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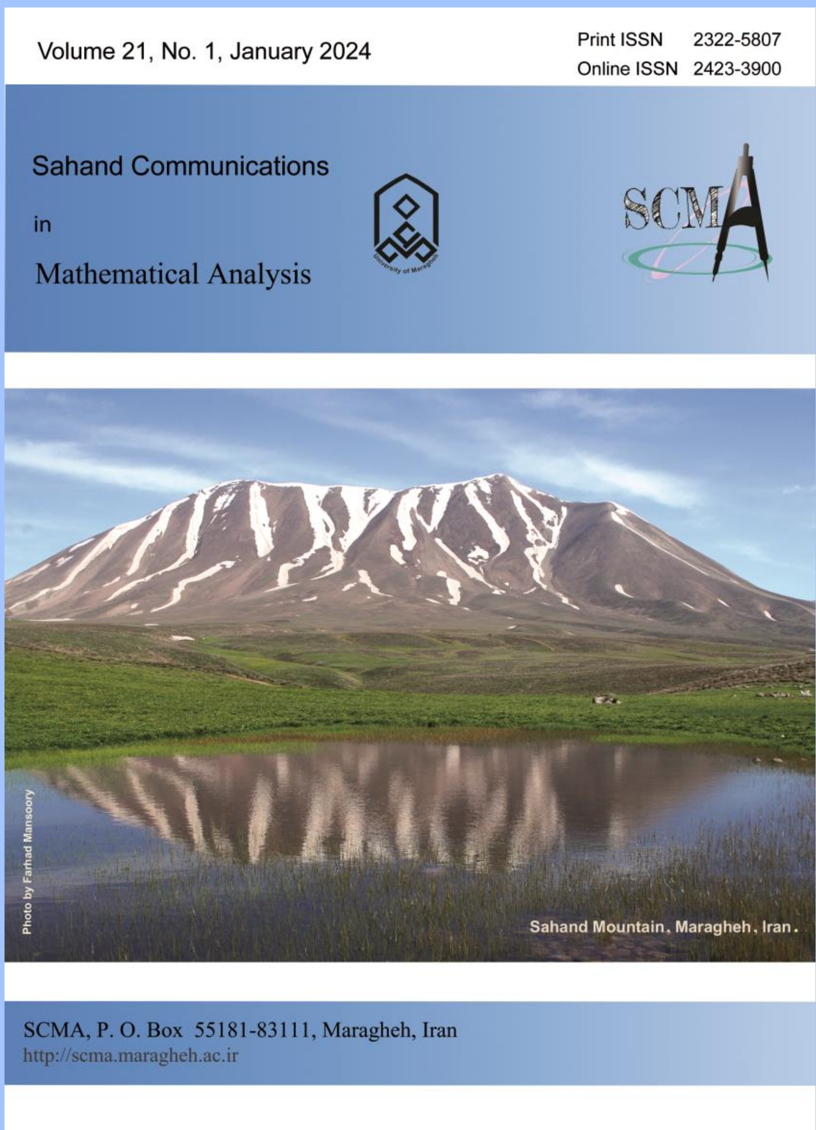

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A Study on a Fractional q-Integro-Differential Inclusion by Quantum Calculus with Numerical and Graphical Simulations

Mehran Ghaderi[1] and Shahram Rezapour[2*]

Abstract. In this paper, we investigate the existence of a solution for the fractional q-integro-differential inclusion with new double sum and product boundary conditions. One of the most recent techniques of fixed point theory, i.e. endpoints property, and inequalities, plays a central role in proving the main results. For a better understanding of the issue and validation of the results, we presented numerical algorithms, tables and some figures. The paper ends with an example.

1. Introduction

As we know, a large number of physical phenomena can be described and modeled by differential equations. Also, the new issue of differential inclusions has a special ability in interpreting physical phenomena with shocks. With the entry of fractional calculus(FC) into this field, the ability of differential equations and consequently boundary value problems(BVP) in modeling faced significant growth. It is clear that the reason for this growth is the generalization of operators from integer order to arbitrary fractional order in FC, which is not a small advantage in the study of systems with memory. This is the same property of non-locality of fractional operators that researchers have recently used to provide modeling in bio-mathematics [5, 13], physics [11, 19, 23, 29, 41], thermodynamics [31, 40]and engineering [35, 36]. In recent decades, the non-locality of fractional calculus has acted as a driving force for research

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* Corresponding author.

in this field. Based on the available results and evidence, modeling by ordinary calculus is not capable of describing the real behavior of phenomena and is often associated with the error of estimating the phenomenon [24]. Of course, it is worth mentioning that the approach of researchers of different sciences to this property of non-locality has not been the same. For example, physicists used it to model viscosity and heat flow and etc., while mathematicians tried to generalize and present new fractional operators [16]. Caputo, ψ-Caputo, Caputo-Fabrizio, Hadamard, Hilfer, Riemann-Liouville and Atangana-Baleanu(AB) can be mentioned among the famous fractional operators that researchers use in their research today. To get more information about these contributions, one can refer to [7–10, 17, 18, 22, 25, 33, 34, 37–39]. On the other hand, the prominent role of computer and software packages in the numerical methods of investigating complex equations and modeling cannot be ignored, which requires a discrete space. In this work, we also provide this space with the help of quantum calculus and time scale.

The history of quantum calculus(QC) dates back to the work of the British mathematician Frank Hilton Jackson. In 1910, he gave a new definition of the derivative, by which the basic principles of quantum calculus were founded [26, 27]. Jackson removed the concept of limit from the definition of the derivative and introduced two types of operators, namely q-derivative and h-derivative. Of course, q-derivative's growth was higher than h-derivative's and it didn't take long for it to enter the field of FC. Fractional q-derivative has both the advantages of FC and due to the discreteness of the space, it provides the possibility of using the computer in solving and simulating complex equations. For the same reason, in the last decade, q-derivative has received a lot of attention from researchers and many articles have been published in this field. For details see [4, 14, 20, 30, 32, 44, 45]. On the other hand, Set-Valued mappings, namely Multifunction, have interesting features whose properties have been investigated from different aspects and recently used in modeling due to their ability to interpret physical phenomena with shock. In 2007 Wlodarczyk et al. studied the existence and uniqueness of endpoint of closed set-valued contractions in metric spaces [42]. Wardowski in 2009, investigated the existence of fixed point and endpoint of multifunction in cone metric space [43]. A year later, Amini-Harandi presented an interesting property for multifunction, which plays the main role in this article [6]. Having said that, here we are going to investigate the existence of the solution for a fractional q-integro-differential inclusion by using the advantages mentioned about fractional and quantum calculus and multifunction.

