}_{ap}$ the space of almost periodic functions.

The space of bounded functions vanishing at infinity, denoted $\mathcal{C}_{+,0}$, is the set of all functions $h \in \mathcal{C}_b$ satisfying $\lim\limits_{x\to+\infty} h(x) = 0$.

Asymptotically almost periodic functions are introduced and studied by M. Fréchet in [18].

Definition 2.2. A function $f \in \mathcal{C}_b$ is said asymptotically almost periodic if there exist $g \in \mathcal{C}_{ap}$ and $h \in \mathcal{C}_{+,0}$ such that $f = g + h$ on $\mathbb{J} := [0, +\infty[$. The space of all asymptotically almost periodic functions is denoted by $\mathcal{C}_{aap}$.

Proposition 2.3. *The decomposition of an asymptotically almost periodic function is unique on* $\mathbb{J}$.

Let $\mathcal{E}(\mathbb{I})$ be the algebra of space of smooth functions on $\mathbb{I} = \mathbb{R}$ or $\mathbb{J}$, and define the space

$$\mathcal{D}_{L^p}(\mathbb{I}) := \left\{\varphi \in \mathcal{E}(\mathbb{I}) : \forall j \in \mathbb{Z}_+,\ \varphi^{(j)} \in L^p(\mathbb{I})\right\}, \quad p \in [1, +\infty],$$

that we endow with the topology defined by the family of semi-norms

$$|\varphi|_{k,p,\mathbb{I}} := \sum_{j\leq k} \left\|\varphi^{(j)}\right\|_{L^p(\mathbb{I})}, \quad k \in \mathbb{Z}_+.$$

So, $\mathcal{D}_{L^p}(\mathbb{I})$ is a Fréchet subalgebra of $\mathcal{E}(\mathbb{I})$. Denote $\mathcal{B}(\mathbb{I}) := \mathcal{D}_{L^\infty}(\mathbb{I})$.

Remark 2.4. We means by $\varphi \in \mathcal{D}_{L^p}(\mathbb{J})$ that $\lim\limits_{\substack{x\to 0\\ >}} \varphi^{(j)}(x)$ exists $\forall j \in \mathbb{Z}_+$.

Definition 2.5. (i) The space of smooth almost periodic functions, denoted by $\mathcal{B}_{ap}$, is the set of all functions $\varphi \in \mathcal{E}(\mathbb{R})$ such that $\forall j \in \mathbb{Z}_+,\ \varphi^{(j)} \in \mathcal{C}_{ap}$.

(ii) The space of smooth asymptotically almost periodic functions, denoted by $\mathcal{B}_{aap}$, is the set of all functions $\varphi \in \mathcal{E}(\mathbb{R})$ such that $\forall j \in \mathbb{Z}_+,\ \varphi^{(j)} \in \mathcal{C}_{aap}$.

Remark 2.6. We endow $\mathcal{B}_{ap}$ and $\mathcal{B}_{aap}$ with the topology induced by $\mathcal{B} := \mathcal{B}(\mathbb{R})$.

We give some properties of the space $\mathcal{B}_{aap}$, see [7].

Proposition 2.7. (i) *The space $\mathcal{B}_{aap}$ is a subalgebra of $\mathcal{B}$ stable under translation and derivation.*
(ii) $\mathcal{B}_{aap} \times \mathcal{B}_{ap} \subset \mathcal{B}_{aap}$.
(iii) $\mathcal{B}_{aap} * L^1 \subset \mathcal{B}_{aap}$.

In order to introduce certain class of spaces, we need some definitions and results from [20].

Let $M = (M_k)_{k\in\mathbb{Z}_+}$ be a sequence of positive numbers, define the following properties

Logarithmic convexity

$$(M.1) \qquad M_k^2 \leq M_{k-1}M_{k+1}, \quad \forall k \in \mathbb{N}.$$

Stability under ultradifferential operators

$$(M.2) \qquad M_{k+l} \leq AH^{k+l}M_kM_l, \quad \exists A > 0,\ \exists H > 0,\ \forall k,\ l \in \mathbb{Z}_+.$$

Strong non quasi-analyticity

$$(M.3) \qquad \sum_{l=k+1}^{+\infty} \frac{M_{l-1}}{M_l} \leq Ak\frac{M_k}{M_{k+1}}, \quad \exists A > 0,\ \forall k \in \mathbb{Z}_+.$$

Non quasi-analyticity

$$(M.3)' \qquad \sum_{k=0}^{+\infty} \frac{M_{k-1}}{M_k} < \infty.$$

Definition 2.8. The associated function of the sequence M is defined by

$$M(t) = \sup_{k\in\mathbb{Z}_+} \ln \frac{t^k M_0}{M_k}, \quad t > 0.$$

Example 2.9. If the sequence M_k is the Gevrey sequence $(k!^\sigma)$, $\sigma > 1$, then it satisfies $(M.1)$, $(M.2)$, $(M.3)$ and it associated function $M(t)$ is equivalent to $t^{\frac{1}{\sigma}}$.

The next result shows that the conditions $(M.1)$ and $(M.2)$ can expressed in the term of the associated function, see Propositions 3.1 and 3.6 of [20].

Proposition 2.10. (i) *The sequence M satisfies $(M.1)$ if and only if*

$$M_k = \sup_{t>0} \frac{t^k M_0}{e^{M(t)}}, \quad \forall k \in \mathbb{Z}_+.$$

(ii) *Let M satisfies $(M.1)$, then M satisfies $(M.2)$ if and only if*

$$2M(t) \leq M(Ht) + \ln(AM_0), \quad \exists A > 0,\ \exists H > 0,\ \forall t > 0.$$

As a consequence of Proposition 2.10-(ii), the following result was obtained in Lemma 4.2 of [9].

Lemma 2.11. *If M satisfies $(M.2)$ then $\exists A > 0,\ \exists H > 0,\ \forall t_1, \cdots, t_n > 0,\ \forall n \in \mathbb{N}$,*

$$M(t_1) + \cdots + M(t_n) \leq M\left(H^{\frac{(n-1)(n+2)}{2n}} \max(t_1, \cdots, t_n)\right) + (n-1)\ln(AM_0).$$

Remark 2.12. Throughout the paper we assume that the sequence M satisfying the conditions $(M.1)$, $(M.2)$ and $(M.3)'$.

We recall from [22] some needed spaces. Let $p \in [1, +\infty]$, $h > 0$, the space

$$\mathcal{D}_{L^p}^{M,h} := \left\{ \varphi \in \mathcal{E}(\mathbb{R}) : \|\varphi\|_{p,h,M} := \sup_{j \in \mathbb{Z}_+} \frac{\left\|\varphi^{(j)}\right\|_{L^p(\mathbb{R})}}{h^j M_j} < \infty \right\},$$

endowed with the norm $\|\cdot\|_{p,h,M}$ is a Banach space.

The space of L^p-Beurling ultradifferentiable functions is

$$\mathcal{D}_{L^p}^{(M)} := proj \lim_{h \to 0} \mathcal{D}_{L^p}^{M,h}.$$

The space of Beurling ultradifferentiable functions

$$\mathcal{D}^{(M)} := \left\{ \begin{array}{c} \varphi \in \mathcal{E}(\mathbb{R}) : \forall K \text{ compact of } \mathbb{R},\ \forall h > 0, \\ \exists c > 0,\ \forall j \in \mathbb{Z}_+,\ \sup_{x \in K} \left|\varphi^{(j)}(x)\right| \leq c h^j M_j \end{array} \right\}$$

is dense in $\mathcal{D}_{L^p}^{(M)}$, $p \in [1, +\infty[$. The space $\dot{\mathcal{B}}^{(M)}$ is the closure of the space $\mathcal{D}^{(M)}$ in $\mathcal{B}^{(M)} := \mathcal{D}_{L^\infty}^{(M)}$.

Definition 2.13. Let $p \in]1, +\infty]$, the space of L^p-Beurling ultradistributions denoted by $\mathcal{D}'_{L^p,(M)}$ is the topological dual of $\mathcal{D}_{L^q}^{(M)}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Denotes by $\mathcal{D}'_{L^1,(M)}$ the topological dual of $\dot{\mathcal{B}}^{(M)}$. The elements of $\mathcal{D}'_{L^\infty,(M)}$ are said bounded ultradistributions.

3. Asymptotically Almost Periodic Generalized Ultradistributions

In this section, we introduce an algebra of asymptotically almost periodic generalized ultradistributions and also we investigate some of their main properties.

Let $I :=]0,1]$ and $(u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^p}(\mathbb{I}))^I$, $p \in [1,+\infty]$, $j \in \mathbb{Z}_+$ and $k > 0$, we mean by the notation

$$\left\| u_\varepsilon^{(j)} \right\|_{L^p(\mathbb{I})} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right), \quad \varepsilon \to 0,$$

that $\exists c > 0,\ \exists \varepsilon_0 > 0,\ \forall \varepsilon < \varepsilon_0,$

$$\left\| u_\varepsilon^{(j)} \right\|_{L^p(\mathbb{I})} \leq c e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Definition 3.1. (i) The space of asymptotically almost periodic moderate elements is denoted and defined by

$$\mathcal{M}_{aap}^M := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aap})^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0 \end{array} \right\}.$$

(ii) The space of asymptotically almost periodic null elements is denoted and defined by

$$\mathcal{N}_{aap}^M := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aap})^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0 \end{array} \right\}.$$

We give some properties of the spaces $\mathcal{M}_{aap}^M$ and $\mathcal{N}_{aap}^M$.

Proposition 3.2. (i) *We have the null characterization of* $\mathcal{N}_{aap}^M$, *i.e.*

$$\mathcal{N}_{aap}^M := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}^M : \forall k > 0, \\ \|u_\varepsilon\|_{L^\infty(\mathbb{R})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0 \end{array} \right\}.$$

(ii) *The space* $\mathcal{M}_{aap}^M$ *is an algebra stable under translation and derivation.*

(iii) *The space* $\mathcal{N}_{aap}^M$ *is an ideal of* $\mathcal{M}_{aap}^M$.

