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Novel Optimal Class of Eighth-Order Methods for Solving Nonlinear Equations and Their Dynamical Aspects

Abdallah Dawoud[1], Malak Khashoqji[2], Tareq Al-hussain[3] and Ibrahim Alsubaihi[4] *

Abstract. In this paper, a novel optimal class of eighth-order convergence methods for finding simple roots of nonlinear equations is derived based on the Predictor-Corrector of Halley method. By combining weight functions and derivative approximations, an optimal class of iterative methods with eighth-order convergence is constructed. In terms of computational cost, the proposed methods require three function evaluations, and the first derivative is evaluated once per iteration. Moreover, the methods have efficiency indices equal to 1.6817. The proposed methods have been tested with several numerical examples, as well as a comparison with existing methods for analyzing efficacy is presented.

1. Introduction

Solving the nonlinear algebraic equations $f(x) = 0$ by using iterative numerical methods is a crucial technique that has tremendous applications in different scientific fields, namely engineering, artificial intelligence and physics, see [1, 2, 5, 7, 8, 11–13]. Consequently, numerous iterative methods were developed to meet this need. One of the long-established methods to solve nonlinear equations is Newton method [14], which is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.1}$$

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Based on Newton method many modifications have been proposed to improve its order of convergence and evince new methods, see for example [1, 3, 5, 7, 8, 12, 13] and references therein. A new numerical method was developed by Halley in [3] which has a cubic convergence, and is written as

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}. \tag{1.2}$$

Noor and Noor [9] increased the order of convergence of Halley's method to six order. The resultant method is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

and

$$x_{n+1} = y_n - \frac{2f(y_n)f'(y_n)}{2f'(y_n)^2 - f(y_n)f''(y_n)}. \tag{1.3}$$

The method presented in (1.3) is called the Predictor-Corrector Halley method, and it is a nonoptimal method of the sixth order of convergence. The efficiency index of (1.3) is given by $IE = r^{\frac{1}{\theta}} = 6^{\frac{1}{5}} \approx 1.4309$, where r is the order of convergence and Θ is the number of function evaluations per iteration, according to the Kung-Traub conjecture [6]. In the following sections, a new class of optimal methods of eighth-order convergence is derived based on the Predictor-Corrector Halley method using derivative approximation and weight functions. Section 2 presents the construction of the new class of iterative methods. Section 3 shows the convergence analysis of the class and presents several families of iterative methods derived from the class by choosing different weight functions. In Section 4, several numerical examples which analyze the efficacy of the novel class of iterative algorithms described in this paper are included. Ultimately, the last section 5 presents the ploynmiographs and the generated basins of attraction of complex polynomials of different degrees through our newly proposed methods and compares them with other iterative methods.

2. Construction of the New Class of Methods

In this section, a new class of optimal eighth-order iterative methods is formulated based on the method developed by Noor and Noor [9] (1.3). To enhance the method to an eighth order, Newton's method is added in third step as follows.

Theorem 2.1 ([10]). *let $\psi_1(x), \psi_2(x), \psi_3(x), \ldots, \psi_n(x)$ be iterative functions with the order $q_1, q_2, q_3, \ldots, q_n$, respectively. Then, the composition* *of iterative functions* $\psi_1(\psi_2(\psi_3(\ldots\psi_n(x))$*, defines the iterative method of the order* $q_1q_2q_3\ldots q_n$*.*

By using theorem 2.1, Newton method is added as a third step, and we obtain the following three-step method

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{2f(y_n)f'(y_n)}{2f'(y_n)^2 - f(y_n)f''(y_n)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}.
\end{aligned} \tag{2.1}
$$

Now, to reduce the number of evaluations per iteration, $f'(y_n)$, $f''(y_n)$ and $f'(z_n)$ must be approximated. Approximations for $f'(y_n)$, $f''(y_n)$ and $f'(z_n)$ have been proved by Muhaijir [8]. The approximate values for $f'(y_n)$, $f''(y_n)$ and $f'(z_n)$ are given as

$$f'(y_n) = 2\left(\frac{f(y_n)-f(x_n)}{y_n-x_n}\right) - f'(x_n) = 2f[x_n,y_n] - f'(x_n), \tag{2.2}$$

$$f''(y_n) = 2\left(\frac{f(y_n)-f(x_n)-(y_n-x_n)f'(x_n)}{(y_n-x_n)^2}\right), \tag{2.3}$$

$$f'(z_n) = f[x_n,z_n] + f[y_n,z_n] - f[x_n,y_n]. \tag{2.4}$$

Equations (2.2), (2.3), and (2.4) were substituted into (2.1) to obtain the seventh-order method below