In 2012, B. Ahmad et al. reviewed the existence and uniqueness of solutions for the following q-difference equations

$$\begin{cases} \mathcal{D}_q^2 \boldsymbol{w}(\kappa) = g(\kappa, \boldsymbol{w}(\kappa)), \quad \kappa \in \mathcal{K}, \\ \boldsymbol{w}(0) = \boldsymbol{w}(K), \ \mathcal{D}_q \boldsymbol{w}(0) = \mathcal{D}_q \boldsymbol{w}(K), \end{cases}$$

such that $g \in \mathcal{C}(\mathcal{K} \times \mathbb{R}, \mathbb{R})$, $\mathcal{K} = [0, K] \cap q^{\mathbb{N}}$, and $q^{\mathbb{N}} := \{q^n : n \in \mathbb{N}\} \cup \{0\}$ [3]. In the same year, Ravi P Agarwal et al. investigated the existence and dimension of the set of mild solutions to the following inclusion problem

$$\begin{cases} {}^C\mathcal{D}^\eta \boldsymbol{w}(\kappa) \in \mathcal{A}\boldsymbol{w}(\kappa) + \mathcal{B}(\kappa, \boldsymbol{w}(\kappa)), \quad \kappa \in [0, K], \ \eta \in (0, 1] \\ \boldsymbol{w}(0) + f(0) = \boldsymbol{w}_0, \end{cases}$$

which $\mathcal{A}$ is a sectorial operator (SO), ${}^C\mathcal{D}^\eta$ is Caputo derivative of fraction order η, and $\mathcal{B} : [0, K] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $f : \mathcal{C}([0, K], \mathbb{R}^n) \to \mathbb{R}^n$ [2]. In 2013, Zhao et al. studied the BVP of fractional q-derivative equation as follows

$$\begin{cases} \mathcal{D}_q^\eta \boldsymbol{w}(\kappa) + \mathcal{B}(\kappa, \boldsymbol{w}(\kappa)) = 0, \quad \kappa \in (0, 1s), \ \eta \in (0, 1] \\ \boldsymbol{w}(0) = 0, \ \boldsymbol{w}(1) = \displaystyle\int_0^\alpha \frac{(\alpha - qp)^{\nu - 1}}{\Gamma_q(\nu)} \boldsymbol{w}(p) d_q p, \end{cases}$$

such that $\eta \in (1, 2]$, $\nu \in (0, 2]$, $\alpha \in (0, 1)$, $\mathcal{B} : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$, and $\mathcal{D}_q^\eta$ is Riemann-Liouville q-derivative of order η [46].

Considering the topics discussed above and getting motivation from previous works, we want here to examine the existence of a solution for the following fractional quantum integro-differential inclusion problem (1.1)

$$ {}^C\mathcal{D}_q^\eta \boldsymbol{w}(\kappa) \in \mathcal{T}\left(\kappa, \boldsymbol{w}(\kappa), {}^C\mathcal{D}_q^\sigma \boldsymbol{w}(\kappa), \int_0^\kappa \boldsymbol{w}(p) dp\right), \quad \kappa \in \mathcal{K} = [0.1]$$

under new sum and product boundary conditions

$$\begin{cases} \boldsymbol{w}(0) + \mathcal{S}\boldsymbol{w'}(1) = 0, \\ \boldsymbol{w'}(d) = \mathcal{P} \end{cases} \tag{1.2}$$

which $\mathcal{S} = \sum_{j=1}^m \nu_j$, $\mathcal{P} = \prod_{j=1}^m u_j$, $\nu_j, u_j \in \mathbb{R}$, and $d \in (0, 1)$. In our problem ${}^C\mathcal{D}_q^\eta$ is Caputo quantum operator of fractional order $1 \leq \eta < 2$, and $\sigma \in (0, 1)$, such that $\mathcal{T} : \mathcal{K} \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$, is multifunction where $\mathcal{P}(\mathbb{R})$ set of all subsets of real numbers. Note that we will continue to do all our calculations on the time scale, namely $TS_{\kappa_0} = \{\kappa_0, \kappa_0 q, \kappa_0 q^2, \dots\} \cup \{0\}$, where $\kappa_0 \in \mathbb{R}$, and $q \in (0, 1)$.

2. Preliminaries

Definition 2.1 ([26])**.** Assume that $v, p \in \mathbb{R}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then the quantum-analogue of v and power function $(v-p)^n$ defined as follows, respectively

$$[v]_q = \frac{1-q^v}{1-q} = 1 + q + \cdots + q^{v-1}.$$

and

$$\begin{cases} (v-p)_q^{(n)} = \prod_{j=0}^{n-1}(v - pq^i) & \text{for } n \geq 1, \\ (v-p)_q^{(0)} = 1. \end{cases}$$

Definition 2.2 ([27])**.** Let $v \in \mathbb{R} - \{0, -1, -2, \dots\}$, then the quantum gamma function formulated as follows

$$\Gamma_q(v) = \frac{(1-q)^{(v-1)}}{(1-q)^{v-1}},$$

also, it is worth mentioning that $\Gamma_q(v+1) = [v]_q \Gamma_q(v)$ holds true.

In the following, we present an algorithm for calculating the quantum gamma function. Also, we computed some values of q in Table 1.

Algorithm 1 The proposed procedure to calculate $\Gamma_q(w)$.

function quantum gamma = qG(q,v,r)
$t = 1$;
for $j = 0 : r$
$t = t * (1 - q^{(j+1)})/(1 - q^{(v+j)})$;
end
$qG = t/(1-q)^{(v-1)}$;
end

TABLE 1. Numerical results for $\Gamma_q(2.25)$ for different value of q

r	$q = 0.2$	$q = 0.45$	$q = 0.69$	$q = 0.77$	$q = 0.89$	$q = 0.95$
			$v = 2.25$			
1	1.0486	1.1997	1.7704	2.3102	4.9645	12.3195
2	1.0413	1.1283	1.4983	1.8719	3.7499	8.9705
3	1.0399	1.0986	1.3513	1.6262	3.0530	7.0484
4	1.0396	1.0858	1.2643	1.4738	2.6063	5.8134
5	1.0395	1.0800	1.2100	1.3730	2.2984	4.9589
6	1.0395	1.0775	1.1749	1.3036	2.0753	4.3355
..	...	...	...	...	...	...
11	1.0395	1.0754	1.1138	1.1581	1.5240	2.7536
12	1.0395	1.0754	1.1105	1.1470	1.4679	2.5843
...	...	...	...	...	...	...
25	1.0395	1.0754	1.1031	1.1125	1.1855	1.6136
26	1.0395	1.0754	1.1031	1.1122	1.1783	1.5808
...	...	...	...	...	...	...
42	1.0395	1.0754	1.1031	1.1113	1.1312	1.2946
43	1.0395	1.0754	1.1031	1.1113	1.1303	1.2853
..	...	...	...	...	...	...
83	1.0395	1.0754	1.1031	1.1113	1.1231	1.1467
84	1.0395	1.0754	1.1031	1.1113	1.1230	1.1458
...	...	...	...	...	...	...
207	1.0395	1.0754	1.1031	1.1113	1.1230	1.1286
208	1.0395	1.0754	1.1031	1.1113	1.1230	1.1285

Definition 2.3 ([1])**.** The quantum derivative of a continuous function as $\boldsymbol{w}(\kappa)$ is as follows

$$(\mathcal{D}_q \boldsymbol{w})(\kappa) = \frac{\boldsymbol{w}(\kappa) - \boldsymbol{w}(q\kappa)}{(1-q)\kappa},$$

in addition, $(\mathcal{D}_q \boldsymbol{w})(0) = \lim_{\kappa \to 0} (\mathcal{D}_q \boldsymbol{w})(\kappa)$. Furthermore, for all $n \in \mathbb{N}$, the relation $(\mathcal{D}_q^n \boldsymbol{w})(\kappa) = \mathcal{D}_q(\mathcal{D}_q^{n-1} \boldsymbol{w})(\kappa)$ holds true.

