Rigidity of Weak Einstein-Randers Spaces

Behnaz Lajmiri, Behroz Bidabad and Mehdi Rafie-Rad

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 1
Pages: 207-220

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.1983170.1218

Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 1 (2024), 207-220
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.1983170.1218

Rigidity of Weak Einstein-Randers Spaces

Behnaz Lajmiri[1], Behroz Bidabad[2*] and Mehdi Rafie-Rad[3]

Abstract. The Randers metrics are popular metrics similar to the Riemannian metrics, frequently used in physical and geometric studies. The weak Einstein-Finsler metrics are a natural generalization of the Einstein-Finsler metrics. Our proof shows that if (M, F) is a simply-connected and compact Randers manifold and F is a weak Einstein-Douglas metric, then every special projective vector field is Killing on (M, F). Furthermore, we demonstrate that if a connected and compact manifold M of dimension $n \geq 3$ admits a weak Einstein-Randers metric with Zermelo navigation data (h, W), then either the S-curvature of (M, F) vanishes, or (M, h) is isometric to a Euclidean sphere $\mathbb{S}^n(\sqrt{k})$, with a radius of $1/\sqrt{k}$, for some positive integer k.

1. Introduction

The study of Randers metrics and weak Einstein-Finsler metrics is a pivotal research domain with extensive implications in both physical and geometric fields. Randers metrics are particularly appealing for their proximity to Riemannian metrics and their simplicity, rendering them a frequently utilized tool in various studies.

This work is motivated by the desire to expand our understanding of the properties and behaviors of Randers metrics and weak Einstein-Finsler metrics. By proving that a simply-connected compact Randers manifold with a weak Einstein-Douglas metric has a special projective vector field that is Killing, we make a significant contribution to the field by providing new insights into the nature of these metrics.

2020 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Projective vector fields, Conformal vector fields, Randers metric, Weak Einstein, S-curvature, Rigidity.
Received: 19 December 2022, Accepted: 14 May 2023.
* The corresponding author.

Furthermore, the finding that a connected and compact manifold of dimension $n \geq 3$, that admits a weak Einstein-Randers metric with Zermelo navigation data, has vanishing S-curvature or is isometric to a Euclidean sphere $\mathbb{S}^n(\sqrt{k})$ of radius $1/\sqrt{k}$, is a major advancement in the field. This result opens up new avenues for exploration and research and inspires further investigation into the relationship between these metrics and the physical and geometric phenomena they describe.

In conclusion, this work stands as a testament to the potency of mathematical inquiry and underscores the significance of delving into the properties of mathematical entities to acquire fresh insights and knowledge.

Historically, it is known that, through Yamabe's work, every Riemannian metric on a compact n-dimensional manifold ($n \geq 2$), can be conformally deformed into the one with constant scalar curvature [18]. The existence of such a conformal deformation relies on the existence of a function that satisfies certain PDE and might give some information on the topological structure of the Riemannian manifold. For instance Obata proves that a compact Einstein manifold of constant scalar curvature k admits a non-constant function ϕ such that $\Delta\phi = nk\phi$, if and only if the manifold is isometric to the sphere $\mathbb{S}^n(\sqrt{k})$, of radius $1/\sqrt{k}$ in the $(n+1)$-dimensional Euclidean space, see [12]. This theorem has been proved and used in the Einstein spaces in [19] , as well as in the space of constant scalar curvature in [9]. The first eigenvalue λ_1 of the Laplacian operator on $(\mathbb{S}^n(\sqrt{k}), h)$, the Euclidean sphere of radius $\frac{1}{\sqrt{k}}$ in $\mathbb{R}^{n+1}$ is nk and the corresponding eigenfunction f satisfies the following system of differential equations:

$$\nabla df + kfh = 0, \quad k > 0, \tag{1.1}$$

see [12]. Tanno studied the following system of third order partial differential equations, see [17]