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= \frac{2f(y_n)(2f[x_n,y_n]-f'(x_n))}{(2f[x_n,y_n]-f'(x_n))^2 - 2f(y_n)\left(\frac{f(y_n)-f(x_n)-(y_n-x_n)f'(x_n)}{(y_n-x_n)^2}\right)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{f[x_n,z_n]+f[y_n,z_n]-f[x_n,y_n]}.
\end{aligned} \tag{2.5}
$$

Where

$$f[x_n,y_n] = \frac{f(y_n)-f(x_n)}{y_n-x_n}, \tag{2.6}$$

$$f[x_n,z_n] = \frac{f(z_n)-f(x_n)}{z_n-x_n}, \tag{2.7}$$

$$f[y_n,z_n] = \frac{f(z_n)-f(y_n)}{z_n-y_n}. \tag{2.8}$$

The resultant modified method (2.5) has a seventh order of convergence. For it to be elevated to an eighth order, it must be multiplied by weighted functions $A(g_1), W(g_2)$, and $H(g_3)$. Where $g_1 = \frac{f(y_n)}{f(x_n)}, g_2 = \frac{f(z_n)}{f(y_n)}$, and $g_3 = \frac{f(z_n)}{f(x_n)}$. Ultimately, the optimal class of iterative methods with an eighth order of convergence will be

$$
\begin{aligned}
(2.9)\quad y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= \frac{2f(y_n)(2f[x_n,y_n]-f'(x_n))}{(2f[x_n,y_n]-f'(x_n))^2 - 2f(y_n)\left(\frac{f(y_n)-f(x_n)-(y_n-x_n)f'(x_n)}{(y_n-x_n)^2}\right)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{f[x_n,z_n]+f[y_n,z_n]-f[x_n,y_n]}(A(g_1)W(g_2)H(g_3)).
\end{aligned}
$$

3. Convergence Analysis and a Description of Iterative Methods

What follows is an analysis of the convergence of order of the optimal eighth-order class presented in (2.9). In particular, we find the conditions to the weight functions in (2.9), which guarantee the required class of optimal eighth order and present some proposed families of iterative methods.

Theorem 3.1. *Let $\alpha \in I$, where I is an open interval, and α be a simple zero of a sufficiently differentiable function $f : \mathrm{I} \subseteq \mathrm{R} \to \mathrm{R}$. If the initial approximation x_0 is relatively close to the simple zero, then the class of iterative methods defined by (2.9) converges to α with an order of eight if the following conditions are satisfied*

$$
\begin{aligned}
&A(0)=1, \quad A'(0)=A''(0)=A'''(0)=0, \quad \left|A^{(4)}(0)\right|<\infty,\\
&W(0)=1, \quad W'(0)=0, \quad |W''(0)|<\infty,
\end{aligned}
$$

and

$$H(0)=H'(0)=1.$$

Proof. Let $e_n = x_n - \alpha$ given that $f(\alpha) = 0$, the Taylor expansion formula for f at α yields the following

$$f(x) = f'(\alpha)\left(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8 + O(e^9)\right). \tag{3.1}$$

Where $c_i = \frac{f^{(i)}(\alpha)}{i!f'(\alpha)}$, $\mathrm{i} = 1, 2, 3\ldots$ and

$$f'(x) = f'(\alpha)\left(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + Oe^8\right). \tag{3.2}$$

Based on the equations (3.1) and (3.2), the following is obtained

$$\frac{f(x)}{f'(x)} = e - c_2e^2 + \cdots + \left(-64c_2^7 + \cdots 7c_8\right)e^8 + O(e^9). \tag{3.3}$$

Furthermore, substituting (3.3) into the first step of (2.5), would yield

$$y_n = \alpha + c_2 e^2 + \cdots + \left(64 c_2^7 - \cdots + 7 c_8\right) e^8 + O\left(e^9\right). \tag{3.4}$$

From (3.4), using the Taylor expansion formula to determine $f\left(y_n\right)$ around $y_n = \alpha$, would result in

$$f\left(y_n\right) = f'(\alpha)\left[c_2 e^2 + \cdots + \left(144 c_2^7 + \cdots + 7 c_8\right) e^8 + O\left(e^9\right)\right]. \tag{3.5}$$

By substituting the equations (3.1), (3.2), (3.4) and (3.5) into the second step of (2.5), the resultant equation will be

$$z_n = \alpha - c_2 c_3 e^4 + \cdots + \left(21 c_2^7 - \cdots - 17 c_4 c_5\right) e^8 + O\left(e^9\right). \tag{3.6}$$