Definition 2.4 ([21])**.** Suppose that $\boldsymbol{w}(\kappa) : [0, \infty] \to \mathbb{R}$, be a continuous function, then its fractional Riemann-Liouville quantum integral and its fractional Caputo quantum derivative are expressed respectively by

$$\mathcal{I}_q^{\eta} \boldsymbol{w}(\kappa) = \frac{1}{\Gamma_q(\eta)} \int_0^{\kappa} (\kappa - qp)^{\eta - 1} \boldsymbol{w}(p) d_q p,$$

and

$${}^c\mathcal{D}^{\eta} \boldsymbol{w}(\kappa) = \frac{1}{\Gamma_q(n-\eta)} \int_0^{\kappa} (\kappa - qp)^{n-\eta-1} \mathcal{D}_q^n \boldsymbol{w}(p) d_q p, \quad n = [\eta] + 1.$$

Lemma 2.5 ([15])**.** *assume that $n = [\eta] + 1$, then the following relation is true*

$$\left({}^{C}\mathcal{I}_q^{\eta}\,{}^{C}\mathcal{D}_q^{\eta}\boldsymbol{w}\right)(\kappa) = \boldsymbol{w}(\kappa) - \sum_{j=0}^{n-1} \frac{w^j}{\Gamma_q(j+1)} (\mathcal{D}_q^j \boldsymbol{w})(0),$$

which is deduced from it, the general solution for ${}^{C}\mathcal{D}_q^{\eta}\boldsymbol{w}(\kappa) = 0$, expressed by

$$\boldsymbol{w}(\kappa) = \ell_0 + \ell_1\kappa + \ell_2\kappa^2 + \cdots + \ell_{n-1}\kappa^{n-1},$$

where $\ell_0, \ldots, \ell_{n-1} \in \mathbb{R}$.

Notation 2.6. Here, we introduce some symbols that are used in the topology of the used space. Consider $(\mathcal{G}, d_{\mathcal{G}})$ be a metric space, also suppose that $\mathcal{P}(\mathcal{G})$ and $2^{\mathcal{G}}$ represent the set of all subset of $\mathcal{G}$ and the set of all non-empty subset of $\mathcal{G}$, respectively. In the sequel, we mean the symbols $\mathcal{P}_{cl}(\mathcal{G})$, $\mathcal{P}_{bd}(\mathcal{G})$, $\mathcal{P}_{cx}(\mathcal{G})$ and $\mathcal{P}_{ct}(\mathcal{G})$ respectively as the class of all closed, bounded, convex and compact subsets of $\mathcal{G}$, respectively.

Definition 2.7 ([6])**.** A fixed point of a multifunction(set-valued map) such as $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$ is an element $\kappa \in \mathcal{K}$, such that $\kappa \in \mathcal{E}(\kappa)$. As well as, if we have $\mathcal{E}(\kappa) = \{\kappa\}$, then this element, namely κ, is called an end point of $\mathcal{E}$.

Definition 2.8 ([6])**.** Let $(\mathcal{G}, d_{\mathcal{G}})$ be a metric space and $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$ is a multifunction, then $\mathcal{E}$, has an approximate property if we have $\inf_{\kappa\in\mathcal{G}} \sup_{r\in\mathcal{E}(\kappa)} d_{\mathcal{G}}(\kappa, r) = 0$.

Definition 2.9 ([12])**.** If $(\mathcal{G}, d_{\mathcal{G}})$ is a metric space, then the Pompeiu-Hausdorff meter, namely $\mathcal{HM} : 2^{\mathcal{G}} \times 2^{\mathcal{G}} \to [0.\infty]$, is defined as follows

$$\mathcal{HM}(\mathcal{W}, \mathcal{Z}) = \left\{ \sup_{w\in\mathcal{W}} d_{\mathcal{G}}(w, \mathcal{Z}), \sup_{z\in\mathcal{Z}} d_{\mathcal{G}}(\mathcal{W}, z) \right\},$$

which $\mathcal{HM}(\mathcal{W}, z) = \inf_{w\in\mathcal{W}} d_{\mathcal{G}}(w.z)$. Then the symbols $(\mathcal{P}_{bd,cl}(\mathcal{G}), \mathcal{HM})$, and $(\mathcal{P}_{cl}(\mathcal{G}), \mathcal{HM})$ represent a metric space and a generalized metric space, respectively.

Definition 2.10 ([12])**.** Assume that $\mathcal{V} = \mathcal{C}(\mathcal{K}, \mathbb{R})$, then define the space

$$\mathcal{G} = \left\{ \boldsymbol{w}(\kappa) : \boldsymbol{w}(\kappa), {}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa} \boldsymbol{w}(p)dp \in \mathcal{V} \right\}$$

equipped with the norm

$$\|\boldsymbol{w}\| = \sup_{\kappa\in\mathcal{K}} |\boldsymbol{w}(\kappa)| + \sup_{\kappa\in\mathcal{K}} \left|{}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa)\right| + \sup_{\kappa\in\mathcal{K}} \left|\int_0^{\kappa} \boldsymbol{w}(p)dp\right|.$$

Now $(\mathcal{G}, \|.\|)$ is a Banach space.

Definition 2.11. Let $\boldsymbol{w} \in \mathcal{G}$, then for all $\kappa \in \mathcal{K}$, define the set of selection of $\mathcal{S}^*$ as follows

$$\mathcal{S}^*_{\mathcal{T},\boldsymbol{w}} = \left\{ \mathfrak{g} \in \mathcal{L}^1(\mathcal{K}) : \boldsymbol{w}(\kappa) \in \mathcal{T}\left(\kappa, \boldsymbol{w}(\kappa), {}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa} \boldsymbol{w}(p)dp\right) \right\},$$

If $\dim(\mathcal{G}) < \infty$, then the above selection is nonempty which is proved in [12].

In 2010, Amini-Harandi introduced the end-point technique, which plays an essential role in proving our main result [6]. Now we will express it here.