Proof. (i) Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}^M$, i.e. $\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,$

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq c_j e^{M\left(\frac{k_j}{\varepsilon}\right)}, \tag{3.1}$$

and $(u_\varepsilon)_\varepsilon$ satisfies the null estimate of zero order, i.e. $\forall k > 0,\ \exists c' > 0,\ \exists \varepsilon_0' \in I,\ \forall \varepsilon < \varepsilon_0',$

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R})} \leq c' e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{3.2}$$

In order to show that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aap}^M$, we use the Landau- Kolmogorov inequality which state that for any $f \in \mathcal{C}^n(\mathbb{R})$, $n \in$

$\mathbb{Z}_+$, we have for every $1 \leq p \leq n$,

$$\left\| f^{(p)} \right\|_{L^\infty(\mathbb{R})} \leq 2\pi \left\| f \right\|_{L^\infty(\mathbb{R})}^{1-\frac{p}{n}} \left\| f^{(n)} \right\|_{L^\infty(\mathbb{R})}^{\frac{p}{n}}.$$

For every $j \in \mathbb{Z}_+$, by using the Landau-Kolmogorov inequality for $p = j$ and $n = 2j$, and due to the estimates (3.1), (3.2), we obtain

$$\begin{aligned} \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} &\leq 2\pi \left\| u_\varepsilon \right\|_{L^\infty(\mathbb{R})}^{1-\frac{1}{2}} \left\| u_\varepsilon^{(2j)} \right\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \\ &\leq 2\pi \left(c' e^{-M\left(\frac{k}{\varepsilon}\right)} \right)^{\frac{1}{2}} \left(c_{2j} e^{M\left(\frac{k_{2j}}{\varepsilon}\right)} \right)^{\frac{1}{2}} \\ &\leq 2\pi \left(c' c_{2j} \right)^{\frac{1}{2}} e^{-\frac{1}{2} M\left(\frac{k}{\varepsilon}\right) + \frac{1}{2} M\left(\frac{k_{2j}}{\varepsilon}\right)}. \end{aligned}$$

By Lemma 2.11, let $k' > 0$ and taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\left(\frac{k_{2j}}{\varepsilon}, \frac{k'}{\varepsilon} \right)$, then

$$e^{-M\left(\frac{k}{\varepsilon}\right) + M\left(\frac{k_{2j}}{\varepsilon}\right)} \leq A M_0 e^{-M\left(\frac{k'}{\varepsilon}\right)}.$$

Consequently, $\forall j \in \mathbb{Z}_+,\ \forall k' > 0,\ \exists C_j = \left(2\pi \left(c' c_{2j} \right)^{\frac{1}{2}} A M_0 \right) > 0,\ \forall \varepsilon < \min\left(\varepsilon_j, \varepsilon_0' \right),$

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq C_j e^{-M\left(\frac{k'}{\varepsilon}\right)},$$

which means that $\left(u_\varepsilon \right)_\varepsilon \in \mathcal{N}_{aap}^M$.

(ii) The stability under translation and derivation of the space $\mathcal{M}_{aap}^M$ is obvious. Let $\left(u_\varepsilon \right)_\varepsilon, \left(v_\varepsilon \right)_\varepsilon \in \mathcal{M}_{aap}^M$, i.e. they satisfy the estimate (3.1), for any $j \in \mathbb{Z}_+$, we have

$$\begin{aligned} \left\| \left(u_\varepsilon v_\varepsilon \right)^{(j)} \right\|_{L^\infty(\mathbb{R})} &\leq \sum_{i+l=j} \frac{j!}{i!l!} \left\| u_\varepsilon^{(i)} \right\|_{L^\infty(\mathbb{R})} \left\| v_\varepsilon^{(l)} \right\|_{L^\infty(\mathbb{R})} \\ &\leq \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l e^{M\left(\frac{k_i}{\varepsilon}\right) + M\left(\frac{k_l}{\varepsilon}\right)}, \end{aligned}$$

due to Lemma 2.11, taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\limits_{i+l=j} \left(\frac{k_i}{\varepsilon}, \frac{k_l}{\varepsilon} \right)$, then

$$e^{M\left(\frac{k_i}{\varepsilon}\right) + M\left(\frac{k_l}{\varepsilon}\right)} \leq A M_0 e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Hence, $\forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \exists C_j = \left(AM_0 \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l \right) > 0,\ \forall \varepsilon < \min_{i+l=j} (\varepsilon_i, \varepsilon_l)$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq C_j e^{M\left(\frac{k}{\varepsilon}\right)},$$

which gives $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$.

(iii) Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{N}^M_{aap}$, i.e. $(u_\varepsilon)_\varepsilon$ satisfies the estimate (3.1) and $(v_\varepsilon)_\varepsilon$ satisfies $\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\left\| v_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq c_j e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{3.3}$$

For every $j \in \mathbb{Z}_+$ and by (3.1) and (3.3),

$$\begin{aligned} \left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} &\leq \sum_{i+l=j} \frac{j!}{i!l!} \left\| u_\varepsilon^{(i)} \right\|_{L^\infty(\mathbb{R})} \left\| v_\varepsilon^{(l)} \right\|_{L^\infty(\mathbb{R})} \\ &\leq \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l e^{M\left(\frac{k_i}{\varepsilon}\right)} e^{-M\left(\frac{k}{\varepsilon}\right)}, \end{aligned}$$

by Lemma 2.11, let $k' > 0$ and taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\left(\frac{k_i}{\varepsilon}, \frac{k'}{\varepsilon}\right)$, then

$$e^{M\left(\frac{ki}{\varepsilon}\right)} e^{-M\left(\frac{k}{\varepsilon}\right)} \leq AM_0 e^{-M\left(\frac{k'}{\varepsilon}\right)}.$$

Hence, $\forall j \in \mathbb{Z}_+,\ \forall k' > 0,\ \exists C_j = \left(AM_0 \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l \right) > 0,\ \forall \varepsilon < \min_{i+l=j} (\varepsilon_i, \varepsilon_l)$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq C_j e^{-M\left(\frac{k'}{\varepsilon}\right)},$$

so, $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}^M_{aap}$.

□

Now we give the definition of asymptotically almost periodic generalized ultradistributions.

Definition 3.3. The set of asymptotically almost periodic generalized ultradistributions, is denoted and defined by the quotient

$$\mathcal{G}^M_{aap} := \frac{\mathcal{M}^M_{aap}}{\mathcal{N}^M_{aap}}.$$

The next result follows from Proposition 3.2.

Proposition 3.4. *The set of asymptotically almost periodic generalized ultradistributions is an algebra.*

Example 3.5. We have $\mathcal{G}_{aap} \subsetneq \mathcal{G}_{aap}^{M}$, where $\mathcal{G}_{aap}$ is the algebra of asymptotically almost periodic generalized functions of [7] defined as the quotient algebra

$$\mathcal{G}_{aap} := \frac{\mathcal{M}_{aap}}{\mathcal{N}_{aap}},$$

where

$$\mathcal{M}_{aap} := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aap})^I : \forall j \in \mathbb{Z}_+, \ \exists k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} = O\left(\varepsilon^{-k}\right), \ \ \varepsilon \to 0 \end{array} \right\}$$

and

$$\mathcal{N}_{aap} := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aap})^I : \forall j \in \mathbb{Z}_+, \ \forall k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} = O\left(\varepsilon^{k}\right), \ \ \varepsilon \to 0 \end{array} \right\}.$$

For more details on $\mathcal{G}_{aap}$ see [7].

Moreover, we have the following canonical embedding of $\mathcal{G}_{aap}$ into $\mathcal{G}_{aap}^{M}$.

Proposition 3.6. *The map*

$$\begin{array}{rlcl} I_{aap}: & \mathcal{G}_{aap} & \longrightarrow & \mathcal{G}_{aap}^{M} \\ & (u_\varepsilon)_\varepsilon + \mathcal{N}_{aap} & \longmapsto & (u_\varepsilon)_\varepsilon + \mathcal{N}_{aap}^{M} \end{array}$$

is a linear embedding.

Proof. It remains to prove $\mathcal{M}_{aap} \subset \mathcal{M}_{aap}^{M}$ and $\mathcal{M}_{aap} \cap \mathcal{N}_{aap}^{M} \subset \mathcal{N}_{aap}$. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}$, i.e. $\forall j \in \mathbb{Z}_+, \ \exists k_j > 0, \ \exists c_j > 0, \ \exists \varepsilon_j \in I, \ \forall \varepsilon < \varepsilon_j$,

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq c_j \varepsilon^{-k_j}. \tag{3.4}$$

Due to Proposition 2.10-(i),

$$M_n = \sup_{t>0} \frac{t^n M_0}{e^{M(t)}}, \quad \forall n \in \mathbb{Z}_+,$$

hence, $\forall n \in \mathbb{Z}_+, \ \forall k > 0, \ \forall \varepsilon > 0$,

$$e^{M\left(\frac{k}{\varepsilon}\right)} \geq \frac{k^n M_0}{M_n} \varepsilon^{-n}. \tag{3.5}$$

By (3.4) and (3.5), taking $k_j' = -\left[k_j\right] - 1$, then $\forall j \in \mathbb{Z}_+, \ \exists k_j > 0, \ \exists c_j > 0, \ \exists \varepsilon_j \in I, \ \forall \varepsilon < \varepsilon_j$,

$$\begin{aligned} \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} &\leq c_j \varepsilon^{-k_j} \\ &\leq c_j \varepsilon^{-[k_j]-1} \end{aligned}$$

$$\leq c_j \frac{M_{k'_j}}{k_j'^{k'_j} M_0} e^{M\left(\frac{k'_j}{\varepsilon}\right)},$$

so, $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}^M$. Thus, $\mathcal{M}_{aap} \subset \mathcal{M}_{aap}^M$. Let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aap}^M$, i.e. $\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq c_j e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{3.6}$$

We have, $\forall n \in \mathbb{Z}_+,\ \forall k > 0,\ \forall \varepsilon > 0$,

$$\varepsilon^n \geq \frac{k^n M_0}{M_n} e^{-M\left(\frac{k}{\varepsilon}\right)},$$

from (3.6), taking $k' = [k]$, then $\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\begin{aligned}\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} &\leq c_j e^{-M\left(\frac{k}{\varepsilon}\right)} \\ &\leq c_j e^{-M\left(\frac{k'}{\varepsilon}\right)} \\ &\leq c_j \frac{M_{k'}}{k'^{k'} M_0} \varepsilon^{k'},\end{aligned}$$

so $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aap}$. Thus, $\mathcal{N}_{aap}^M \subset \mathcal{N}_{aap}$. Moreover, $\mathcal{M}_{aap} \cap \mathcal{N}_{aap}^M \subset \mathcal{N}_{aap}^M \subset \mathcal{N}_{aap}$. □

Now, we give other important examples of asymptotically almost periodic generalized ultradistributions.

Definition 3.7. Let $T \in \mathcal{D}'_{L^\infty,(M)}$ such that there exist an ultradifferential operator P of class (M), $f \in \mathcal{C}_{aap}$ and $g \in \mathcal{C}_{aap}$ such that $T = P(D) f + g$. We denote by $E'_{aap,(M)}$ the space of such ultradistributions.

Let $\rho \in \mathcal{D}_{L^1}^{\{N\}} := ind \lim_{h \to +\infty} \mathcal{D}_{L^p}^{N,h}$ and set $\rho_\varepsilon(\cdot) = \frac{1}{\varepsilon}\rho\left(\frac{\cdot}{\varepsilon}\right),\ \varepsilon > 0$.

Proposition 3.8. *Let M and N be two sequences satisfying $(M.1)$, $(M.2)$ and $(M.3)'$, then the map*

$$\begin{array}{cccc} J_{aap}: & E'_{aap,(MN)} & \longrightarrow & \mathcal{G}_{aap}^M \\ & T & \longmapsto & (T * \rho_\varepsilon)_\varepsilon + \mathcal{N}_{aap}^M \end{array}$$

is a linear embedding.