Applying the Taylor expansion formula of $f\left(z_n\right)$ around $z_n = \alpha$ would yield

$$f\left(z_n\right) = f'(\alpha)\left[-c_2 c_3 e^4 + \cdots + \left(21 c_2^7 - \cdots - 17 c_4 c_5\right) e^8 + O\left(e^9\right)\right]. \tag{3.7}$$

From (3.1) and (3.4)-(3.7) the following could be obtained

$$f\left[x_n, z_n\right] = f'(\alpha)\left[1 + c_2 e + \cdots + \left(21 c_2^2 c_3 c_4 - \cdots + c_8\right) e^7 + O\left(e^8\right)\right]. \tag{3.8}$$

$$f\left[y_n, z_n\right] = f'(\alpha)\left[1 + c_2^2 e^2 + \cdots + \left(80 c_2^2 c_3 c_4 - \cdots - 44 c_2^7\right) e^7 + O\left(e^8\right)\right]. \tag{3.9}$$

$$f\left[x_n, y_n\right] = f'(\alpha)\left[1 + c_2 e + \left(c_2^2 + c_3\right) e^2 + \cdots + O\left(e^8\right)\right]. \tag{3.10}$$

Substituting (3.6), (3.7) and (3.8)-(3.10) into the last step of (2.5) would give

$$x_{n+1} = \alpha + c_2^2 c_3^2 e^7 + \left(-3 c_2^3 c_3^2 + 3 c_2^2 c_3 c_4 + 4 c_2 c_3^3\right) e^8 + O\left(e^9\right). \tag{3.11}$$

This indicates that the method (2.5) has a seventh order of convergence. Now, to transform (2.5) into an eighth order of convergence $A(g_1), W(g_2)$ and $H(g_3)$ will be expanded using Taylor expansion at $g_1 = g_2 = g_3 = 0$, which would result in

$$A(g_1) = A(0) + A'(0) g_1 + \frac{1}{2} A''(0) g_1^2 + A'''(0) \frac{g_1^3}{3!} + A^{(4)}(0) \frac{g_1^4}{4!} + O\left(g_1\right)^5. \tag{3.12}$$

$$W(g_2) = W(0) + W'(0) g_2 + \frac{1}{2} W''(0) g_2^2 + O(g_2)^3. \tag{3.13}$$

$$H(g_3) = H(0) + H'(0) g_3 + \frac{1}{2} H''(0) g_3^2 + O(g_3)^3. \tag{3.14}$$

Eventually, using (3.12), (3.13), (3.14), (2.5) and the conditions $A(0) = 1, A'(0) = A''(0) = A'''(0) = 0, \left|A^{(4)}(0)\right| < \infty$, $W(0) = 1, W'(0) =$

$0, |W''(0)| < \infty$, and $H(0) = H'(0) = 1$, to transform the method into an eighth order method as shown in (2.9). The error expression is obtained from $e_{n+1} = x_{n+1} - \alpha$ as

(3.15)
$$e_{n+1} = \left(\frac{1}{2}c_2c_3^3W''(0) - c_2^2c_3c_4 + 2c_2^3c_3^2 + \frac{1}{24}A^{(4)}(0)c_2^5c_3\right)e^8 + O(e^9).$$

This proves that the method listed in (2.9) has an optimal eighth order of convergence, which concludes the proof of the theorem. □

Now, a construction for different families of iterative methods has been established based on the optimal class given in (2.9) by choosing weight functions that satisfy the conditions mentioned in theorem 3.1.

Method 1 (DSM1): Let

$A(g_1) = g_1^a + g_1^b + 1, \quad a, b \in R$ and $a \geq 5, b \geq 4$
$W(g_2) = \cos(g_2) + g_2^\mu, \quad \mu \in R$ and $\mu > 1$
$H(g_3) = \sin(g_3) + 1.$

The functions $A(g_1), W(g_2)$ and $H(g_3)$ satisfy the conditions and yield a family of three-parameter eighth-order methods

(3.16)
$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= \frac{2f(y_n)(2f[x_n, y_n] - f'(x_n))}{(2f[x_n, y_n] - f'(x_n))^2 - 2f(y_n)\left(\frac{f(y_n)-f(x_n)-(y_n-x_n)f'(x_n)}{(y_n-x_n)^2}\right)},\\
x_{n+1} &= z_n - \frac{f(z_n)(g_1^a + g_1^b + 1)(\cos(g_2) + g_2^\mu)(\sin(g_3) + 1)}{f[x_n, z_n] + f[y_n, z_n] - f[x_n, y_n]}.
\end{aligned}$$

Where $a, b, \mu \in R$ and $a \geq 5, b \geq 4, \mu > 1$.