Lemma 2.12 ([6]). *Suppose that $(\mathcal{G}, d_{\mathcal{G}})$ is a complete metric space, also consider two map Ψ and $\mathcal{E}$ with the following properties*

- $\Psi : [0,\infty) \to [0,\infty)$ *is upper semi continuous (USC), which $\forall \kappa > 0$ we have $\Psi(\kappa) < \kappa$, and $\liminf_{\kappa\to\infty}(\kappa - \Psi(\kappa)) > 0$.*
- *$\forall w, z \in \mathcal{G}$, for the set-valued map $\mathcal{E} : \mathcal{G} \to \mathcal{P}_{cl,bd}(\mathcal{G})$, the inequality $\mathcal{HM}\big(\mathcal{E}(w), \mathcal{E}(z)\big) \le \Psi\big(d_{\mathcal{G}}(w,z)\big)$ holds true.*

Then the set-valued map $\mathcal{E}$, has a unique endpoint iff $\mathcal{E}$ has an approximate end-point property.

3. Main Results

Now we have provided the prerequisites necessary to express our main results, and only one lemma remains, which we prove here.

Lemma 3.1. *The unique solution for the fractional q-differential problem ${}^{c}\mathcal{D}_q^{\eta}\boldsymbol{w}(\kappa) = g(\kappa)$ under boundary conditions* (1.2) *expressed by*

$$\begin{aligned}\boldsymbol{w}(\kappa) &= \mathcal{I}_q^{\eta} g(\kappa) + \ell_0 + \ell_1\kappa \\ &= \mathcal{I}_q^{\eta} g(\kappa) + \mathcal{S}\left[\mathcal{I}_q^{\eta-1}g(d) - \mathcal{I}_q^{\eta-1}g(1) - \mathcal{P}\right] + \left[\mathcal{P} - \mathcal{I}_q^{\eta-1}g(d)\right]\kappa.\end{aligned}$$

such that $\eta \in [1,2)$, and $g(\kappa) \in \mathcal{V}$.

Proof. In view of Lemma 2.5, the problem ${}^{C}\mathcal{D}_q^{\eta}\boldsymbol{w}(\kappa) = g(\kappa)$, has a unique solution which acquired by

$$\boldsymbol{w}(\kappa) = \mathcal{I}_q^{\eta} g(\kappa) + \ell_0 + \ell_1 t, \tag{3.1}$$

which $\ell_0, \ell_1 \in \mathbb{R}$. To apply the boundary conditions, it is necessary to calculate the first order derivative, namely $\boldsymbol{w}'(\kappa) = \ell_1 + \mathcal{I}_q^{\eta-1}g(\kappa)$. Now with regard to boundary conditions (1.2), we get

$$\begin{cases} \ell_0 + \mathcal{S} + \mathcal{I}_q^{\eta-1}g(1) + \mathcal{S}\ell_1 = 0, \\ \mathcal{I}_q^{\eta-1}g(d) + \ell_1 - \mathcal{P} = 0. \end{cases}$$

By performing simple calculations, the values of ℓ_0 and ℓ_1 will be as follows

$$\begin{cases} \ell_0 = \mathcal{S}\left[\mathcal{I}_q^{\eta-1}g(d) - \mathcal{I}_q^{\eta-1}g(1) - \mathcal{P}\right], \\ \ell_1 = \mathcal{P} - \mathcal{I}_q^{\eta-1}g(d). \end{cases}$$

Placing coefficients ℓ_0 and ℓ_1 in equation (3.1) provides the desired result. □

To obtain the result in our inclusion problem, it is necessary to apply the following hypotheses.

$\mathcal{A}_1$) Since $\mathcal{T} : \mathcal{K}\times\mathbb{R}^3 \to P_{cp}(\mathbb{R})$ is integrable and bounded, therefore $\mathcal{T}(.,a,b,c) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable.

$\mathcal{A}_2$) For $\Psi : [0,\infty) \to [0,\infty)$, which is nondecreasing and (USC), $\forall p > 0$ we have $\liminf_{p\to\infty}(p - \Psi(p)) > 0$ and $\Psi(p) < p$.

$\mathcal{A}_3$) For all $\kappa \in \mathcal{K}$, and $w_j, z_j \in \mathbb{R}$, $j = 1,2,3$, there exist $\boldsymbol{\Omega} \in \mathcal{C}(\mathcal{K},[0,\infty))$, where

$$\mathcal{HM}\left(\mathcal{T}(\kappa,w_1,w_2,w_3),\mathcal{T}(\kappa,z_1,z_2,z_3)\right) \le \frac{\boldsymbol{\Omega}(\kappa)}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\Psi\left(\sum_{j=1}^{3}|w_j - z_j|\right),$$

such that

$$\boldsymbol{\delta_1} = \|\boldsymbol{\Omega}\|\left[\frac{1}{\Gamma_q(\eta+1)} + \frac{|\mathcal{S}|d^{\eta-1}}{\Gamma(\eta)} - \frac{|\mathcal{S}|}{\Gamma_q(\eta)} + \frac{d^{\eta-1}}{\Gamma_q(\eta)}\right],$$

$$\boldsymbol{\delta_2} = \|\boldsymbol{\Omega}\|\left[\frac{1}{\Gamma_q(\eta-\sigma+1)} + \frac{d^{\eta-1}}{\Gamma_q(2-\sigma)}\right],$$

$$\boldsymbol{\delta_3} = \|\boldsymbol{\Omega}\|\left[\frac{1}{\Gamma_q(\eta+2)} + \frac{|\mathcal{S}|}{\Gamma_q(\eta)}\left(d^{\eta-1}+1+|\mathcal{P}|\right) + \frac{d^{\eta-1}}{\Gamma_q(\eta)} + |\mathcal{P}|\right].$$

$\mathcal{A}_4$) Suppose that $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$, be a operator which for $\mathfrak{g} \in \mathcal{S}^*_{\mathcal{T},\boldsymbol{w}}$ read as follows:

$$\begin{aligned}\mathcal{E}(\hbar) &= \{\hbar \in \mathcal{G} : \hbar(\kappa) = \mathcal{I}_q^{\eta}\mathfrak{g}(\kappa) + \mathcal{S}\left[\mathcal{I}_q^{\eta-1}\mathfrak{g}(d) - \mathcal{I}_q^{\eta-1}\mathfrak{g}(1) - \mathcal{P}\right] \\ &\quad + \left[\mathcal{P} - \mathcal{I}_q^{\eta-1}\mathfrak{g}(d)\right]\kappa, \forall\kappa \in \mathcal{K}\}\end{aligned}$$

Theorem 3.2. *Let conditions $\mathcal{A}_1 - \mathcal{A}_4$ are holds true. If the set-valued map $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$, has the approximate endpoint property, then the inclusion q-integro-differential problem mentioned in* (1.1)-(1.2) *has a solution.*

Proof. To show that our problem (1.1)-(1.2) has a solution, we go to find the endpoint of $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$. This end point is the solution to our inclusion. We do this in two steps.