$$\nabla_h \nabla_j \nabla_i f + k\left(2\nabla_h f g_{ji} + \nabla_j f g_{ih} + \nabla_i f g_{hj}\right) = 0, \tag{1.2}$$

where k is a positive constant. Originally, the aforementioned differential equation arises from an investigation of the Laplacian operator on a Euclidean sphere $\mathbb{S}^n(\sqrt{k})$ with constant curvature k. The first eigenvalue of the Laplacian on $\mathbb{S}^n(\sqrt{k})$ is mk and the corresponding eigenfunction h satisfies the following second order PDE:

$$\nabla_j \nabla_i h + khg_{ji} = 0. \tag{1.3}$$

Many mathematicians have studied the above differential equation in the context of Finsler geometry, see for instance, [1, 4, 6, 11]. In the present work we extend the above results for Randers metric in the following sense. On a Randers metric on M, we denote by ∇ the Levi-Civita connection of α, and use the usual symbolic conventions for the general (α, β)-metrics, [8]. Zermelo's navigation problem involves determining the shortest time paths on a Riemannian manifold (M, h), with an external force W. It turns out that the shortest paths are the geodesics of the Randers metric $F = \alpha + \beta$ on M, expressed in terms of a Riemannian metric h and a vector field W called the navigation data of F. Also, we know that the weak Einstein-Finsler metrics are quite natural generalization of the Einstein-Finsler metrics whose Ricci scalar takes the form $\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right) F^2$, where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i(x)\, y^i$ is a 1-form on M. To elaborate more precisely, we show the following theorems in Finsler geometry.

Theorem 1.1. *Let (M,F) be a simply connected and compact Randers manifold. If F is a weak Einstein-Douglas metric then every special projective vector field on (M, F) is Killing.*

Theorem 1.2. *Let (M, F) be a connected and compact weak Einstein-Randers space of dimension $n \geq 3$ having the navigation data (h, W). Then, the S-curvature of (M, F) vanishes or (M, h) is isometric to the Euclidean sphere $\mathbb{S}^n(\sqrt{k})$, for some positive number k.*

2. Preliminaries and Notations

Let M be an n-dimensional C^∞ connected manifold and T_xM the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \bigcup_{x \in M} T_xM$. We will denote the elements of TM by (x, y) where $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM_0 \to M$ is given by $\pi(x, y) := x$. A Finsler structure on M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of the tangent bundle TM, and (iii) the Hessian of F^2 with the components $g_{ij}(x, y) := \frac{1}{2}[F^2(x, y)]_{y^i y^j}$ is a positive definite matrix on TM_0. The pair (M, F) is then called a Finsler space. Throughout this paper, we denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)\, y^i y^j}$ and a 1-form by $\beta = b_i(x)\, y^i$.

A Randers metric is a type of Finsler metric, which is a generalization of Riemannian metrics. The Finsler metric assigns a length to a tangent vector at any direction at each point on a manifold, while a Riemannian metric only assigns a length to a tangent vector at each point. A Randers metric is defined by a 1-form β and a Riemannian metric α, and is given by the following formula: $F(x, y) = \alpha + \beta$, where $(x, y) \in TM$ represents the position and tangent vectors on the manifold, respectively.

An example of a Randers metric is the Zermelo navigation problem, which involves finding the shortest path between two points on a smooth surface, taking into account the wind direction and speed. The Randers metric for this problem is given by the following formula: $F(x,y) = \sqrt{h_x(y,y)} + W_x(y)$, where h is the Riemannian metric on the surface, and W is the 1-form representing the wind direction and speed.

A weak Einstein-Finsler metric is a generalization of the Einstein-Finsler metric, which is a Finsler metric that satisfies a version of the Einstein equation in Finsler geometry. The weak Einstein-Finsler metric is a Finsler metric that satisfies a weakened form of the Einstein equation, and is characterized by its curvature properties.