Proof. Let $T \in E'_{aap,(MN)}$ then $T = P(D) f + g$, where P is an ultradifferential operator of class (MN), $f \in \mathcal{C}_{aap}$ and $g \in \mathcal{C}_{aap}$. Due to Young inequality, we have $\forall j \in \mathbb{Z}_+,\ \forall x \in \mathbb{R}$,

$$\left| (T * \rho_\varepsilon)^{(j)}(x) \right| \leq \left| f * P(D) \rho_\varepsilon^{(j)}(x) \right| + \left| g * \rho_\varepsilon^{(j)}(x) \right|$$

$$\leq \|f\|_{L^\infty(\mathbb{R})} \sum_{i\in\mathbb{Z}_+} |a_i| \frac{1}{\varepsilon^{i+j}} \int_{\mathbb{R}} \left|\rho^{(i+j)}(y)\right| dy$$
$$+ \|g\|_{L^\infty(\mathbb{R})} \frac{1}{\varepsilon^j} \int_{\mathbb{R}} \left|\rho^{(j)}(y)\right| dy.$$

On the other hand, as $\rho \in \mathcal{D}_{L^1}^{\{N\}}$ then $\exists h > 0$ such that $\|\rho\|_{1,h,N} < \infty$. As $P(D) = \sum_{i\in\mathbb{Z}_+} a_i D^i$, is an ultradifferential operator of class (MN), so $\exists L > 0$ and $\exists c > 0$ such that $\forall i \in \mathbb{Z}_+,\ |a_i| \leq cL^i (M_i N_i)^{-1}$. It follows that

$$\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leq c\|f\|_{L^\infty(\mathbb{R})} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_i N_i} \frac{h^{i+j}}{\varepsilon^{i+j}} \frac{\left\|\rho^{(i+j)}\right\|_{L^1(\mathbb{R})}}{h^{i+j}}$$
$$+ \|g\|_{L^\infty(\mathbb{R})} \frac{h^j}{\varepsilon^j} \frac{\left\|\rho^{(j)}\right\|_{L^1(\mathbb{R})}}{h^j}.$$

Since M and N satisfy $(M.2)$, there exist $A,\ A' > 0$ and $H,\ H' > 0$ such that $M_{i+j} \leq AH^{i+j} M_i M_j$ and $N_{i+j} \leq A' H'^{i+j} N_i N_j$, which give

$$\frac{1}{M_i N_i} \leq \frac{AA'(HH')^{i+j}}{M_{i+j} N_{i+j}} M_j N_j.$$

Therefore,

$$\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leq cAA'\|f\|_{L^\infty(\mathbb{R})} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_{i+j} N_{i+j}}$$
$$\times \left(HH'\right)^{i+j} \frac{h^{i+j}}{\varepsilon^{i+j}} M_j N_j \frac{\left\|\rho^{(i+j)}\right\|_{L^1(\mathbb{R})}}{h^{i+j}}$$
$$+ \|g\|_{L^\infty(\mathbb{R})} \frac{h^j}{\varepsilon^j} \frac{\left\|\rho^{(j)}\right\|_{L^1(\mathbb{R})}}{h^j},$$

hence,

$$\frac{1}{M_j N_j} \left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leq cAA'\|f\|_{L^\infty(\mathbb{R})} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_{i+j} N_{i+j}}$$
$$\times \left(HH'\right)^{i+j} \frac{h^{i+j}}{\varepsilon^{i+j}} \frac{\left\|\rho^{(i+j)}\right\|_{L^1(\mathbb{R})}}{h^{i+j}}$$
$$+ \|g\|_{L^\infty(\mathbb{R})} \frac{h^j}{\varepsilon^j} \frac{1}{M_j N_j} \frac{\left\|\rho^{(j)}\right\|_{L^1(\mathbb{R})}}{h^j}$$
$$\leq cAA'\|f\|_{L^\infty(\mathbb{R})} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_{i+j}}$$

$$\times \left(HH'\right)^{i+j} \frac{h^{i+j}}{\varepsilon^{i+j}} \frac{\left\|\rho^{(i+j)}\right\|_{L^1(\mathbb{R})}}{h^{i+j} N_{i+j}}$$

$$+ \|g\|_{L^\infty(\mathbb{R})} \frac{h^j}{\varepsilon^j} \frac{1}{M_j} \frac{\left\|\rho^{(j)}\right\|_{L^1(\mathbb{R})}}{h^j N_j}$$

$$\leq cAA' \|f\|_{L^\infty(\mathbb{R})} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_{i+j}}$$

$$\times \left(HH'\right)^{i+j} \frac{h^{i+j}}{\varepsilon^{i+j}} \|\rho\|_{1,h,N}$$

$$+ \|g\|_{L^\infty(\mathbb{R})} \frac{h^j}{\varepsilon^j} \frac{1}{M_j} \|\rho\|_{1,h,N}.$$

Thus,

$$\frac{(2L)^j}{M_j N_j} \left\|(T * \rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leq \|\rho\|_{1,h,N} \Big(cAA' \|f\|_{L^\infty(\mathbb{R})} \times \sum_{i\in\mathbb{Z}_+} 2^{-i} \frac{(2L)^{i+j}}{M_{i+j}} \left(HH'\right)^{i+j} \frac{h^{i+j}}{\varepsilon^{i+j}} + \|g\|_{L^\infty(\mathbb{R})} (2L)^j \frac{h^j}{\varepsilon^j} \frac{1}{M_j} \Big),$$

i.e.

$$\frac{(2L)^j}{M_j N_j} \left\|(T * \rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leq \|\rho\|_{1,h,N} \Big(cAA' \|f\|_{L^\infty(\mathbb{R})} \times \sum_{i\in\mathbb{Z}_+} 2^{-i} \frac{\left(\frac{2LHH'h}{\varepsilon}\right)^{i+j}}{M_{i+j}} + \|g\|_{L^\infty(\mathbb{R})} \frac{\left(\frac{2Lh}{\varepsilon}\right)^j}{M_j} \Big).$$

The fact that M satisfies $(M.1)$, so Proposition 2.10-(i), gives

$$\frac{\left(\frac{2LHH'h}{\varepsilon}\right)^{i+j}}{M_{i+j}} \leq \frac{1}{M_0} e^{M\left(\frac{2LHH'h}{\varepsilon}\right)}$$

and

$$\frac{\left(\frac{2Lh}{\varepsilon}\right)^j}{M_j} \leq \frac{1}{M_0} e^{M\left(\frac{2Lh}{\varepsilon}\right)}.$$

Consequently,

$$\begin{aligned}\frac{(2L)^j}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{M_0}\|\rho\|_{1,h,N}\Big(2cAA'\|f\|_{L^\infty(\mathbb{R})}\\ &\quad\times e^{M\left(\frac{2LHH'h}{\varepsilon}\right)}+\|g\|_{L^\infty(\mathbb{R})}e^{M\left(\frac{2Lh}{\varepsilon}\right)}\Big)\\ &\leq \frac{C}{M_0}\left(e^{M\left(\frac{2LHH'h}{\varepsilon}\right)}+e^{M\left(\frac{2Lh}{\varepsilon}\right)}\right),\end{aligned}$$

where $C=\|\rho\|_{1,h,N}\max\Big(2cAA'\|f\|_{L^\infty(\mathbb{R})},\|g\|_{L^\infty(\mathbb{R})}\Big)$. Hence,

$$\frac{(2L)^j}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}\leq\frac{2C}{M_0}e^{M\left(\frac{2LHH'h}{\varepsilon}\right)+M\left(\frac{2Lh}{\varepsilon}\right)}.$$

Due to Lemma 2.11, let $k>0$ such that $\frac{k}{\varepsilon}=H\max\left(\frac{2LHH'h}{\varepsilon},\frac{2Lh}{\varepsilon}\right)$ and $C'_j=\left(2AC\frac{M_jN_j}{(2L)^j}\right)>0$, we get

$$\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}\leq C'_je^{M\left(\frac{k}{\varepsilon}\right)},$$

which means that $(T*\rho_\varepsilon)\in\mathcal{M}^M_{aap}$. The linearity follows from the fact that the convolution is linear. Let $\rho\in\mathcal{D}^{\{N\}}_{L^1}$ such that $\int_{\mathbb{R}}\rho(x)\,dx=1$. If $(T*\rho_\varepsilon)\in\mathcal{N}^M_{aap}$, then $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0$,

$$\|T*\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{3.7}$$

Let $\psi\in\mathcal{D}^{(MN)}_{L^1}$, we have

$$\langle T,\psi\rangle=\lim_{\varepsilon\to0}\int_{\mathbb{R}}(T*\rho_\varepsilon)(x)\,\psi(x)\,dx.$$

From (3.7), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}}(T*\rho_\varepsilon)(x)\,\psi(x)\,dx\right| &\leq \|\psi\|_{L^1(\mathbb{R})}\|T*\rho_\varepsilon\|_{L^\infty(\mathbb{R})}\\ &\leq c\|\psi\|_{L^1(\mathbb{R})}e^{-M\left(\frac{k}{\varepsilon}\right)},\end{aligned}$$

let $\varepsilon\to0$, thus $\langle T,\psi\rangle=0,\forall\psi\in\mathcal{D}^{(MN)}_{L^1}$. Hence, J_{aap} is injective. □

Remark 3.9. In addition, if the sequence M satisfies the condition $(M.3)$, then due to Theorem 3 of [21], the space $E'_{aap,(M)}$ coincides with the space of asymptotically almost periodic Beurling ultradistributions studied in [21]. Therefore, in view of Proposition 3.8, the space of asymptotically almost periodic Beurling ultradistributions is embedded into the algebra of asymptotically almost periodic generalized ultradistributions.

In order to establish some properties of the algebra $\mathcal{G}_{aap}^{M}$, we recall some needed algebras of generalized ultradistributions of [9]. The algebra of almost periodic generalized ultradistributions is denoted and defined by

$$\mathcal{G}_{ap}^{M} := \frac{\mathcal{M}_{ap}^{M}}{\mathcal{N}_{ap}^{M}},$$

where

$$\mathcal{M}_{ap}^{M} := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)} \right),\ \ \varepsilon \to 0 \end{array} \right\}$$

and

$$\mathcal{N}_{ap}^{M} := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)} \right),\ \ \varepsilon \to 0 \end{array} \right\}.$$

The algebra of L^p-generalized ultradistributions on $\mathbb{I},\ p \in [1, +\infty]$, is denoted and defined by the quotient algebra

$$\mathcal{G}_{L^p}^{M}(\mathbb{I}) := \frac{\mathcal{M}_{L^p}^{M}(\mathbb{I})}{\mathcal{N}_{L^p}^{M}(\mathbb{I})},$$

where

$$\mathcal{M}_{L^p}^{M}(\mathbb{I}) := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^p}(\mathbb{I}))^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^p(\mathbb{I})} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)} \right),\ \ \varepsilon \to 0 \end{array} \right\}$$

and

$$\mathcal{N}_{L^p}^{M}(\mathbb{I}) := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^p}(\mathbb{I}))^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^p(\mathbb{I})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)} \right),\ \ \varepsilon \to 0 \end{array} \right\}.$$

Notation 1. Denote by $\mathcal{G}_{\mathcal{B}}^{M}(\mathbb{I}) := \mathcal{G}_{L^\infty}^{M}(\mathbb{I})$, $\mathcal{G}_{\mathcal{B}}^{M} := \mathcal{G}_{L^\infty}^{M}(\mathbb{R})$ and $\mathcal{G}_{L^1}^{M} := \mathcal{G}_{L^1}^{M}(\mathbb{R})$.