Step I.: we shall show for all $\hbar \in \mathcal{G}$, $\mathcal{E}(\hbar) \subset \mathcal{G}$ which $\mathcal{E}(\hbar)$ is closed. Since the map $\kappa \mapsto \mathcal{T}(\kappa, \boldsymbol{w}(\kappa), {}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa}\boldsymbol{w}(p)dp)$, is measurable and closed value map , for all $\hbar \in \mathcal{G}$. Therefore, such a map has a non-empty measurable selection, namely $\mathcal{S}^*_{\mathcal{T},\boldsymbol{w}} \neq \phi$. Now assume that $\{t_n\}_{n\geq 1}$ be a sequence in $\mathcal{E}(\hbar)$, which $t_n \to t$. Choose $\mathfrak{g}_n \in \mathcal{S}^*_{\mathcal{T},\boldsymbol{w}}$, which for all $\kappa \in \mathcal{K}$ and $n \geq 1$

$$t_n = \mathcal{I}_q^{\eta}\mathfrak{g}_n(\kappa) + \mathcal{S}\left[\mathcal{I}_q^{\eta-1}\mathfrak{g}_n(d) - \mathcal{I}_q^{\eta-1}\mathfrak{g}_n(1) - \mathcal{P}\right] + \left[\mathcal{P} - \mathcal{I}_q^{\eta-1}\mathfrak{g}_n(d)\right]\kappa.$$

Compactness of $\mathcal{T}$, implies that $\mathfrak{g}_n$ has a subsequence(show this again with $\mathfrak{g}_n$), which converges to some $\mathfrak{g} \in \mathcal{L}^1[0,1]$. it is easy to check that $\mathfrak{g} \in \mathcal{S}^*_{\mathcal{T},\boldsymbol{w}}$, and for all $\kappa \in \mathcal{K}$

$$t_n(\kappa) \to t(\kappa) = \mathcal{I}_q^{\eta}\mathfrak{g}(\kappa) + \mathcal{S}\left[\mathcal{I}_q^{\eta-1}\mathfrak{g}(d) - \mathcal{I}_q^{\eta-1}\mathfrak{g}(1) - \mathcal{P}\right] + \left[\mathcal{P} - \mathcal{I}_q^{\eta-1}\mathfrak{g}(d)\right]\kappa.$$

It can be concluded from this $t \in \mathcal{E}(\hbar)$, thus $\mathcal{G}$ is closed values. In addition, from the compactness of the value of $\mathcal{T}$, it follows that $\in \mathcal{E}(\hbar)$ is bounded.

Step II.: Our goal at this step is to establish the following inequality: $\mathcal{HM}\big(\mathcal{E}(w), \mathcal{E}(z)\big) \leq \Psi\big(\|w - z\|\big)$. To do this, let $w, z \in \mathcal{G}$, $\hbar_1 \in \mathcal{E}(z)$, and choose $\mathfrak{g}_1 \in \mathcal{S}^*_{\mathcal{T},\boldsymbol{w}}$ such that for almost $\kappa \in \mathcal{K}$, we can write

$$\hbar_1 = \mathcal{I}_q^{\eta}\mathfrak{g}_1(\kappa) + \mathcal{S}\left[\mathcal{I}_q^{\eta-1}\mathfrak{g}_1(d) - \mathcal{I}_q^{\eta-1}\mathfrak{g}_1(1) - \mathcal{P}\right] + \left[\mathcal{P} - \mathcal{I}_q^{\eta-1}\mathfrak{g}_1(d)\right]\kappa.$$

But, in view of hypothesis $\mathcal{A}_3$

$$\begin{aligned}
&\mathcal{H}_{\mathfrak{d}}\left(\mathcal{T}(\kappa, w_1, w_2, w_3), \mathcal{T}(\kappa, z_1, z_2, z_3)\right)\\
&\quad \leq \frac{1}{\boldsymbol{\delta_1} + \boldsymbol{\delta_2} + \boldsymbol{\delta_3}}\boldsymbol{\Omega}(\kappa)\Psi\Big(|w_1(\kappa) - z_1(\kappa)| + \left|{}^{C}\mathcal{D}_q^{\eta}w_2(\kappa) - {}^{C}\mathcal{D}_q^{\eta}z_2(\kappa)\right|\\
&\qquad + \left|\int_0^{\kappa} w_3(p)dp - \int_0^{\kappa} z_3(p)dp\right|\Big),
\end{aligned}$$

hence, $\exists s \in \mathcal{T}\big(\kappa, \boldsymbol{w}(\kappa), {}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa}\boldsymbol{w}(p)dp\big)$, where $\forall \kappa \in \mathcal{K}$:

$$|\mathfrak{g}_1 - s| \leq \frac{1}{\boldsymbol{\delta_1} + \boldsymbol{\delta_2} + \boldsymbol{\delta_3}}\boldsymbol{\Omega}(\kappa)\Psi\left(\sum_{j=1}^{3}|w_j - z_j|\right).$$

Now consider the map $\mathcal{F} : \mathcal{K} \to \mathcal{P}(\mathbb{R})$, such that

$$\mathcal{F}(\kappa) = \left\{ s \in \mathbb{R} : |\mathfrak{g}_1 - s| \leq \frac{1}{\boldsymbol{\delta_1} + \boldsymbol{\delta_2} + \boldsymbol{\delta_3}}\boldsymbol{\Omega}(\kappa)\Psi\left(\sum_{j=1}^{3}|w_j - z_j|\right)\right\}$$

Forasmuch as $\frac{1}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\boldsymbol{\Omega}(\kappa)\Psi\left(\sum_{j=1}^{3}|w_j-z_j|\right)$, and $\mathfrak{g}_1$ are measurable, so the set-valued map

$$\mathcal{F}(.)\cap\mathcal{T}\left(.,\boldsymbol{w}(.),{}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(.),\int_0^{.}\boldsymbol{w}(p)dp\right)$$

is measurable.

Take $\mathfrak{g}_2(\kappa)\in\mathcal{T}\big(\kappa,\boldsymbol{w}(\kappa),{}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa),\int_0^{\kappa}\boldsymbol{w}(p)dp\big)$, which for all $\kappa\in\mathcal{K}$, we have

$$|\mathfrak{g}_1(\kappa)-\mathfrak{g}_2(\kappa)|\leq\frac{1}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\boldsymbol{\Omega}(\kappa)\Psi\left(\sum_{j=1}^{3}|w_j-z_j|\right).$$

Now, $\forall\kappa\in\mathcal{K}$, assume that $\mathfrak{g}_2\in\mathcal{E}(\hbar)$, with

$$\hbar_2=\mathcal{I}_q^{\eta}\mathfrak{g}_2(\kappa)+\mathcal{S}\left[\mathcal{I}_q^{\eta-1}\mathfrak{g}_2(d)-\mathcal{I}_q^{\eta-1}\mathfrak{g}_2(1)-\mathcal{P}\right]+\left[\mathcal{P}-\mathcal{I}_q^{\eta-1}\mathfrak{g}_2(d)\right]\kappa.$$