Method 2 (DSM2): Let

$A(g_1) = 1 + g_1^m, \quad m \in R$ and $m \geq 4$
$W(g_2) = \cosh(g_2)$
$H(g_3) = \cos(g_3) + \sin(g_3).$

It can be seen that the functions meet the conditions of Theorem 3.1 and by substituting them into (2.9) a new family of one parameter eighth order methods are given by

(3.17)
$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= \frac{2f(y_n)(2f[x_n, y_n] - f'(x_n))}{(2f[x_n, y_n] - f'(x_n))^2 - 2f(y_n)\left(\frac{f(y_n)-f(x_n)-(y_n-x_n)f'(x_n)}{(y_n-x_n)^2}\right)},
\end{aligned}$$

$$x_{n+1} = z_n - \frac{f(z_n)(1+{g_1}^m)(\cosh(g_2))(\cos(g_3)+\sin(g_3))}{f[x_n,z_n]+f[y_n,z_n]-f[x_n,y_n]}.$$

Where $m \in R$ and $m \geq 4$.

Method 3 (DSM3): Let

$A(g_1) = 1 + g_1^m, \quad m \in R$ and $m \geq 4$
$W(g_2) = 1 - g_2^S, \quad S \in R$ and $S \geq 2$
$H(g_3) = e^{g_3} g_3 + 1$

From the new functions which satisfy the conditions in theorem 3.1, a new family of two-parameter eighth-order methods is obtained as

(3.18)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = \frac{2f(y_n)(2f[x_n,y_n]-f'(x_n))}{(2f[x_n,y_n]-f'(x_n))^2 - 2f(y_n)\left(\frac{f(y_n)-f(x_n)-(y_n-x_n)f'(x_n)}{(y_n-x_n)^2}\right)},$$

$$z_n - \frac{f(z_n)(1+g_1^m)(e^{g_3}g_3+1)(1-g_2^s)}{f[x_n,z_n]+f[y_n,z_n]-f[x_n,y_n]}$$

where $m, s \in R$ and $m \geq 4$, $s \geq 2$

4. Numerical Examples

In this section, several nonlinear equations have been selected to test the performance and efficiency of the proposed methods. The proposed methods DSM1 (3.16), DSM2 (3.17), and DSM3 (3.18) have been compared with the Predictor-Corrector Halley method (1.3), and the following eighth order methods.

The method proposed by Sharma [12] (SHM):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4.1}$$

$$z_n = y_n - \frac{f(x_n)f(y_n)}{f(x_n)-2f(y_n)f'(y_n)},$$

$$x_{n+1} = z_n - \left[1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n)}\right)^2\right]\left(\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}\right).$$

The method proposed by Abbas and AlSubaihi [1] (HSM):

$$y_n = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}, \tag{4.2}$$

$$z_n = x_n + (\beta - 1)c_1 - \beta c_2,$$

$$x_{n+1} = z_n - \frac{f(z_n)}{a_1 + 2a_2(z_n - x_n) + 3a_3(z_n - x_n)^2}.$$

Where

$\beta = 2$,

$a_1 = f'(x_n)$,

$a_2 = \dfrac{f[y_n, x_n, x_n](z_n - x_n) - f[z_n, x_n, x_n](y_n - x_n)}{z_n - y_n}$,

$a_3 = \dfrac{f[z_n, x_n, x_n] - f[y_n, x_n, x_n]}{z_n - y_n}$,

$c_1 = \dfrac{f(x_n)(f(x_n) - f(y_n))}{f'(x_n)(f(x_n) - 2f(y_n))}$,

$c_2 = \dfrac{f(x_n)}{f'(x_n)} + \dfrac{f(y_n)(f(x_n)^3 + f(y_n)^2 f(x_n) + \frac{1}{2} f(y_n)^3)(f(x_n) + f(y_n))^2}{f'(x_n) f(x_n)^5}$,

$f[y_n, x_n, x_n] = \dfrac{f[y_n, x_n] - f(x_n)}{y_n - x_n}$,

and

$$f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}.$$

The method proposed by Liu and Wang [7] (LWM):

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \left(\frac{f(y_n) f(x_n)}{f'(x_n) f(x_n) - 2f'(x_n) f(y_n)} \right), \\ x_{n+1} &= z_n - \left(\frac{f(z_n)}{f'(x_n)} \right) k_1. \end{aligned} \tag{4.3}$$

Where,

$k_1 = \left(\left(\dfrac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \right)^2 + \left(\dfrac{f(z_n)}{f(y_n) - f(z_n)} \right) + \left(4 \dfrac{f(z_n)}{f(x_n) + f(z_n)} \right) \right)$,

With fifteen decimal digits, Table 1 shows the nonlinear equations used for testing.