The following result summarise some properties of $\mathcal{G}_{aap}^{M}$.

Proposition 3.10. (i) *$\mathcal{G}_{aap}^{M}$ is a subalgebra of $\mathcal{G}_{\mathcal{B}}^{M}$ stable under translation and derivation.*
(ii) $\mathcal{G}_{aap}^{M} \times \mathcal{G}_{ap}^{M} \subset \mathcal{G}_{aap}^{M}$.
(iii) $\mathcal{G}_{aap}^{M} * \mathcal{G}_{L^1}^{M} \subset \mathcal{G}_{aap}^{M}$.

Proof. (i) It follows from Proposition 3.2-(ii) that $\mathcal{G}_{aap}^{M}$ is an algebra stable under translation and derivation. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}^{M}$, so

$(u_\varepsilon)_\varepsilon$ satisfies (3.1) and since $\mathcal{B}_{aap} \subset \mathcal{B}$, then $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathcal{B}}$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{aap}$, so $(u_\varepsilon)_\varepsilon$ satisfies (3.3) and then $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathcal{B}}$.

(ii) Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{aap}$ and $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{ap}$, if $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{ap}$, then in view of Proposition 2.7-(ii), it follows that $\forall \varepsilon \in I,\ u_\varepsilon v_\varepsilon \in \mathcal{B}_{aap}$. As $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{ap}$, so they satisfy the estimate (3.1). For every $j \in \mathbb{Z}_+$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l e^{M\left(\frac{k_i}{\varepsilon}\right)+M\left(\frac{k_l}{\varepsilon}\right)},$$

by Lemma 2.11, taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\limits_{i+l=j} \left(\frac{k_i}{\varepsilon}, \frac{k_l}{\varepsilon}\right)$, then

$$e^{M\left(\frac{k_i}{\varepsilon}\right)+M\left(\frac{k_l}{\varepsilon}\right)} \leq AM_0 e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Thus, $\forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \exists C_j = \left(AM_0 \sum\limits_{i+l=j} \frac{j!}{i!l!} c_i c_l\right) > 0,\ \forall \varepsilon < \min\limits_{i+l=j} (\varepsilon_i, \varepsilon_l)$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq C_j e^{M\left(\frac{k}{\varepsilon}\right)},$$

which gives $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$. It is easy to show that the product $\widetilde{u} \times \widetilde{v}$ does not depend on the representatives $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$.

(iii) Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^1}$ be a respective representatives of $\widetilde{u} \in \mathcal{G}^M_{aap}$ and $\widetilde{v} \in \mathcal{G}^M_{L^1}$. If $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$ so it satisfies the estimate (3.1) and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^1}$, i.e. satisfies $\forall j \in \mathbb{Z}_+,\ \exists k'_j > 0,\ \exists c'_j > 0,\ \exists \varepsilon'_j \in I,\ \forall \varepsilon < \varepsilon'_j$,

$$\left\| v_\varepsilon^{(j)} \right\|_{L^1(\mathbb{R})} \leq c'_j e^{M\left(\frac{k'_j}{\varepsilon}\right)}.$$

In view of Proposition 2.7-(iii), $\forall \varepsilon \in I,\ (u_\varepsilon * v_\varepsilon) \in \mathcal{B}_{aap}$ and due to Young inequality, we obtain for every $j \in \mathbb{Z}_+$,

$$\begin{aligned} \left\| (u_\varepsilon * v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} &\leq \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \| v_\varepsilon \|_{L^1(\mathbb{R})} \\ &\leq c_j c'_0 e^{M\left(\frac{k_j}{\varepsilon}\right)} e^{M\left(\frac{k'_0}{\varepsilon}\right)}, \end{aligned}$$

by Lemma 2.11, let $k > 0$ such that $\frac{k}{\varepsilon} = H \max\left(\frac{k_j}{\varepsilon}, \frac{k'_0}{\varepsilon}\right)$, then

$$e^{M\left(\frac{k_j}{\varepsilon}\right)} e^{M\left(\frac{k'_0}{\varepsilon}\right)} \leq AM_0 e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Consequently, $\forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \exists C_j = (c_j c_0' A M_0) > 0,\ \forall \varepsilon < \min(\varepsilon_j, \varepsilon_0'),$

$$\left\| (u_\varepsilon * v_\varepsilon)^{(j)} \right\|_{L^\infty(\mathbb{R})} \leq C_j e^{M\left(\frac{k}{\varepsilon}\right)},$$

this gives that $(u_\varepsilon * v_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}^M$. It is easy to prove that the convolution does not depend on the representatives $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$.

□

4. Extension of Generalized Ultradistributions

The uniqueness of the decomposition of an asymptotically almost periodic generalized ultradistribution requires an extension result in the context of bounded generalized ultradistributions. Which is a generalization of the Seeley theorem, see [24].

Lemma 4.1. *There are two sequences of real numbers* $(a_l)_{l \in \mathbb{Z}_+}$ *and* $(b_l)_{l \in \mathbb{Z}_+}$ *such that*

(i) $b_l < 0,\ \forall l \in \mathbb{Z}_+$.

(ii) $\sum\limits_{l=0}^{+\infty} |a_l| \, |b_l|^n < +\infty,\ \forall n \in \mathbb{Z}_+$.

(iii) $\sum\limits_{l=0}^{+\infty} a_l b_l^n = 1,\ \forall n \in \mathbb{Z}_+$.

(iv) $b_l \to -\infty,\ l \to +\infty$.

Proof. See [24]. □

Define the space

$$\mathcal{B}_{+,0}(\mathbb{I}) := \left\{ \varphi \in \mathcal{B}(\mathbb{I}) : \forall j \in \mathbb{Z}_+,\ \lim_{x \to +\infty} \varphi^{(j)}(x) = 0 \right\}.$$

The algebra of bounded generalized ultradistributions vanishing at infinity on $\mathbb{I}$, is defined as the quotient algebra

$$\mathcal{G}_{+,0}^M(\mathbb{I}) := \frac{\mathcal{M}_{+,0}^M(\mathbb{I})}{\mathcal{N}_{+,0}^M(\mathbb{I})},$$

where

$$\mathcal{M}_{+,0}^M(\mathbb{I}) := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{+,0}(\mathbb{I}))^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{I})} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)} \right),\ \ \varepsilon \to 0 \end{array} \right\}$$

and

$$\mathcal{N}_{+,0}^M(\mathbb{I}) := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{+,0}(\mathbb{I}))^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0, \\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{I})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)} \right),\ \ \varepsilon \to 0 \end{array} \right\}.$$

Theorem 4.2. *The linear extension operator* $\widetilde{E}:\mathcal{G}_{\mathcal{B}}^{M}(\mathbb{J})\longrightarrow\mathcal{G}_{\mathcal{B}}^{M}(\mathbb{R})$, $\widetilde{u}=[(u_{\varepsilon})_{\varepsilon}]\longmapsto\widetilde{E}\widetilde{u}=[(Eu_{\varepsilon})_{\varepsilon}]$, *where*

$$Eu_{\varepsilon}(x):=\begin{cases} u_{\varepsilon}(x), & \text{if} \quad x\geq 0,\\ \sum\limits_{l=0}^{+\infty}a_{l}u_{\varepsilon}(b_{l}x), & \text{if} \quad x<0,\end{cases}$$

is well defined and we have $\widetilde{E}\widetilde{u}_{|\mathbb{J}}=\widetilde{u}$. *In particular,* $\forall\widetilde{u}\in\mathcal{G}_{+,0}^{M}(\mathbb{J})$, $\widetilde{E}\widetilde{u}\in\mathcal{G}_{+,0}^{M}(\mathbb{R})$.

Proof. Let $\widetilde{u}=[(u_{\varepsilon})_{\varepsilon}]\in\mathcal{G}_{\mathcal{B}}^{M}(\mathbb{J})$, and $(u_{\varepsilon})_{\varepsilon}\in\mathcal{M}_{\mathcal{B}}^{M}(\mathbb{J})$ be a representative of $\widetilde{u}$. Then $\forall\varepsilon\in I$, $Eu_{\varepsilon}\in\mathcal{B}(\mathbb{R})$ and $Eu_{\varepsilon|\mathbb{J}}=u_{\varepsilon}$. Indeed, due to Lemma 4.1-(i) for every $x<0$ we get $b_{l}x>0$, $\forall l\in\mathbb{Z}_{+}$. For any $\varepsilon\in I$, $j\in\mathbb{Z}_{+}$, $x<0$, we have

$$(Eu_{\varepsilon})^{(j)}(x)=\sum_{l=0}^{+\infty}a_{l}b_{l}^{j}u_{\varepsilon}^{(j)}(b_{l}x),\tag{4.1}$$

as $u_{\varepsilon}\in\mathcal{B}(\mathbb{J})$, $\forall\varepsilon\in I$, and in view of Lemma 4.1-(ii), we obtain $\forall j\in\mathbb{Z}_{+}$, $\forall\varepsilon\in I$, $\forall x<0$,

$$\left|(Eu_{\varepsilon})^{(j)}(x)\right|\leq\left\|u_{\varepsilon}^{(j)}\right\|_{L^{\infty}(\mathbb{J})}\sum_{l=0}^{+\infty}|a_{l}|\,|b_{l}|^{j}<+\infty,\tag{4.2}$$

consequently, $\forall j\in\mathbb{Z}_{+}$, the series given in (4.1) are absolutely converge. Moreover, by Lemma 4.1-(iii),

$$\begin{aligned}\lim_{\substack{x\to 0\\ <}}(Eu_{\varepsilon})^{(j)}(x)&=\sum_{l=0}^{+\infty}a_{l}b_{l}^{j}\lim_{\substack{x\to 0\\ <}}u_{\varepsilon}^{(j)}(b_{l}x)\\ &=u_{\varepsilon}^{(j)}(0)\sum_{l=0}^{+\infty}a_{l}b_{l}^{j}\\ &=u_{\varepsilon}^{(j)}(0),\end{aligned}$$

hence, $\forall\varepsilon\in I$, $Eu_{\varepsilon}\in\mathcal{E}(\mathbb{R})$. From (4.2), it holds $\forall\varepsilon\in I$, $\forall j\in\mathbb{Z}_{+}$,

$$\left\|(Eu_{\varepsilon})^{(j)}\right\|_{L^{\infty}(\mathbb{R}\setminus\mathbb{J})}\leq\left\|u_{\varepsilon}^{(j)}\right\|_{L^{\infty}(\mathbb{J})}\sum_{l=0}^{+\infty}|a_{l}|\,|b_{l}|^{j}$$

also

$$\left\|(Eu_{\varepsilon})^{(j)}\right\|_{L^{\infty}(\mathbb{J})}=\left\|u_{\varepsilon}^{(j)}\right\|_{L^{\infty}(\mathbb{J})}.$$

Consequently,

$$\left\|(Eu_{\varepsilon})^{(j)}\right\|_{L^{\infty}(\mathbb{R})}\leq\max\left(1,\sum_{l=0}^{+\infty}|a_{l}|\,|b_{l}|^{j}\right)\left\|u_{\varepsilon}^{(j)}\right\|_{L^{\infty}(\mathbb{J})},\tag{4.3}$$

as $\forall\varepsilon \in I,\ \forall j \in \mathbb{Z}_+,\ u_\varepsilon^{(j)} \in L^\infty(\mathbb{J})$ and by (4.3) we obtain $\forall\varepsilon \in I,\ \forall j \in \mathbb{Z}_+,\ (Eu_\varepsilon)^{(j)} \in L^\infty(\mathbb{R})$. If $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{B}}^M(\mathbb{J})$ then it follows from (4.3) that $(Eu_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{B}}^M(\mathbb{R})$. The operator $\widetilde{E}\widetilde{u}$ is well-defined. Indeed, if $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ are representatives of $\widetilde{u}$, thus $\forall\varepsilon \in I,\ \forall j \in \mathbb{Z}_+,$

$$\left\|(Eu_\varepsilon - Ev_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \leq \max\left(1, \sum_{l=0}^{+\infty} |a_l|\,|b_l|^j\right)\left\|(u_\varepsilon - v_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{J})},$$

since $(u_\varepsilon - v_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{B}}^M(\mathbb{J})$ then $\forall j \in \mathbb{Z}_+,\ \forall k > 0,$

$$\left\|(Eu_\varepsilon - Ev_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right), \quad \varepsilon \to 0,$$

which means that $(Eu_\varepsilon - Ev_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{B}}^M(\mathbb{R})$.