An instance of a weak Einstein-Finsler metric is the Berwald metric, which is a type of Finsler metric that satisfies a weakened form of the Einstein equation. The Berwald metric is given by the following formula: $F(x,y) = \sqrt{h_x(y,y)} + \frac{1}{2}\left\langle \nabla_{h_x(y,\cdot)} y, y \right\rangle$, where ∇ is the covariant derivative, and h is a Riemannian metric.

A globally defined vector field $\mathbf{G}$ induced by F on TM_0, which in a standard coordinate $\left(x^i, y^i\right)$ on TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i(x,y)$ are the local functions on TM_0 satisfying $G^i(x,\lambda y) = \lambda^2 G^i(x,y)$, $\lambda > 0$. Consider the following conventions:

$$G^i_j := \frac{\partial G^i}{\partial y^i}, \qquad G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}, \qquad G^i_{jkl} := \frac{\partial G^i_{jk}}{\partial y^l}.$$

Notice that, the local functions G^i_{jk} give rise to a torsion-free connection on π^*TM called the Berwald connection, denoted here by D,and it serves as a practical connection utilized within this paper.

Given a Finsler structure F on an n-dimensional manifold M, the Busemann-Hausdorff volume form is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}\left(\mathbb{B}^n(1)\right)}{\mathrm{Vol}\left\{(y^i) \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i}|_x\right) < 1\right\}},$$

where $dV_F = \sigma_F(x)\, dx^1 \cdots dx^n$.

Define $\underline{g} := \det\left(g_{ij}(x,y)\right)$ and $\tau(x,y) := \ln \frac{\sqrt{g}}{\sigma_F(x)}$, and let $y \in T_xM$, be a tangent vector and $\gamma(t), -\epsilon < t < \epsilon$, denotes the geodesic with the initials $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. The function $S(x,y) := \frac{d}{dt}[\tau(\gamma(t), \dot{\gamma}(t))]|_{t=0}$ is called the S-curvature with respect to the Busemann-Hausdorff volume form. A Finsler space is said to be of isotropic S-curvature if there is a function $c = c(x)$ defined on M such that $S = (n+1)c(x)F$. It is called a Finsler space of constant S-curvature once c is a constant.

Let F be a Finsler structure on an n-manifold and G^i denote the geodesic coefficients of F. Define $\mathbf{R}_y = R^i{}_k(x,y)\, dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \rightarrow$

T_xM by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature. The Ricci scalar is defined by $\mathbf{Ric} := R^k{}_k$. The Ricci scalar $\mathbf{Ric}$ is a generalization of the Ricci tensor in Riemannian geometry, in the sense that, a Finsler structure F on an n-dimensional manifold M, is called a weak Einstein metric if $\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2$, where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i(x)\, y^i$ is a 1-form on M. It is called an Einstein metric if $\theta = 0$ in the above equation, that is, $\mathbf{Ric} = (n-1)\,\sigma(x)\, F^2$. In Finsler geometry, almost all geometric objects are defined on TM, and thus depend on both the position x and the direction y. Additionally, the Lie derivatives of these objects in the direction of a vector field X on M must be considered concerning to the complete lift vector field $\hat{X}$ on TM. A diffeomorphism between the two Finsler manifolds (M, F) and $(M, \bar{F})$ is called a projective transformation if it takes every forward (resp. backward) geodesic to a forward (resp. backward) geodesic. A projective transformation is called an affine transformation if it leaves invariant the connection coefficients.

A smooth vector field X is called a projective vector field or affine vector field on (M, F) if the associated local flow is a projective or affine transformation, respectively.

Lemma 2.1 ([1]). *A vector field X on the Finsler manifold (M, F) is a projective vector field if and only if there is a function $\Psi = \Psi(x, y)$ on TM_0, positively 1-homogeneous on y, such that*

$$\pounds_{\hat{X}} G^i = \Psi(x, y)\, y^i, \tag{2.1}$$

where G^i is the spray coefficients. X is an affine vector field if and only if $\Psi(x, y) = 0$.