Table 1. Test functions and their roots

Functions	Roots (α)
$f_1(x) = \sin(x) - \left(\frac{1}{3}\right)x$	2.278862660075828
$f_2(x) = \sin^2(x) - x^2 + 1$	1.404491648215341
$f_3(x) = x^3 + 4x^2 - 10$	1.365230013414097
$f_4(x) = (x+2)e^x - 1$	-0.442854401002389
$f_5(x) = \ln(x) + \sqrt{x} - 5$	8.309432694231572

All calculations have been performed using MATLAB (2018a) software with 1500 digits of accuracy and with $\varepsilon = 10^{-300}$ tolerance. The stopping criteria have two conditions. Those conditions are $|x_n - \alpha| \leq \varepsilon$ and $|f(x_n)| \leq \varepsilon$. Moreover, Tables 2-6 show a comparison of the number of iterations, the absolute value of the function, the absolute+ error, and the computational order of convergence. The computational order of convergence is calculated via the following equation

$$\text{COC} = \frac{\ln\left|\frac{x_{n+1}-\alpha}{x_n-\alpha}\right|}{\ln\left|\frac{x_n-\alpha}{x_{n-1}-\alpha}\right|}. \tag{4.4}$$

For the methods HM (1.3), DSM1 (3.16), DSM2 (3.17), DSM3 (3.18), SHM (4.1), HSM (4.2) and LWM (4.3), some initial values are given to find the solution for the functions in Table 1. Note that the sign ($-$) in Table 5 shows that the method could not find the root of the function. Also, the computational order of convergence is calculated via the equation

Remark: The values for a, b and μ in DSM1 (3.16) are set to 7, 6 and 2, respectively. For both DSM2 (3.17) and DSM3 (3.18) $m = 4$. In DSM3 (3.18), $s = 2$.

Table 2. Comparison of various iterative methods for $f_5(x)$

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - \alpha\rvert$	COC
		$f_1(x), x_0 = 2$		
HM	4	9.98783e-1085	1.01533e-1084	6
SHM	3	9.83341e-386	9.99636e-386	8
LWM	3	4.68288e-334	4.76048e-334	8
HSM	3	6.02354e-321	6.12335e-321	8
DSM1	3	7.98921e-494	8.12159e-494	8
DSM2	3	5.29551e-447	5.38326e-447	8
DSM3	3	2.72451e-448	2.76965e-448	8

TABLE 3. Comparison of various iterative methods for $f_2(x)$.

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n-\alpha\rvert$	COC
		$f_2(x), x_0 = 1.6$		
HM	4	6.90052e-1146	2.7797e-1146	6
SHM	3	1.1127e-423	4.48221e-424	8
LWM	3	2.02765e-382	8.16787e-383	8
HSM	3	3.23698e-365	1.30394e-365	8
DSM1	3	1.39734e-552	5.62883e-553	8
DSM2	3	5.85586e-535	2.35888e-535	8
DSM3	3	3.77563e-535	1.52091e-535	8

TABLE 4. Comparison of various iterative methods for $f_3(x)$

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n-\alpha\rvert$	COC
		$f_3(x), x_0 = 1.2$		
HM	4	1.81461e-1411	1.09887e-1412	6
SHM	3	3.52828e-531	2.13661e-532	8
LWM	3	3.07766e-480	1.86374e-481	8
HSM	3	4.27956e-345	2.59157e-346	8
DSM1	3	8.81976e-620	5.34097e-621	8
DSM2	3	9.95068e-568	6.02582e-569	8
DSM3	3	7.86454e-569	4.76252e-570	8

TABLE 5. Comparison of various iterative methods for $f_4(x)$

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n-\alpha\rvert$	COC
		$f_4(x), x_0 = 0.5$		
HM	4	4.76686e-477	2.90273e-477	6
SHM	-	-	-	-
LWM	4	1.00172e-1169	6.09986e-1170	8
HSM	4	1.46334e-778	8.91085e-779	8
DSM1	4	8.33573e-508	5.07595e-508	8
DSM2	4	8.28661e-496	5.04604e-496	8
DSM3	4	8.82393e-594	5.37323e-594	8