The fact that $\widetilde{E}\widetilde{u} = [(Eu_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{B}}^M(\mathbb{R})$ and $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{B}}^M(\mathbb{J})$ gives $\widetilde{E}\widetilde{u}_{|\mathbb{J}} - \widetilde{u} := \left[\left(Eu_{\varepsilon|\mathbb{J}}\right)_\varepsilon\right] - \widetilde{u} = [(u_\varepsilon)_\varepsilon] - \widetilde{u} = \widetilde{u} - \widetilde{u} = 0$ in $\mathcal{G}_{\mathcal{B}}^M(\mathbb{J})$. Consequently, $\widetilde{E}\widetilde{u}_{|\mathbb{J}} = \widetilde{u}$ in $\mathcal{G}_{\mathcal{B}}^M(\mathbb{J})$.

If $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{+,0}^M(\mathbb{J}) \subset \mathcal{G}_{\mathcal{B}}^M(\mathbb{J})$, then $\widetilde{E}\widetilde{u} = [(Eu_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{B}}^M(\mathbb{R})$. We have, $\forall\varepsilon \in I,\ Eu_\varepsilon \in \mathcal{B}(\mathbb{R})$. As $\forall\varepsilon \in I,\ \forall j \in \mathbb{Z}_+,\ (Eu_\varepsilon)^{(j)} = u_\varepsilon^{(j)}$ on $\mathbb{J}$, so $\lim\limits_{x\to+\infty}(Eu_\varepsilon)^{(j)}(x) = 0$, i.e. $\forall\varepsilon \in I,\ Eu_\varepsilon \in \mathcal{B}_{+,0}(\mathbb{R})$. Thus, $\widetilde{E}\widetilde{u} \in \mathcal{G}_{+,0}^M(\mathbb{R})$. □

5. The Decomposition

In this section we show that an asymptotically almost periodic generalized ultradistribution is uniquely decomposed. The following results are needed in the sequel, see [12].

Lemma 5.1. *Let $f \in \mathcal{B}_{aap}$ such that $f = g + h$ on $\mathbb{J}$ and for $j \in \mathbb{Z}_+,\ f^{(j)} = g_j + h_j$ on $\mathbb{J}$. Then $g_j = (g)^{(j)}$ on $\mathbb{R}$ and $h_j = (h)^{(j)}$ on $\mathbb{J}$.*

Lemma 5.2. *If $f = (g+h) \in \mathcal{C}_{aap}$, then $\|g\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{J})}$. Furthermore, if $f \in \mathcal{C}_{ap}$ and $\omega \in \mathbb{R}$, $\|f\|_{L^\infty(\mathbb{R})} = \sup\limits_{x\geq\omega}|f(x)|$.*

Theorem 5.3. *Let $\widetilde{u} \in \mathcal{G}_{aap}^M(\mathbb{R})$ then there exist $\widetilde{v} \in \mathcal{G}_{ap}^M(\mathbb{R})$ and $\widetilde{w} \in \mathcal{G}_{+,0}^M(\mathbb{R})$ such that $\widetilde{u} = \widetilde{v} + \widetilde{w}$ on $\mathbb{J}$, and the decomposition is unique on $\mathbb{J}$.*

Proof. Let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{aap}^M$, then $\forall\varepsilon \in I,\ \forall j \in \mathbb{Z}_+,\ u_\varepsilon^{(j)} \in \mathcal{C}_{aap}$. So there exist $v_{\varepsilon,j} \in \mathcal{C}_{ap},\ w_{\varepsilon,j} \in \mathcal{C}_{+,0}$, such that $\forall j \in \mathbb{N},\ u_\varepsilon^{(j)} = (v_{\varepsilon,j} + w_{\varepsilon,j}) \in \mathcal{C}_{aap}$ on $\mathbb{J}$, and for $j = 0$, $u_\varepsilon = v_\varepsilon + w_\varepsilon$ on $\mathbb{J}$. By Lemma 5.1, it holds that $\forall j \in \mathbb{N}$, $v_{\varepsilon,j} = (v_\varepsilon)^{(j)}$ on $\mathbb{R}$ and $w_{\varepsilon,j} = (w_\varepsilon)^{(j)}$ on $\mathbb{J}$, which gives $v_\varepsilon \in \mathcal{B}_{ap}$

and $w_\varepsilon \in \mathcal{B}_{+,0}(\mathbb{J})$. Now, we show $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}^M$. If $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aap}^M$, then $\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \le c_j e^{M\left(\frac{k_j}{\varepsilon}\right)}. \tag{5.1}$$

Due to Lemma 5.2, we obtain, $\forall j \in \mathbb{Z}_+$,

$$\left\| v_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \le \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})}, \tag{5.2}$$

hence, $\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\left\| v_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \le c_j e^{M\left(\frac{k_j}{\varepsilon}\right)}, \tag{5.3}$$

which gives $(v_\varepsilon)_\varepsilon \in \mathcal{M}_{ap}^M$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aap}^M$, so $\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{R})} \le c_j e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{5.4}$$

It follows due to (5.2) and (5.4) that $(v_\varepsilon)_\varepsilon \in \mathcal{N}_{ap}^M$. Therefore, $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{ap}^M$. On the other hand, it is easy to see that $\forall j \in \mathbb{Z}_+$,

$$\left\| w_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})} \le \left\| u_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})} + \left\| v_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})}. \tag{5.5}$$

The estimates (5.1) ,(5.3) and (5.5) give $\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j$,

$$\left\| w_\varepsilon^{(j)} \right\|_{L^\infty(\mathbb{J})} \le 2c_j e^{M\left(\frac{k_j}{\varepsilon}\right)},$$

hence $(w_\varepsilon)_\varepsilon \in \mathcal{M}_{+,0}^M(\mathbb{J})$. If $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aap}^M$, then we get $(w_\varepsilon)_\varepsilon \in \mathcal{N}_{+,0}^M(\mathbb{J})$ from (5.4) and (5.5). Consequently, $\widetilde{w} = [(w_\varepsilon)_\varepsilon] \in \mathcal{G}_{+,0}^M(\mathbb{J})$. Due to Theorem 4.2 extending $\widetilde{w} \in \mathcal{G}_{+,0}^M(\mathbb{J})$ to $\widetilde{E}\widetilde{w} \in \mathcal{G}_{+,0}^M(\mathbb{R})$ with $\widetilde{E}\widetilde{w} = \widetilde{w}$ on $\mathbb{J}$. Finally, $\widetilde{u} = \widetilde{v} + \widetilde{w}$ on $\mathbb{J}$.

Now, we show that the decomposition $\widetilde{u} = \widetilde{v} + \widetilde{w}$ on $\mathbb{J}$ is unique. Indeed, suppose that there exist $\widetilde{v}_1,\ \widetilde{v}_1 \in \mathcal{G}_{ap}^M$ and $\widetilde{w}_1,\ \widetilde{w}_2 \in \mathcal{G}_{+,0}^M := \mathcal{G}_{+,0}^M(\mathbb{R})$ such that

$$\widetilde{u} = \widetilde{v}_i + \widetilde{w}_i \ \text{ on } \ \mathbb{J}, \quad i = 1,\ 2.$$

To prove that the decomposition is unique and in order to facilitate the calculation we need the following null characterization

$$\mathcal{N}_{\mathcal{B}}^M(\mathbb{I}) := \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{B}}^M(\mathbb{I}) : \forall k > 0, \\ \|u_\varepsilon\|_{L^\infty(\mathbb{I})} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon \to 0 \end{array} \right\}, \tag{5.6}$$

which is proved due the classical Landau-Kolmogorov inequality in the same way as Proposition 3.2-(i). Let $(v_{i,\varepsilon})_\varepsilon \in \mathcal{M}_{ap}^M$ and $(w_{i,\varepsilon})_\varepsilon \in \mathcal{M}_{+,0}^M$

be respectively representatives of $\widetilde{v}_i$ and $\widetilde{w}_i,\ i=1,\ 2.$ So $\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)_\varepsilon+\left(w_{1,\varepsilon}-w_{2,\varepsilon}\right)_\varepsilon\in\mathcal{N}_{\mathcal{B}}^{M}\left(\mathbb{J}\right)$, i.e. $\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)_\varepsilon+\left(w_{1,\varepsilon}-w_{2,\varepsilon}\right)_\varepsilon\in\mathcal{M}_{\mathcal{B}}^{M}\left(\mathbb{J}\right)$ and $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0,$

$$\left\|v_{1,\varepsilon}-v_{2,\varepsilon}+w_{1,\varepsilon}-w_{2,\varepsilon}\right\|_{L^\infty(\mathbb{J})}\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}, \tag{5.7}$$

which gives $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0,\ \forall x\geq 0,$

$$\left|v_{1,\varepsilon}\left(x\right)-v_{2,\varepsilon}\left(x\right)+w_{1,\varepsilon}\left(x\right)-w_{2,\varepsilon}\left(x\right)\right|\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{5.8}$$

For any real sequence $\left(s_m\right)_{m\in\mathbb{N}}$, such that $s_m\rightarrow+\infty$ there exist $\left(s_{m_{l(\varepsilon)}}\right)_l$ a subsequence of $\left(s_m\right)_{m\in\mathbb{N}}$ such that taking the translate at $s_{m_{l(\varepsilon)}}-s_{m_{p(\varepsilon)}}$ in (5.8) and let $l,\ p\rightarrow+\infty$ we obtain $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0,\ \forall x\geq 0,$

$$\left|v_{1,\varepsilon}\left(x\right)-v_{2,\varepsilon}\left(x\right)\right|\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}.$$