Subsequently, let $\sup_{\kappa\in\mathcal{K}}|\boldsymbol{\Omega}(\kappa)|=\|\boldsymbol{\Omega}\|$, therefore

$$\begin{aligned}\hbar_1(\kappa)-\hbar_2(\kappa)&=\mathcal{I}_q^{\eta}[\mathfrak{g}_1-\mathfrak{g}_2](\kappa)+\mathcal{S}\mathcal{I}_q^{\eta-1}[\mathfrak{g}_1-\mathfrak{g}_2](d)-\mathcal{S}\mathcal{I}_q^{\eta-1}[\mathfrak{g}_1-\mathfrak{g}_2](1)\\&\quad+\left(\mathcal{I}_q^{\eta-1}[\mathfrak{g}_1-\mathfrak{g}_2](d)\right)\kappa,\end{aligned}$$

which yields

$$\begin{aligned}|\hbar_1-\hbar_2|&\leq\frac{1}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\|\boldsymbol{\Omega}\|\Psi\big(\|w-z\|\big)\\&\quad\times\left[\frac{1}{\Gamma_q(\eta+1)}+\frac{|\mathcal{S}|d^{\eta-1}}{\Gamma_q(\eta)}-\frac{|\mathcal{S}|}{\Gamma_q(\eta)}+\frac{d^{\eta-1}}{\Gamma_q(\eta)}\right]\\&=\frac{\boldsymbol{\delta_1}}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\Psi\big(\|w-z\|\big).\end{aligned}$$

Also,

$$\begin{aligned}\left|{}^{C}\mathcal{D}_q^{\sigma}\hbar_1-{}^{C}\mathcal{D}_q^{\sigma}\hbar_2\right|&\leq\frac{1}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\|\boldsymbol{\Omega}\|\Psi\big(\|w-z\|\big)\\&\quad\times\left[\frac{1}{\Gamma_q(\eta-\sigma+1)}+\frac{d^{\eta-1}}{\Gamma_q(2-\sigma)}\right]\\&=\frac{\boldsymbol{\delta_2}}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\Psi\big(\|w-z\|\big),\end{aligned}$$

and

$$\left|\int_0^{\kappa}\hbar_1(p)dp-\int_0^{\kappa}\hbar_2(p)dp\right|\leq\frac{1}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\|\boldsymbol{\Omega}\|\Psi\big(\|w-z\|\big)$$

$$\times\left[\frac{1}{\Gamma_q(\eta+2)}+\frac{|\mathcal{S}|}{\Gamma_q(\eta)}\big(d^{\eta-1}+1+|\mathcal{P}|\big)+\frac{d^{\eta-1}}{\Gamma_q(\eta)}+|\mathcal{P}|\right]$$

$$=\frac{\boldsymbol{\delta_3}}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\Psi\big(\|w-z\|\big).$$

It can be inferred from the above relationships that

$$\|\hbar_1-\hbar_2\|=\sup_{\kappa\in\mathcal{K}}|\hbar_1(\kappa)-\hbar_2(\kappa)|+\sup_{\kappa\in\mathcal{K}}|{}^{C}\mathcal{D}_q^{\sigma}\hbar_1(\kappa)-{}^{C}\mathcal{D}_q^{\sigma}\hbar_2(\kappa)|$$

$$+\sup_{\kappa\in\mathcal{K}}\left|\int_0^{\kappa}\hbar_1(p)dp-\int_0^{\kappa}\hbar_2(p)dp\right|$$

$$\leq\frac{1}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\Psi\big(\|w-z\|\big)(\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3})$$
$$=\Psi\big(\|w-z\|\big).$$

Thus,for all $w,z\in\mathcal{G}$, we have

$$\mathcal{HM}\big(\mathcal{E}(w),\mathcal{E}(z)\big)\leq\Psi\big(\|w-z\|\big).$$

Now, according to Lemma 2.12, and the endpoint property of $\mathcal{E}$, $\exists w^*\in\mathcal{G}$, where $\mathcal{E}(w^*)=\{w^*\}$. Hence, w^* is a solution for the fractional q-inclusion problem mentioned in (1.1)-(1.2). □

4. Examples

Example 4.1. Regard the following fractional quantum integro-differential inclusion problem

$$\begin{cases}{}^{c}\mathcal{D}_q^{\frac{5}{4}}\boldsymbol{w}(\kappa)\in\mathcal{T}\Big[0,\dfrac{2+\cos(\kappa)}{23\kappa^2+\kappa^3}+\dfrac{7}{23(2+\sqrt{\kappa})}|\boldsymbol{w}(\kappa)|+\dfrac{7\kappa}{46}\displaystyle\int_0^{\kappa}\frac{\boldsymbol{w}(p)dp}{1+p}\\ \quad+\frac{7}{46}e^{\left|{}^{c}\mathcal{D}_q^{\frac{3}{5}}\boldsymbol{w}(\kappa)\right|}\Big],\\ \boldsymbol{w}(0)+2.4125\boldsymbol{w'}(1)=0,\\ \boldsymbol{w'}\left(\dfrac{3}{20}\right)=\dfrac{1}{81}.\end{cases}\tag{4.1}$$

where $\kappa\in\mathcal{K}=[0,1]$. Here, we put: $\eta=\frac{5}{4}$, $\sigma=\frac{3}{5}$, $d=\frac{3}{20}$, $\mathcal{S}=\sum_{j=1}^{4}\nu_j=2.4125$ with $\nu_1=\frac{7}{10}$, $\nu_2=\frac{9}{8}$, $\nu_3=\frac{2}{5}$, $\nu_4=\frac{3}{16}$, and $\mathcal{P}=\prod_{j=1}^{j=4}u_j=\frac{1}{81}$ with $u_j=\frac{1}{3}$. we choose $\boldsymbol{\Omega}:[0,1]\to[0,\infty)$ by $\boldsymbol{\Omega}(\kappa)=\frac{8}{67}\kappa$, $\|\boldsymbol{\Omega}\|=\frac{8}{67}$, and

$\Psi(\kappa)=\frac{\kappa}{23}$. Obviously Ψ is non-decreasing and (USC) on $\mathcal{K}$.

Consider the set-valued map $\mathcal{T}:\mathcal{K}\times\mathbb{R}^3\to\mathcal{P}_{ct}(\mathbb{R})$ as follows:

$$\mathcal{T}(t,w_1,w_2,w_3)=\Bigg[0,\frac{2+\cos(\kappa)}{23\kappa^2+\kappa^3}+\frac{7}{23(2+\sqrt{\kappa})}|\boldsymbol{w}(\kappa)|+\frac{7\kappa}{46}\int_0^\kappa\frac{\boldsymbol{w}(p)dp}{1+p}$$
$$+\frac{7}{46}e^{\left|{}^{c}\mathcal{D}_q^{\frac{3}{5}}\boldsymbol{w}(\kappa)\right|}\Bigg].$$

Nevertheless, the values of $\boldsymbol{\delta_1},\boldsymbol{\delta_2},\boldsymbol{\delta_3}$ are calculated for $q=\frac{1}{5},\frac{7}{20},\frac{4}{5}$ in Table 2.

TABLE 2. Numerical result of $\boldsymbol{\delta_1},\boldsymbol{\delta_2},\boldsymbol{\delta_3}$, for different values of q.