TABLE 6. Comparison of various iterative methods for $f_5(x)$

Method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - \alpha\rvert$	COC
		$f_5(x), x_0 = 11.9$		
HM	4	2.23394e-1144	7.60362e-1144	6
SHM	3	3.92439e-492	1.33574e-491	8
LWM	3	2.98533e-398	1.01611e-397	8
HSM	3	2.76523e-342	9.41198e-342	8
DSM1	3	3.81916e-439	1.29992e-438	8
DSM2	3	9.32007e-409	3.17226e-408	8
DSM3	3	7.12008e-454	2.42345e-453	8

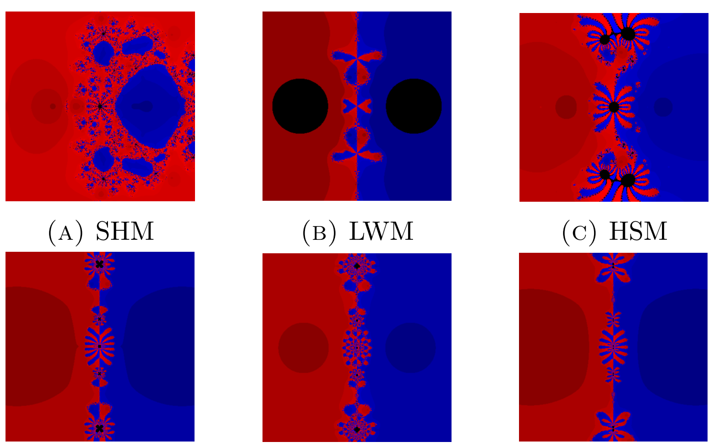
The numerical results in Tables 2-6 show that when DSM1 (3.16), DSM2 (3.17), and DSM3 (3.18) are compared to the method in (1.3), they require less iterations than the Predictor-Corrector Halley method. Moreover, they yield similar results obtained by similar methods of the same order of convergence, such as SHM (4.1), HSM (4.2), and LWM (4.3). In addition to that, it shows that in most of the cases, the proposed methods have a smaller absolute value of $f(x_n)$ compared to the other methods. Therefore, the new proposed class of methods is of practical interest and competes with other methods with the same order of convergence.

5. POLYNOMIOGRAPHY

There are several areas of mathematics where polynomials play a significant role. A huge part of the history of mathematics revolves around finding the roots of polynomials. By introducing polynomiography, Kalantari [4] took polynomials' root-finding techniques to the next level. Polynomiography is the science and art of visualising the approximation of the roots of complex polynomials, using fractal, and nonfractal graphs created by applying the mathematical convergence properties of iteration functions [4]. When compared with the classical approach of numerically comparing iterative methods in the real domain, the study of convergence and stability of the iterative method based on its polynomiograph gives detailed information and a deep understanding of its behavior at a glance. Therefore, it is an essential and effective tool to use. To obtain the ploynomiograph of the root in terms of fractal graphs, consider a square $\mathbb{R} \times \mathbb{R} = [-2, 2] \times [-2, 2]$ in which we take $400 \times 400 = 160000$ initial points which contain all the roots $(z_n, j = 1, 2, 3, \ldots)$ of the concerned complex polynomial and we apply the methods starting at every initial point z_0 in the square. If the sequence generated by the iterative method converges to a root z_n of the polynomial with a tolerance $|f(z_{n+1})| < 10^{-3}$ and a maximum of 20 iterations, we decide that z_0 is in the basins of attraction of this root. A brighter color means fewer iterations and a darker color is assigned to show more iterations. If the iterative method starting be at z_0 and reaches a root in K iterations ($K \leq 20$), then this point z_0 is assigned with different colors if $|z_{n+1} - z_n| < 10^{-3}$. It is assumed that the starting point has diverged if $K > 20$ and is assigned a black color. It is important to note that the presence of black dots in the polynmiograph does not necessarily indicate that the method is nonconvergent at these points or that it is unable to find a root at these points; nevertheless, it indicates that the method failed to find the solution under the conditions established for convergence, such as the number of steps and tolerance.