Due to Lemma 5.2, it holds $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0,$

$$\left\|v_{1,\varepsilon}-v_{2,\varepsilon}\right\|_{L^\infty(\mathbb{R})}\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{5.9}$$

As $\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)_\varepsilon\in\mathcal{M}_{ap}^{M}\subset\mathcal{M}_{\mathcal{B}}^{M}\left(\mathbb{R}\right)$ and by (5.9), so in view of (5.6) it follows that $\left(v_{1,\varepsilon}-v_{2,\varepsilon}\right)_\varepsilon\in\mathcal{N}_{\mathcal{B}}^{M}\left(\mathbb{R}\right)$, then $\widetilde{v}_1=\widetilde{v}_2$ in $\mathcal{G}_{\mathcal{B}}^{M}\left(\mathbb{R}\right)$. Due to (5.7), (5.9) and as $\left(w_{1,\varepsilon}-w_{2,\varepsilon}\right)_\varepsilon\in\mathcal{M}_{\mathcal{B}}^{M}\left(\mathbb{J}\right)$ it holds in view of (5.6) that $\left(w_{1,\varepsilon}-w_{2,\varepsilon}\right)_\varepsilon\in\mathcal{N}_{\mathcal{B}}^{M}\left(\mathbb{J}\right)$, i.e. $\widetilde{w}_1=\widetilde{w}_2$ in $\mathcal{G}_{\mathcal{B}}^{M}\left(\mathbb{J}\right)$. □

Notation 2. Let $\widetilde{u}\in\mathcal{G}_{aap}^{M}$ and $\widetilde{u}=\widetilde{v}+\widetilde{w}$ on $\mathbb{J}$, where $\widetilde{v}\in\mathcal{G}_{ap}^{M}$ and $\widetilde{w}\in\mathcal{G}_{+,0}^{M}$, then $\widetilde{v}$ and $\widetilde{w}$ are called respectively the principal term and the corrective term of $\widetilde{u}$ and we denote them respectively $\widetilde{u}_{ap}$ and $\widetilde{u}_{cor}$. Also $\widetilde{u}=\left(\widetilde{u}_{ap}+\widetilde{u}_{cor}\right)\in\mathcal{G}_{aap}^{M}$ means that $\widetilde{u}_{ap}\in\mathcal{G}_{ap}^{M}$, $\widetilde{u}_{cor}\in\mathcal{G}_{+,0}^{M}$ and $\widetilde{u}=\widetilde{u}_{ap}+\widetilde{u}_{cor}$ on $\mathbb{J}$.

6. Non-linear Operation

This section shows that the composition of a tempered generalized ultradistribution with an asymptotically almost periodic generalized ultradistribution is an asymptotically almost periodic generalized ultradistribution. First, recall from [9], the algebra of tempered generalized ultradistributions on $\mathbb{C}$, denoted and defined as the quotient algebra

$$\mathcal{G}_{\tau}^{M}\left(\mathbb{C}\right):=\frac{\mathcal{M}_{\tau}^{M}\left(\mathbb{C}\right)}{\mathcal{N}_{\tau}^{M}\left(\mathbb{C}\right)},$$

where

$$\mathcal{M}_{\tau}^{M}\left(\mathbb{C}\right):=\left\{\begin{array}{c}\left(f_\varepsilon\right)_\varepsilon\in\left(\mathcal{E}\left(\mathbb{R}^2\right)\right)^I:\forall j\in\mathbb{Z}_+,\ \exists k>0,\\ \sup\limits_{x\in\mathbb{R}^2}\left(1+\left|x\right|\right)^{-k}\left|f_\varepsilon^{(j)}\left(x\right)\right|=O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon\rightarrow0\end{array}\right\}$$

and

$$\mathcal{N}_{\tau}^{M}\left(\mathbb{C}\right) := \left\{ \begin{array}{c} (f_{\varepsilon})_{\varepsilon} \in \left(\mathcal{E}\left(\mathbb{R}^{2}\right)\right)^{I} : \forall j \in \mathbb{Z}_{+},\ \exists m>0,\ \forall k>0, \\ \sup_{x\in\mathbb{R}^{2}} (1+|x|)^{-m}\left|f_{\varepsilon}^{(j)}(x)\right| = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon\to 0 \end{array} \right\}.$$

Proposition 6.1. *Let* $\widetilde{u}=[(u_{\varepsilon})_{\varepsilon}]\in\mathcal{G}_{aap}^{M}$ *and* $\widetilde{F}=[(f_{\varepsilon})_{\varepsilon}]\in\mathcal{G}_{\tau}^{M}(\mathbb{C})$ *then*

$$\widetilde{F}\circ\widetilde{u} := [(f_{\varepsilon}\circ u_{\varepsilon})_{\varepsilon}]$$

is well-defined element of $\mathcal{G}_{aap}^{M}$*. The principal term and the corrective term of* $\widetilde{F}\circ\widetilde{u}$ *are respectively* $\widetilde{F}(\widetilde{u}_{ap})$ *and* $\widetilde{F}(\widetilde{u}_{ap}+\widetilde{u}_{cor})-\widetilde{F}(\widetilde{u}_{ap})$*, where* $\widetilde{u}=\widetilde{u}_{ap}+\widetilde{u}_{cor}$ *on* $\mathbb{J}$*.*

Proof. Let $(u_{\varepsilon})_{\varepsilon}\in\mathcal{M}_{aap}^{M}$ and $(f_{\varepsilon})_{\varepsilon}\in\mathcal{M}_{\tau}^{M}(\mathbb{C})$ be a respective representatives of $\widetilde{u}$ and $\widetilde{F}$, then $\forall j\in\mathbb{Z}_{+},\ \exists k_{j}>0,\ \exists c_{j}>0,\ \exists\varepsilon_{j}\in I,\ \forall\varepsilon<\varepsilon_{j}$,

$$\left\|u_{\varepsilon}^{(j)}\right\|_{L^{\infty}(\mathbb{R})} \le c_{j}e^{M\left(\frac{k_{j}}{\varepsilon}\right)} \tag{6.1}$$

and $\forall j\in\mathbb{Z}_{+},\ \exists k_{j}'>0,\ \exists c_{j}'>0,\ \exists\varepsilon_{j}'\in I,\ \forall\varepsilon<\varepsilon_{j}'$,

$$\left\|f_{\varepsilon}^{(j)}(u_{\varepsilon})\right\|_{L^{\infty}(\mathbb{R})} \le c_{j}'e^{M\left(\frac{k_{j}'}{\varepsilon}\right)}\left\|1+u_{\varepsilon}\right\|_{L^{\infty}(\mathbb{R})}^{k_{j}'}. \tag{6.2}$$

By using the classical Faà di Bruno formula, we have $\forall j\in\mathbb{Z}_{+}$,

$$\frac{(f_{\varepsilon}\circ u_{\varepsilon})^{(j)}}{j!} = \sum_{\substack{l_{1}+2l_{2}+\cdots+jl_{j}=j \\ r=l_{1}+\cdots+l_{j}}} \frac{f_{\varepsilon}^{(r)}(u_{\varepsilon}(x))}{l_{1}!\cdots l_{j}!}\prod_{i=1}^{j}\left(\frac{u_{\varepsilon}^{(i)}(x)}{i!}\right)^{l_{i}}, \tag{6.3}$$

as $\forall j\in\mathbb{Z}_{+},\ \forall\varepsilon\in I,\ u_{\varepsilon}^{(j)}\in\mathcal{C}_{aap}$ and f_{ε} is of class $\mathcal{E}$ on $\mathbb{C}$, it follows from the classical result on composition of a continuous function with an asymptotically almost periodic function is also an asymptotically almost periodic function that $\forall\varepsilon\in I,\ \forall r\in\mathbb{Z}_{+},\ f_{\varepsilon}^{(r)}(u_{\varepsilon})\in\mathcal{C}_{aap}$ and since $\mathcal{C}_{aap}$ is an algebra then $\forall j\in\mathbb{Z}_{+},\ \forall\varepsilon\in I,\ (f_{\varepsilon}\circ u_{\varepsilon})^{(j)}\in\mathcal{C}_{aap}$. From (6.1), (6.2) and (6.3), we get

$$\begin{aligned}\frac{\left\|(f_{\varepsilon}\circ u_{\varepsilon})^{(j)}\right\|_{L^{\infty}(\mathbb{R})}}{j!} \le \sum_{\substack{l_{1}+2l_{2}+\cdots+jl_{j}=j \\ r=l_{1}+\cdots+l_{j}}} &\frac{\left\|f_{\varepsilon}^{(r)}(u_{\varepsilon})\right\|_{L^{\infty}()}}{l_{1}!\cdots l_{j}!} \\ &\times\prod_{i=1}^{j}\left(\frac{\left\|u_{\varepsilon}^{(i)}\right\|_{L^{\infty}(\mathbb{R})}}{i!}\right)^{l_{i}}\end{aligned}$$

$$\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{c'_r e^{M\left(\frac{k'_r}{\varepsilon}\right)} \|1+u_\varepsilon\|^{k'_r}_{L^\infty(\mathbb{R})}}{l_1!\cdots l_j!} \times \prod_{i=1}^{j} \left(\frac{\left\| u_\varepsilon^{(i)} \right\|_{L^\infty(\mathbb{R})}}{i!} \right)^{l_i},$$

so

$$\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} \leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{c'_r e^{M\left(\frac{k'_r}{\varepsilon}\right)} \left(1+c_0 e^{M\left(\frac{k_0}{\varepsilon}\right)}\right)^{k'_r}}{l_1!\cdots l_j!} \times \prod_{i=1}^{j} \left(\frac{c_i e^{M\left(\frac{k_i}{\varepsilon}\right)}}{i!} \right)^{l_i},$$

hence there exists $C_r > 0$,

$$\begin{aligned}\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!} e^{M\left(\frac{k'_r}{\varepsilon}\right)} e^{k'_r M\left(\frac{k_0}{\varepsilon}\right)} \times \prod_{i=1}^{j} \left(\frac{c_i}{i!}\right)^{l_i} e^{l_i M\left(\frac{k_i}{\varepsilon}\right)} \\ &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!} e^{M\left(\frac{k'_r}{\varepsilon}\right)} e^{([k'_r]+1) M\left(\frac{k_0}{\varepsilon}\right)} \times \prod_{i=1}^{j} \left(\frac{c_i}{i!}\right)^{l_i} e^{l_i M\left(\frac{k_i}{\varepsilon}\right)}.\end{aligned}$$

Set $m = [k'_r] + 1$, due to Lemma 2.11, we have

$$e^{([k'_r]+1)M\left(\frac{k_0}{\varepsilon}\right)} \leq (AM_0)^{m-1}\, e^{M\left(\frac{k_0}{\varepsilon} H^{\frac{(m-1)(m+2)}{2m}}\right)}$$

and

$$e^{l_i M\left(\frac{k_i}{\varepsilon}\right)} \leq (AM_0)^{l_i-1}\, e^{M\left(\frac{k_i}{\varepsilon} H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)}.$$