Definition 2.2. Two Finsler structures F and $\tilde{F}$ are said to be specially projectively equivalent if $\tilde{G}^i = G^i + \Psi y^i$, where, $\Psi(x, y)$ is linear with respect to $y \in T_xM$, see [16].

2.1. **Weak Einstein-Randers Metric.** Let α be a Riemannian metric and $\beta = b_i(x)\, y^i$ a 1-form on M such that $\|\beta\|_x := \sup\frac{\beta(y)}{\alpha(y)} < 1$. Define $\nabla_j b_i$ by $(\nabla_j b_i)\,\theta^j := db_i - b_j\theta^j_i$, where $\theta^i := dx^i$, and $\theta^j_i := \Gamma^j_{ik}dx^k$, denote the Levi-Civita connection 1-form and ∇ the associate covariant derivative of α, respectively. Let us put

$$r_{ij} := \frac{1}{2}\left(\nabla_j b_i + \nabla_i b_j\right), \qquad s_{ij} := \frac{1}{2}\left(\nabla_j b_i - \nabla_i b_j\right), \tag{2.2}$$

$$s^i_j := a^{ih}s_{hj}, \qquad s_j := b_i s^i_j, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i. \tag{2.3}$$

Denote the geodesic spray coefficients of α and F by G^i_α. The relation between G^i_α and the coefficients G^i are given by

$$G^i = G^i_\alpha + \left(\frac{e_{00}}{2F} - s_0\right) y^i + \alpha s^i_0, \tag{2.4}$$

where $e_{00} := e_{ij}y^iy^j, s_0 := s_iy^i$, and $s^i_0 := s^i_jy^j$, see [16]. Notice that the S-curvature of a Randers metric $F = \alpha + \beta$ is given by

$$S = (n+1)\left\{\frac{e_{00}}{F} - s_0 - \rho_0\right\},$$

where $\rho = \ln\sqrt{1 - \|\beta\|}$ and $\rho_0 = \frac{\partial\rho}{\partial x^k}y^k$, see [16]. It is well-known that a Randers metric F is of isotropic S-curvature $S = (n+1)\,cF$, if and only if $e_{00} = 2c\,(x)\left(\alpha^2 - \beta^2\right)$, see [13, p.2]. In 1931, E. Zermelo studied the following problem: Imagine a scenario where a ship is sailing on an open sea and a gentle wind begins to blow. How should the ship's navigation be optimized to reach a predetermined destination in the least amount of time? Randers metrics can be expressed as the solution to the Zermelo navigation problem on some Riemannian manifold (M, h) with a wind vector field W. Concretely, assume that $h = \sqrt{h_{ij}\,(x)\,y^iy^j}$ and $W = W^i\frac{\partial}{\partial x^i}$ with $\|W\|_h < 1$, then the Randers structure F is obtained by solving the following equation:

$$h\left(x, \frac{y}{F\,(x,y)} - W_x\right) = 1.$$

Here the Randers metric is given by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \tag{2.5}$$

where $W_0 := W_iy^i = h\,(y, W_x)\,, \qquad W_i := h_{ij}W^j, \qquad \lambda := 1 - \|W\|^2_h > 0$ and $\|W_x\|_h = \|\beta_x\|_\alpha$, see [7]. The condition $\|W\|_h < 1$, is essential for obtaining a positive definite Randers metric for Zermelo's navigation problem. In this case, we call (h, W) the navigation data of the Randers metric $F = \alpha + \beta$. A Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is closed. For Douglas metric in (2.4), s_0 and s^i_0 are zero [5].