What follows is the polynomiographs of different complex polynomials for HM (1.3), SHM (4.1), HSM (4.2), LWM (4.3), and the newly developed iterative methods DSM1 (3.16), DSM2 (3.17) and DSM3 (3.18). The polynomiographs present the basins of attraction of the following polynomials:

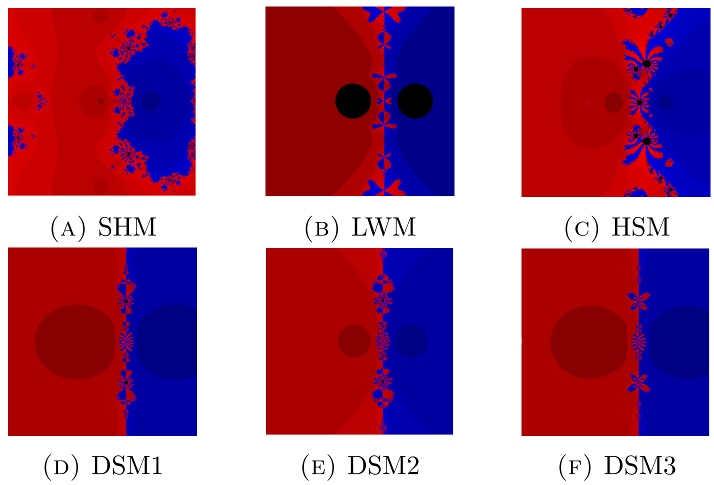
$$P_1(z) = z^2 - 1, \quad P_2(z) = z^2 - z, \quad P_3(z) = z^3 - 4z^2 - 10,$$
$$P_4(z) = z^3 - z, \quad P_5(z) = z^4 - z.$$



(A) SHM (B) LWM (C) HSM

(D) DSM1 (E) DSM2 (F) DSM3

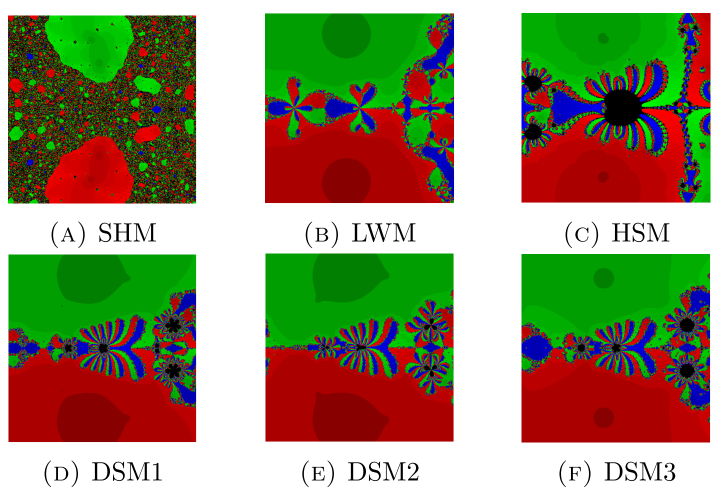
FIGURE 1. Basins of attraction of $P_1(z)$ for HM, SHM, LWM, HSM, DSM1, DSM2 and DSM3.



(A) SHM (B) LWM (C) HSM

(D) DSM1 (E) DSM2 (F) DSM3

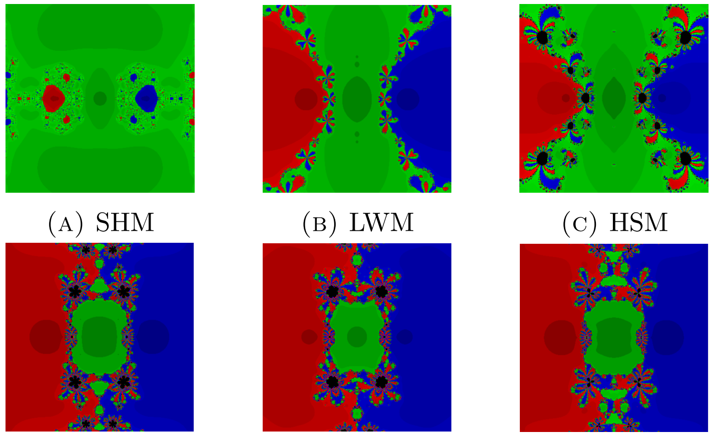
FIGURE 2. Basins of attraction of $P_2(z)$ for HM, SHM, LWM, HSM, DSM1, DSM2 and DSM3.



(A) SHM (B) LWM (C) HSM

(D) DSM1 (E) DSM2 (F) DSM3

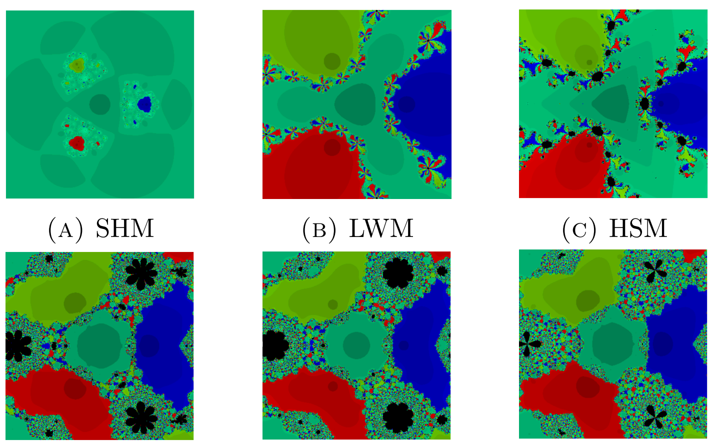
FIGURE 3. Basins of attraction of $P_3(z)$ for HM, SHM, LWM, HSM, DSM1, DSM2 and DSM3.