Therefore,

$$\begin{aligned}
\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} \leq & \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\left(AM_0\right)^{m-1} e^{M\left(\frac{k_r'}{\varepsilon}\right)} \\
& \times e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}\left(AM_0\right)^{l_i-1} \\
& \times e^{M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)} \\
\leq & \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\left(AM_0\right)^{m-1} e^{M\left(\frac{k_r'}{\varepsilon}\right)} \\
& \times e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}\left(AM_0\right)^{r-j} \\
& \times e^{\sum_{i=1}^{j} M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i} \\
\leq & \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\left(AM_0\right)^{m+r-j-1} \\
& \times e^{M\left(\frac{k_r'}{\varepsilon}\right)} e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)} \\
& \times e^{\sum_{i=1}^{j} M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}.
\end{aligned}$$

By Lemma 2.11, we get

$$\begin{aligned}
\sum_{i=1}^{j} M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right) \leq & M\left(H^{\frac{(j-1)(j+2)}{2j}} \max_{1\leq i\leq j}\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)\right) \\
& + (j-1)\ln\left(AM_0\right) \\
\leq & M\left(\frac{m_j}{\varepsilon}\right) + (j-1)\ln\left(AM_0\right),
\end{aligned}$$

where $m_j := H^{\frac{(j-1)(j+2)}{2j}} \max\limits_{1\le i\le j}\left(k_i H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)$, so

$$\frac{\left\|(f_\varepsilon\circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} \le \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\,(AM_0)^{m+r-j-1}\, e^{M\left(\frac{k'_r}{\varepsilon}\right)}$$
$$\times e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}\,(AM_0)^{j-1}\,e^{M\left(\frac{m_j}{\varepsilon}\right)}$$
$$\times\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i},$$

by using again Lemma 2.11, let $N_j>0$, such that

$$\frac{N_j}{\varepsilon} = H^{\frac{5}{3}}\max\left(\frac{k'_r}{\varepsilon},\frac{m_j}{\varepsilon},\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right),$$

thus

$$e^{M\left(\frac{k'_r}{\varepsilon}\right)}e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}e^{M\left(\frac{m_j}{\varepsilon}\right)} \le (AM_0)^2\,e^{M\left(\frac{N_j}{\varepsilon}\right)}.$$

It follows

$$\frac{\left\|(f_\varepsilon\circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})}}{j!} \le \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\,(AM_0)^{m+r-j-1}$$
$$\times (AM_0)^{j-1}\,(AM_0)^2\,e^{M\left(\frac{N_j}{\varepsilon}\right)}\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}$$
$$\le \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\,(AM_0)^{m+r}\,e^{M\left(\frac{N_j}{\varepsilon}\right)}$$
$$\times\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}.$$

Finally, we obtain

$$\left\|(f_\varepsilon\circ u_\varepsilon)^{(j)}\right\|_{L^\infty(\mathbb{R})} \le j!\sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}\,(AM_0)^{m+r}$$
$$\times e^{M\left(\frac{N_j}{\varepsilon}\right)}\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}$$

$$\leq C'_j e^{M\left(\frac{N_j}{\varepsilon}\right)},$$

where $C'_j := j! \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!} (AM_0)^{m+r} \prod_{i=1}^{j} \left(\frac{c_i}{i!}\right)^{l_i}$. Then, we deduce that $(f_\varepsilon \circ u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aap}$. It is easy to prove that the composition $\widetilde{F} \circ \widetilde{u}$ is independent on the representatives $(u_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$. Let $\widetilde{u} = (\widetilde{u}_{ap} + \widetilde{u}_{cor}) \in \mathcal{G}^M_{aap}$. Since $\widetilde{F} \circ \widetilde{u} = \widetilde{F}(\widetilde{u}_{ap}) + \left(\widetilde{F}(\widetilde{u}) - \widetilde{F}(\widetilde{u}_{ap})\right)$, then $\widetilde{F} \circ \widetilde{u} = \widetilde{F}(\widetilde{u}_{ap}) + \left(\widetilde{F}(\widetilde{u}_{ap} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{ap})\right)$ on $\mathbb{J}$. In view of ([9], Proposition 6.1) we obtain $\widetilde{F}(\widetilde{u}_{ap}) \in \mathcal{G}^M_{ap}$. As $\mathcal{G}^M_{aap}$ and $\mathcal{G}^M_{ap}$ are subalgebras of $\mathcal{G}^M_{\mathcal{B}}$ then $\widetilde{F}(\widetilde{u}_{ap} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{ap}) \in \mathcal{G}^M_{\mathcal{B}}$. It suffices to show that $\forall \varepsilon \in I,\ f_\varepsilon(u_{ap,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{ap,\varepsilon}) \in \mathcal{B}_{+,0}$, where $(f_\varepsilon)_\varepsilon,\ (u_{ap,\varepsilon})_\varepsilon$ and $(u_{cor,\varepsilon})_\varepsilon$ are respective representatives of $\widetilde{F},\ \widetilde{u}_{ap}$ and $\widetilde{u}_{cor}$. The classical result on composition of asymptotically almost periodic function with continuous function shows that the corrective term of $f_\varepsilon(u_{ap,\varepsilon} + u_{cor,\varepsilon})$ is $f_\varepsilon(u_{ap,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{ap,\varepsilon})$ and the fact that $\widetilde{F}(\widetilde{u}_{ap} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{ap}) \in \mathcal{G}^M_{\mathcal{B}}$ gives $\forall \varepsilon \in I$, $(f_\varepsilon(u_{ap,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{ap,\varepsilon})) \in \mathcal{B}$. By ([12], Proposition $5-(5)$), we have $\forall \varepsilon \in I,\ f_\varepsilon(u_{ap,\varepsilon} + u_{cor,\varepsilon}) - f_\varepsilon(u_{ap,\varepsilon}) \in \mathcal{C}_{+,0} \cap \mathcal{B} = \mathcal{B}_{+,0}$. Therefore, $\widetilde{F}(\widetilde{u}_{ap} + \widetilde{u}_{cor}) - \widetilde{F}(\widetilde{u}_{ap}) \in \mathcal{G}^M_{+,0}$. □

7. Linear Difference Differential Systems

We consider the following linear difference differential systems

$$L\widetilde{u} = \sum_{i=0}^{p} \sum_{j=0}^{q} \widetilde{A}_{ij} \left(\tau_{\omega_j} \widetilde{u}\right)^{(i)} = \widetilde{f}, \tag{7.1}$$

where $\widetilde{f} = \left(\widetilde{f}_1, \cdots, \widetilde{f}_n\right) \in \left(\mathcal{G}^M_{\mathcal{B}}\right)^n$, $\omega = (\omega_j)_{0\leq j\leq q} \subset \mathbb{R}^q_+$ and $\widetilde{A} = \left(\widetilde{A}^{rl}_{ij}\right)_{1\leq r,l\leq n}$ is a square matrix of almost periodic generalized ultradistributions. The unknown generalized ultradistribution $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_n)$.

Remark 7.1. If $\widetilde{u} \in \left(\mathcal{G}^M_{aap}\right)^n$ then in view of Proposition 3.10 we obtain $L\widetilde{u} \in \left(\mathcal{G}^M_{aap}\right)^n$.

Definition 7.2. A generalized ultradistribution $\widetilde{u} \in \left(\mathcal{G}^M_{\mathcal{B}}(\mathbb{R})\right)^n$ is said a generalized solution of (7.1) on $\mathbb{I}$ if

$$\left(\sum_{i=0}^{p} \sum_{j=0}^{q} A_{ij,\varepsilon} \left(\tau_{\omega_j} u_\varepsilon\right)^{(i)} - f_\varepsilon\right)_\varepsilon \in \left(\mathcal{N}^M_{\mathcal{B}}(\mathbb{I})\right)^n,$$

i.e. for $r = 1, \cdots, n$,

$$\left(\sum_{l=0}^{n} \left(\sum_{i=0}^{p} \sum_{j=0}^{q} A_{ij,\varepsilon}^{rl} \left(\tau_{\omega_j} u_{l,\varepsilon} \right)^{(i)} \right) - f_{r,\varepsilon} \right)_{\varepsilon} \in \mathcal{N}_{\mathcal{B}}^{M} \left(\mathbb{I} \right),$$

where $(u_{\varepsilon})_{\varepsilon} = \left((u_{1,\varepsilon})_{\varepsilon}, \cdots (u_{n,\varepsilon})_{\varepsilon} \right)$, $(f_{\varepsilon})_{\varepsilon} = \left((f_{1,\varepsilon})_{\varepsilon}, \cdots, (f_{n,\varepsilon})_{\varepsilon} \right)$ and $(A_{ij,\varepsilon})_{\varepsilon} = \left(\left(A_{ij,\varepsilon}^{rl} \right)_{\varepsilon} \right)_{1 \leq r,l \leq n}$ are respective representatives of $\widetilde{u} = (\widetilde{u}_1, \cdots, \widetilde{u}_n)$, $\widetilde{f} = \left(\widetilde{f}_1, \cdots, \widetilde{f}_n \right)$ and $\widetilde{A}_{ij} = \left(\widetilde{A}_{ij}^{rl} \right)_{1 \leq r,l \leq n}$.

Theorem 7.3. *Let* $\widetilde{f} = \left(\widetilde{f}_{ap} + \widetilde{f}_{cor} \right) \in \left(\mathcal{G}_{aap}^{M} \right)^{n}$, *the equation* (7.1) *admits a generalized solution* $\widetilde{u} \in \left(\mathcal{G}_{aap}^{M} \right)^{n}$ *on* $\mathbb{J}$ *if and only if there exist* $\widetilde{v} \in \left(\mathcal{G}_{ap}^{M} \right)^{n}$ *and* $\widetilde{w} \in \left(\mathcal{G}_{+,0}^{M} \right)^{n}$ *such that*

$$L\widetilde{v} = \widetilde{f}_{ap} \ \text{on} \ \mathbb{R}, \tag{7.2}$$

and

$$L\widetilde{w} = \widetilde{f}_{cor} \ \text{on} \ \mathbb{J}. \tag{7.3}$$

Proof. If $\widetilde{u} = (\widetilde{u}_{ap} + \widetilde{u}_{cor}) \in \left(\mathcal{G}_{aap}^{M} \right)^{n}$ is a generalized solution of (7.1) on $\mathbb{J}$, then for $r = 1, \cdots, n$,

$$\begin{pmatrix} \sum\limits_{l=0}^{n} \left(\sum\limits_{i=0}^{p} \sum\limits_{j=0}^{q} A_{ij,\varepsilon}^{rl} \left(\tau_{\omega_j} u_{ap,l,\varepsilon} \right)^{(i)} \right) - f_{ap,r,\varepsilon} \\ + \sum\limits_{l=0}^{n} \left(\sum\limits_{i=0}^{p} \sum\limits_{j=0}^{q} A_{ij,\varepsilon}^{rl} \left(\tau_{\omega_j} u_{cor,l,\varepsilon} \right)^{(i)} \right) - f_{cor,r,\varepsilon} \end{pmatrix}_{\varepsilon} \in \mathcal{N}_{\mathcal{B}}^{M} \left(\mathbb{J} \right), \tag{7.4}$$