	$q=\frac{1}{5}$	$q=\frac{7}{20}$	$q=\frac{4}{5}$
$\boldsymbol{\delta_1}$	0.0789	0.2020	0.6638
$\boldsymbol{\delta_2}$	0.0759	0.2058	0.6092
$\boldsymbol{\delta_3}$	0.0695	0.2137	0.6512
$(\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3})^{-1}\|\boldsymbol{\Omega}\|$	0.5323	0.1921	0.0621

Now, it is easy to examine that

$$\mathcal{HM}\left(\mathcal{T}(\kappa,w_1,w_2,w_3),\mathcal{T}(\kappa,z_1,z_2,z_3)\right)\leq\frac{\boldsymbol{\Omega}(\kappa)}{\boldsymbol{\delta_1}+\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}\Psi\left(\sum_{j=1}^{3}|w_j-z_j|\right),$$

and $\inf_{w\in\mathcal{G}}\left(\sup_{z\in\mathcal{E}(z)}\|w-z\|\right)=0$. Now all the conditions of Theorem 3.2 are satisfied. Thanks to endpoint property and Theorem 3.2, our problem which is formulated in (4.1) has a solution. Also, the graphs of some functions are presented in Figures 1,2 and 3.

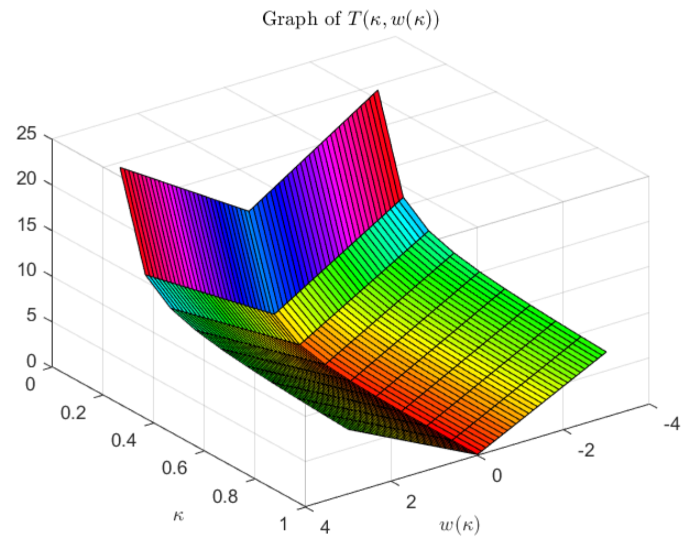

FIGURE 1. The graph of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa))$ in Example 4.1.

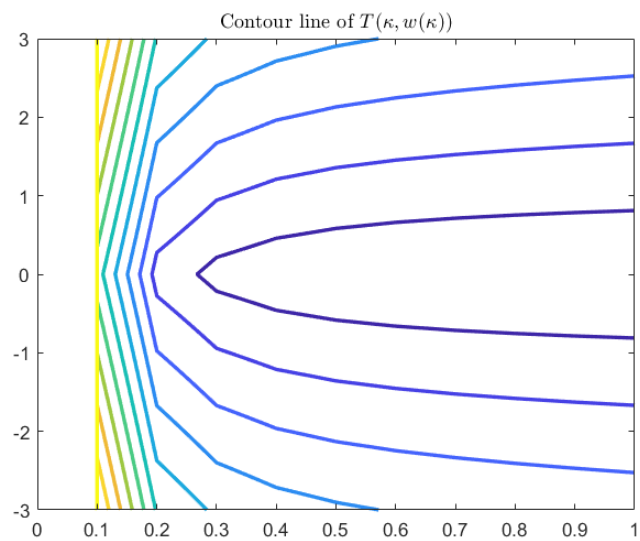

FIGURE 2. The contour line of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa))$ in Example 4.1.

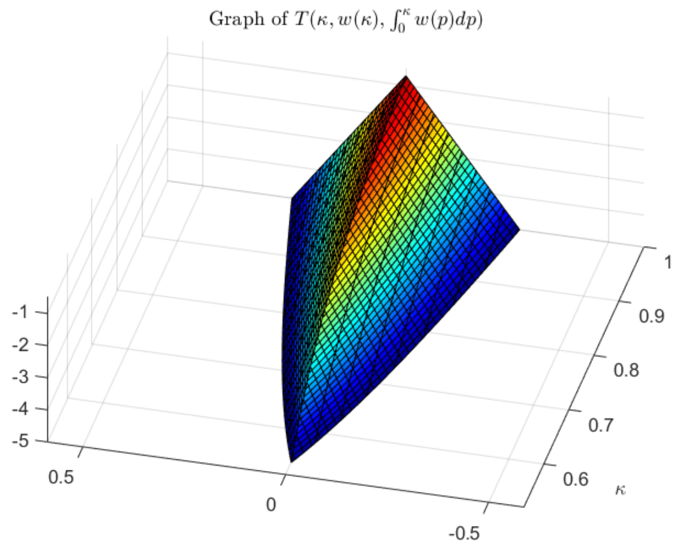

FIGURE 3. The graph of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa), \int_0^\kappa \boldsymbol{w}(p)dp)$ in Example 4.1.

5. Conclusion

In the last decade, increasing the ability to model natural and physical phenomena has become one of the important topics of interdisciplinary research. In this regard, the usual modeling methods have undergone important changes. For example, differential inclusions, which have a special ability to model physical phenomena with multiple shocks, are considered serious competitors for differential equations. In this work, we examined a fractional quantum integro-differential inclusion under sum and product boundary conditions with the numerical method. The existing derivation operators in our problem are of q-Caputo type. To prove the existence of the solution, we used the endpoint feature of the set-valued mappings. With the use of quantum calculus, we provided the right space for the use of computers in calculations and solutions. An example, algorithms and numerical results are also provided to validate our results.

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References

1. C.R. Adams, *The General Theory of a Class of Linear Partial q-Difference Equations*, Trans. Amer. Math. Soc., 26(3) (1924) , pp. 283-312.
2. R.P. Agarwal, B. Ahmad, A. Alsaedi and N. Shahzad, *Existence and dimension of the set of mild solutions to semilinear fractional differential inclusions*, Adv. Difference Equ., 74 (2012) pp. 1-10.
3. B. Ahmad, A. Alsaedi and S.K. Ntouyas, *A study of second-order q-difference equations with boundary conditions*, Nonlinear Anal., 35 (2012), pp. 1-10.
4. H. Akca, J. Benbourenane and H. Eleuch, *The q-derivative and differential equation*, J. Phys. Conf. Ser., 1411(1) (2019), 012002.
5. A. Alalyani and S. Saber, *Stability analysis and numerical simulations of the fractional COVID-19 pandemic model*, Int. J. Nonlinear Sci. Numer. Simul., 24(3) (2022), pp. 989-1002.
6. A. Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal., 72(1) (2010), pp. 132-134.
7. A. Atangana and D. Baleanu, *New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model*, Therm. Sci., 20(2) (2016), pp. 763-769.
8. M. Bohner, O. Tunc and C. Tunc, *Qualitative analysis of caputo fractional integro-differential equations with constant delays*, Comp. Appl. Math., 6(40) (2021), pp. 214.
9. A. Boutiara, J. Alzabut, M. Ghaderi and Sh. Rezapour, *On a coupled system of fractional (p, q)-differential equation with Lipschitzian matrix in generalized metric space*, AIMS Math., 8(1) (2022), pp. 1566-1591.
10. M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Prog. Fract. Differentiation App., 1(2) (2015), pp. 73-85.
11. A. Carpinteri and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, Springer-Verlag Wien, New York, 1997.
12. H. Covitz and S.B. Nadler, *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math., 8 (1970), pp. 5-11.
13. A. Din, Y. Li and M.A. Shah, *The complex dynamics of hepatitis B infected individuals with optimal control*, J. Syst. Sci. Complex., 34(4) (2021), pp. 1301-1323.
14. F.Z. El-Emam, *Convolution conditions for two subclasses of analytic functions defined by Jackson q-difference operator*, J. Egypt. Math. Soc., 7 (2022), pp. 1-10.
15. M. El-Shahed and F.M. Al-Askar, *Positive Solutions for Boundary Value Problem of Nonlinear Fractional q-Difference Equation*,