(A) SHM (B) LWM (C) HSM

(D) DSM1 (E) DSM2 (F) DSM3

FIGURE 4. Basins of attraction of $P_4(z)$ for HM, SHM, LWM, HSM, DSM1, DSM2 and DSM3.



(A) SHM (B) LWM (C) HSM

(D) DSM1 (E) DSM2 (F) DSM3

FIGURE 5. Basins of attraction of $P_5(z)$ for HM, SHM, LWM, HSM, DSM1, DSM2 and DSM3.

6. CONCLUSION

A novel optimal class of eighth-order of convergence methods for finding simple roots of nonlinear equations based on the Predictor-Corrector Halley method has been evinced. The order of convergence of the class is eight and consists of three function evaluations and one evaluation of the first derivative per iteration, so the class of optimal methods has an efficiency index equal to $8^{\frac{1}{4}} = 1.6817$. Numerical examples have been carried out to test the performance and efficiency of the novel methods. Furthermore, in most of the results obtained from the numerical examples and the basins of attraction, the proposed methods proved efficient and yielded better results than those of the same order of convergence.

References

1. H.M. Abbas and I.A. Al-Subaihi, *A New Family of Optimal Eighth-Order Iterative Method for Solving Nonlinear Equations*, Quest Journals JRAM, 8 (2022), pp. 10-17.
2. F. Akutsah, A.A. Mebawondu, P. Pillay, O.K. Narain and C.P. Igiri, *A New Iterative Method for Solving Constrained Minimization, Variational Inequality and Split Feasibility Problems in the Framework of Banach Spaces*, Sahand Commun. Math. Anal., 20 (2023), pp. 147-172.
3. E. Halley, *A new, exact, and easy method of finding the roots of any equations generally, and that without any previous reduction,(ABP. IDÇED, by C. Hutton, G. Shaw, R. Pearson, prevod sa latin-skog)*, Phil. Trans. Roy. Soc. London III, (1809).
4. B. Kalantari, *Method of creating graphical works based on polynomials*, U.S. Patent, 6 (2005), pp. 894-705.
5. H. Khandani and F. Khojasteh, *The Krasnoselskii's Method for Real Differentiable Functions*, Sahand Commun. Math. Anal., 20 (2023), pp. 95-106.
6. H.T. Kung and J.F. Traub, *Optimal order of one-point and multipoint iteration*, J. Assoc. Comput. Mach., 21 (1974), pp. 643-651.
7. L. Liu and X. Wang, *Eighth-order methods with high efficiency index for solving nonlinear equations*, Appl. Math. Comput., 9 (215) (2010), pp. 3449-3454.
8. M.N. Muhaijir, M. Soleh and E. Safitri, *Modification of Chebyshevs Method with Seventh-Order Convergence*, Appl. Math. Sci., Ruse, 11 (2017), pp. 2341-2350.
9. K.I. Noor and M.A. Noor, *Predictorcorrector Halley method for nonlinear* , Appl. Math. Comput., 188 (2007), pp. 1587-1591.
10. M.S. Petkovi and L.D. Petkovi, Families of optimal multipoint methods for solving nonlinear equations: a survey, Appl. Anal. Discrete Math., 4 (2010), pp. 1-22.
11. W. Rahou, A. Salim, J.E. Lazreg and M. Benchohra *On Fractional Differential Equations with Riesz-Caputo Derivative and Non-Instantaneous Impulses*, Sahand Commun. Math. Anal., 20 (2023), pp. 109-132.

12. J.R. Sharma and R. Sharma, *A new family of modified Ostrowskis methods with accelerated eighth order convergence*, Numer. Algorithms, 54 (2010), pp. 445-458.
13. P. Sivakumar, K. Madhu and J.Jayaraman, *Optimal eighth and sixteenth order iterative methods for solving nonlinear equation with basins of attraction*, Appl. Math. E-Notes., 21 (2021), pp. 320-343.
14. J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, NJ, (1964).

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