where $(u_{ap,l,\varepsilon})_{\varepsilon}$, $(u_{cor,l,\varepsilon})_{\varepsilon}$, $(f_{ap,l,\varepsilon})_{\varepsilon}$, $(f_{cor,l,\varepsilon})_{\varepsilon}$ and $\left(A_{ij,\varepsilon}^{rl} \right)_{\varepsilon}$ are respective representatives of $\widetilde{u}_{ap,l}$, $\widetilde{u}_{cor,l}$, $\widetilde{f}_{ap,l}$, $\widetilde{f}_{cor,l}$ and $\widetilde{A}_{ij}^{rl}$ for $1 \leq l \leq n$. In view of (5.6), this means that

$$\begin{pmatrix} \sum\limits_{l=0}^{n} \left(\sum\limits_{i=0}^{p} \sum\limits_{j=0}^{q} A_{ij,\varepsilon}^{rl} \left(\tau_{\omega_j} u_{ap,l,\varepsilon} \right)^{(i)} \right) - f_{ap,r,\varepsilon} \\ + \sum\limits_{l=0}^{n} \left(\sum\limits_{i=0}^{p} \sum\limits_{j=0}^{q} A_{ij,\varepsilon}^{rl} \left(\tau_{\omega_j} u_{cor,l,\varepsilon} \right)^{(i)} \right) - f_{cor,r,\varepsilon} \end{pmatrix}_{\varepsilon} \in \mathcal{M}_{\mathcal{B}}^{M} \left(\mathbb{J} \right),$$

and $\forall k > 0,\ \exists c > 0,\ \exists \varepsilon_0 \in I,\ \forall \varepsilon < \varepsilon_0,\ \forall x \geq 0$,

$$\begin{vmatrix} \sum\limits_{l=0}^{n} \left(\sum\limits_{i=0}^{p} \sum\limits_{j=0}^{q} A_{ij,\varepsilon}^{rl} (x) \left(\tau_{\omega_j} u_{ap,l,\varepsilon} (x) \right)^{(i)} \right) - f_{ap,r,\varepsilon} (x) \\ + \sum\limits_{l=0}^{n} \left(\sum\limits_{i=0}^{p} \sum\limits_{j=0}^{q} A_{ij,\varepsilon}^{rl} (x) \left(\tau_{\omega_j} u_{cor,l,\varepsilon} (x) \right)^{(i)} \right) - f_{cor,r,\varepsilon} (x) \end{vmatrix} \leq c e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{7.5}$$

For any real sequence $(s_m)_{m\in\mathbb{N}}$, such that $s_m \to +\infty$ there exist a subsequence $\left(s_{m_{p_1(\varepsilon)}}\right)_{p_1}$ of $(s_m)_{m\in\mathbb{N}}$ such that taking the translate at $s_{m_{p_1(\varepsilon)}} - s_{m_{p_2(\varepsilon)}}$ in (7.5) and let $p_1,\ p_2 \to +\infty$ we obtain $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0,\ \forall x\geq 0,$

$$\left|\sum_{l=0}^{n}\left(\sum_{i=0}^{p}\sum_{j=0}^{q}A_{ij,\varepsilon}^{rl}\left(x\right)\left(\tau_{\omega_j}u_{ap,l,\varepsilon}\left(x\right)\right)^{(i)}\right)-f_{ap,r,\varepsilon}\left(x\right)\right|\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}.$$

Due to Lemma 5.2, we obtain $\forall k>0,\ \exists c>0,\ \exists\varepsilon_0\in I,\ \forall\varepsilon<\varepsilon_0,$

$$\left\|\sum_{l=0}^{n}\left(\sum_{i=0}^{p}\sum_{j=0}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{ap,l,\varepsilon}\right)^{(i)}\right)-f_{ap,r,\varepsilon}\right\|_{L^{\infty}(\mathbb{R})}\leq ce^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{7.6}$$

As

$$\left(\sum_{l=0}^{n}\left(\sum_{i=0}^{p}\sum_{j=0}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{ap,l,\varepsilon}\right)^{(i)}\right)-f_{ap,r,\varepsilon}\right)_{\varepsilon}\in\mathcal{M}_{ap}^{M}\subset\mathcal{M}_{\mathcal{B}}^{M}\left(\mathbb{R}\right),$$

and by (5.6) ,(7.6), it follows that

$$\left(\sum_{l=0}^{n}\left(\sum_{i=0}^{p}\sum_{j=0}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{ap,l,\varepsilon}\right)^{(i)}\right)-f_{ap,r,\varepsilon}\right)_{\varepsilon}\in\mathcal{N}_{\mathcal{B}}^{M}\left(\mathbb{R}\right), \tag{7.7}$$

this means that $\widetilde{u}_{ap}=(\widetilde{u}_{ap,1},\cdots,\widetilde{u}_{ap,n})\in\left(\mathcal{G}_{ap}^{M}\right)^{n}$ is a generalized solution of (7.2) on $\mathbb{R}$. From (7.4) and (7.7) we obtain

$$\left(\sum_{l=0}^{n}\left(\sum_{i=0}^{p}\sum_{j=0}^{q}A_{ij,\varepsilon}^{rl}\left(\tau_{\omega_j}u_{cor,l,\varepsilon}\right)^{(i)}\right)-f_{cor,r,\varepsilon}\right)_{\varepsilon}\in\mathcal{N}_{\mathcal{B}}^{M}\left(\mathbb{J}\right),$$

this means that $\widetilde{u}_{cor}=(\widetilde{u}_{cor,1},\cdots,\widetilde{u}_{cor,n})\in\left(\mathcal{G}_{+,0}^{M}\right)^{n}$ is a generalized solution of (7.3) on $\mathbb{J}$. If there exist $\widetilde{v}\in\left(\mathcal{G}_{ap}^{M}\right)^{n}$ and $\widetilde{w}\in\left(\mathcal{G}_{+,0}^{M}\right)^{n}$ such that (7.2) and (7.3) hold, then it is easy to see that $\widetilde{u}:=(\widetilde{v}+\widetilde{w})\in\left(\mathcal{G}_{aap}^{M}\right)^{n}$ is a generalized solution of (7.1) on $\mathbb{J}$. □

Corollary 7.4. *A generalized ultradistribution* $\widetilde{U}=\left(\widetilde{U}_{ap}+\widetilde{U}_{cor}\right)\in\mathcal{G}_{aap}^{M}$ *is a primitive on* $\mathbb{J}$ *of the generalized ultradistribution* $\widetilde{u}\in\mathcal{G}_{aap}^{M}$ *such that* $\widetilde{u}=(\widetilde{u}_{ap}+\widetilde{u}_{cor})$ *on* $\mathbb{J}$ *if and only if*

$$\widetilde{U}_{ap}\text{ is a primitive of }\widetilde{u}_{ap}\text{ on }\mathbb{R},$$

and

$$\widetilde{U}_{cor}\text{ is a primitive of }\widetilde{u}_{cor}\text{ on }\mathbb{J}.$$

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References

1. K. Benmeriem and C. Bouzar, *Generalized Gevrey ultradistributions*, New York J. Math., 15 (2009), pp. 37-72.
2. K. Benmeriem and C. Bouzar, *An algebra of generalized Roumieu ultradistributions*, Rend. Semin. Mat. Politec. Torino, 70 (2012), pp. 101-109.
3. S. Bochner, *A new approach to almost periodicity*, Proc. Natl. Acad. Sci. USA., 48 (1962), pp. 2039-2043.
4. H. Bohr, *Almost periodic functions*, Chelsea, 1947.
5. C. Bouzar and M.T. Khalladi, *Almost periodic generalized functions*, Novi Sad J. Math., 41 (2011), pp. 33-42.
6. C. Bouzar and M.T. Khalladi, *Linear differential equations in the algebra of almost periodic generalized functions*, Rend. Semin. Mat. Univ. Politec. Torino., 70 (2012), pp. 111-120.
7. C. Bouzar and M.T. Khalladi, *Asymptotically almost periodic generalized functions*, Oper. Theory Adv. Appl., 231 (2013), pp. 261-272.
8. C. Bouzar and M.T. Khalladi, *On asymptotically almost periodic generalized solutions of differential equations*, Oper. Theory Adv. Appl., 245 (2015), pp. 35-43.
9. C. Bouzar and F. Ouikene, *Almost periodic generalized ultradistributions*, Filomat, 33 (2019), pp. 5407-5425.
10. C. Bouzar and F. Ouikene, *Almost periodic generalized solutions of differential equations*, Math. Bohem., 146 (2021), pp. 121-131.
11. C. Bouzar and M. Slimani, *Asymptotic almost automorphy for algebras of generalized functions*, (2022), arXiv : 2201.06441.
12. C. Bouzar and F.Z. Tchouar, *Asymptotic almost automorphy of functions and distributions*, Ural Math. J., 6 (2020), pp. 54-70.
13. R. D. Carmichael, A. Kaminski and S. Pilipović, *Boundary values and convolution in ultradistribution spaces*, World Scientific, 2007.
14. I. Cioranescu, *Asymptotically almost periodic distributions*, Appl. Anal., 34 (1990), pp. 251-259.
15. I. Cioranescu, *The characterization of the almost periodic ultradistributions of Beurling type*, Proc. Amer. Math. Soc., 116 (1992), pp. 127-134.
16. J.F. Colombeau, *New generalized functions and multiplication of distributions*, North Holland. 1984.
17. A.M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics 377. Springer, Berlin, 1974.

18. M. Fréchet, *Les fonctions asymptotiquement presque périodiques*, Revue Sci., 79 (1941), pp. 341-354.
19. M.C. Gómez-Collado, *Almost periodic ultradistributions of Beurling and Roumieu type*, Proc. Amer. Math. Soc., 129 (2001), pp. 2319-2329.
20. H. Komatsu, *Ultradistributions I. Structure theorems and a characterization*, J. Fac. Sci. Tokyo, Ser. IA, 20 (1973), pp. 25-105.
21. M. Kostić, *Asymptotically almost periodic and asymptotically almost automorphic vector-valued generalized functions*, Bul. Acad. Ştiinţe Repub. Mold. Mat., 91 (2019), pp. 34-53.
22. S. Pilipović, *Characterization of bounded sets in spaces of ultradistributions*, Proc. Amer. Math. Soc., 120 (1994), pp. 1191-1206.
23. L. Schwartz, *Théorie des distributions*, Hermann, 2ieme Ed., 1966.
24. R.T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc., 15 (1964), pp. 625-626.
25. S.L. Sobolev, *Applications of functional analysis in mathematical physics*, American Mathematical Society, 1963.

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[1]Laboratory of Mathematical Analysis and Applications, University of Oran 1, Ahmed Ben Bella, 31000, Oran, Algeria.

Email address: meryemslimani@yahoo.com

[2]Université des Sciences et de la Technologie d'Oran Mohamed-Boudiaf USTOMB El Mnaouar, BP 1505, Bir El Djir 31000, Oran, Algérie and Laboratory of Mathematical Analysis and Applications. University of Oran 1, Ahmed Ben Bella, 31000, Oran, Algeria.

Email address: aitouikene@yahoo.fr