ISRN Math. Anal., (2011), 385459.

16. M. Fabrizio, C. Giorgi and V. Pata, *A new approach to equations with memory*, Arch. Rational Mech. Anal., 198(1) (2010), pp. 189-232.
17. R. George, F. Al-shammari, M. Ghaderi and Sh. Rezapour, *On the boundedness of the solution set for the ψ-Caputo fractional pantograph equation with a measure of non-compactness via simulation analysis*, AIMS Math., 8(9) (2023), pp. 20125-20142.
18. R. George, M. Aydogan, F.M. Sakar, M. Ghaderi and Sh. Rezapour, *A study on the existence of numerical and analytical solutions for fractional integrodifferential equations in Hilfer type with simulation*, AIMS Math., 8(5) (2023), pp. 10665-10684.
19. R. George, M. Houas, M. Ghaderi, Sh. Rezapour and S.K. Elagan, *On a coupled system of pantograph problem with three sequential fractional derivatives by using positive contraction-type inequalities*, Results Phys., 39 (2022), pp. 105687.
20. F. Guo, Sh. Kang and F. Chen, *Existence and uniqueness results to positive solutions of integral boundary value problem for fractional q-derivatives*, Adv. Difference Equ., 379 (2018), pp. 1-15.
21. J. R. Graef and L. Kong, *Positive solutions for a class of higher order boundary value problems with fractional q-derivatives*, Trans. Amer. Math. Soc., 218(19) (1924), pp. 9682-9689.
22. J. Hadamard, *Essai sur l'étude des fonctions, données par leur développement de Taylor*, Gauthier-Villars, 1892.
23. Z. Heydarpour, M.N. Parizi, R. Ghorbanian, M. Ghaderi, Sh. Rezapour and A. mosavi, *A study on a special case of the Sturm-Liouville equation using the Mittag-Leffler function and a new type of contraction*, AIMS Math., 7(10) (2022), pp. 10665-10684.
24. R. Hilfer, *Experimental evidence for fractional time evolution in glass forming materials*, Chem. phys., 284(1-2) (2002), pp. 399-408.
25. R. Hilfer, *Applications of fractional calculus in physics.* World Scientific, Singapore, 2000.
26. F.H. Jackson, *q-Difference equation*, Amer. J. Math., 32(4) (1910), pp. 305-314.
27. F.H. Jackson, *On Q-Definite Integrals*, Quart. J. Pure Appl. Math., 41 (1910), pp. 193-203.
28. M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic, Dordrecht, 1991.
29. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

30. Z.G. Lio, *On the q-derivative and q-series expansions*, Inter. J. Number Theory, 9(8) (2013), pp. 2069-2089.
31. R.P. Meilanov and R.A. Magomedov, *Thermodynamics in fractional calculus*, J. Eng. phys. thermophy., 87(6) (2014), pp. 1521-1531.
32. Sh. Mahmood, M. Jabeen, S.N. Malik, H.M. Srivastava, R. Manzoor and S. M. J. Riaz, *Some Coefficient Inequalities of q-Starlike Functions Associated with Conic Domain Defined by q-Derivative*, J. Funct. Spaces, (2018), 8492072.
33. K.M. Owolabi, *Modelling and simulation of a dynamical system with the Atangana-Baleanu fractional derivative*, Eur. Phys. J. Plus, 133(1) (2018), pp. 1-13.
34. L. Podlubny, *Fractional differential equations*, AcademicPress, San Diego, 1999.
35. H. Sun, Y. Zhang, D. Baleanu, W. Chen and Y. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. Numer. Simul., 64 (2018), pp. 213-231.
36. J.A. Tenreiro Machado, F.M. Silva, S.R. Barbosa, I.S. Jesus, M.C. Reis, M.G. Marcos and A.F. Galhano, *Some applications of fractional calculus in engineering*, Math. probl. Eng., (2010), 639801.
37. C, Tunc and O. Tunc, *On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation*, RACSAM., 115(3) (2021), pp. 115.
38. O. Tunc, O. Atan, C, Tunc and J.C. Yao, *Qualitative Analyses of Integro-Fractional Differential Equations with Caputo Derivatives and Retardations via the Lyapunov–Razumikhin Method*, Axioms , 10(2) (2021), pp. 58.
39. O. Tunc and C. Tunc, *Solution estimates to Caputo proportional fractional derivative delay integro-differential equations*, RACSAM., 117(1) (2023), pp. 12.
40. L. Vazquez, J.J. Trujillo and M. Pilar Velasco, *Fractional heat equation and the second law of thermodynamics*, Fract. Calc. Appl. Anal., 14 (2011), pp. 334-342.
41. B.J. West and P. Grigolini, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 1998.
42. K. Wlodarczyk, D. Klim and R. Plebaniak, *Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces*, J. Math. Anal. Appl., 328(1) (2007), pp. 46-57.
43. D. Wardowski, *Endpoints and fixed points of set-valued contractions in cone metric spaces*, Nonlinear Anal., 71(1-2) (2009), pp. 512-516.

44. Ch. Yo and J. Wang, *Existence of solutions for nonlinear second-order q-difference equations with first-order q-derivatives*, Adv. Diff. Equ., 124 (2013), pp. 1-11.
45. Y. Zhao, H. Chen and Q. Zhang, *Existence and multiplicity of positive solutions for nonhomogeneous boundary value problems with fractional q-derivatives*, Bound. Value Probl., 103 (2013), pp. 1-16.
46. Y. Zhao, H. Chen and Q. Zhang, *Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions*, Adv. Diff. Equ., 74 (2013), pp. 1-15.

[1] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

Email address: mehran.ghaderi@azaruniv.ac.ir & ghaderimehr@gmail.com

[2] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

Email address: sh.rezapour@azaruniv.ac.ir & rezapourshahram@yahoo.ca