Theorem 2.3 ([8])**.** *Let F be a Randers metric on a manifold M. If F is a weak Einstein metric, then it has isotropic S-curvature.*

Akbar-Zadeh in 1986, proved that if (M, g) is a complete and simply connected Finslerian manifold of dimension n with constant strictly positive Ricci directional curvature while possessing an admitting projective vector field, then M is both compact and homeomorphic to a sphere, as outlined in [2]. He proved that if a simply connected and complete

Einstein-Finsler manifold admits a special projective vector field, then the following PDE possesses a non-trivial solution f.

$$D_0 D_0 D_0 f + (n-1)\,\sigma(x)\,F^2 \nabla_0 f = 0, \tag{2.6}$$

where, $\mathbf{Ric} = (n-1)\,\sigma(x)\,F^2$, D denotes the Berwald connection and $D_0 = y^i D_i$ stands for the covariant derivative along the canonical geodesic spray. Notice that, the equation (2.6) is not invariant to the change of affinely equivalent connections such as Chern, Berwald or Cartan connections.

The technique employed here, as presented in the subsequent sections, initially surfaced in [3] and has since found application in various other studies. It seems to be a powerful technique for the (α, β)-metrics since then, so that many interesting results are now proved, see [14]. This algebraic technique is based on the translation of the original equation into the form $Rat + \alpha Irrat = 0$, where Rat and $Irrat$ are polynomials in terms of the components of the tangent vectors in a given coordinate system. The equation $Rat + \alpha Irrat = 0$ is itself equivalent to the equation system $Rat = 0, Irrat = 0$. That is, for any point $x \in M$ the irreducible polynomial $h^2, \alpha^2 \in R\left[y^1, ..., y^2\right]$ are divisible.

Lemma 2.4. *Let us suppose that (M, F) is a Douglas Randers metric with isotropic S-curvature. If F is a weak Einstein metric with Ricci curvature $\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2$, then f satisfies the following equation*

$$D_0 D_0 D_0 f + (n-1)\left(\frac{3\theta}{F} + \sigma\right)F^2 \nabla_0 f = 0,$$

if and only if it satisfies the following equations

$$\nabla_0\nabla_0\nabla_0 f - 2\beta\nabla_0 c\nabla_0 f + 3(n-1)\,\theta\beta\nabla_0 f + (n-1)\,\nabla_0 f\sigma\left(\alpha^2 + \beta^2\right) = 0, \tag{2.7}$$

and

$$-2\nabla_0 c\nabla_0 f - 4c\nabla_0\nabla_0 f + 3(n-1)\,\nabla_0 f\theta + 2(n-1)\,\nabla_0 f\sigma\beta = 0. \tag{2.8}$$

Proof. Denote the geodesic spray coefficients of α and F by G^i_α and G^i

$$G^i = G^i_\alpha + \left(\frac{e_{00}}{2F} - S_0\right)y^i + \alpha s^i_0. \tag{2.9}$$

Since F is Douglas metric, $s_0 = 0$, and $s^i_0 = 0$. By theorem (2.3), if F is a weak Einstein metric, then it has isotropic S-curvature. Also F is of isotropic S-curvature if and only if $e_{00} = 2c\left(\alpha^2 - \beta^2\right)$, where $c = c(x)$ is a function on M. Therefore (2.9) reduces to

$$G^i = G^i_\alpha + c(\alpha - \beta)\,y^i. \tag{2.10}$$

On the ther hand, from [15, p.386] we get

$$D_0 D_0 f = \nabla_0\nabla_0 f + 2F s^i_0 \nabla_i f + F^2 s^i \nabla_i f - 2cF\nabla_0 f. \tag{2.11}$$

Since F is Douglas metric (2.11) reduces to

$$D_0D_0f = \nabla_0\nabla_0 f - 2cF\nabla_0 f. \tag{2.12}$$

In order to compute $D_0D_0D_0f$, we assume that $D_0D_0f = \varphi$ and compute the derivative of φ, as follows

$$\begin{aligned} D_0\varphi &= y^i\delta_i\varphi \\ &= y^i\left(\frac{\partial}{\partial x^i} - G_i^k\frac{\partial}{\partial y^k}\right)\varphi \\ &= y^i\frac{\partial\varphi}{\partial x^i} - 2G^k\frac{\partial\varphi}{\partial y^k}. \end{aligned} \tag{2.13}$$

Replacing $G^k = G_\alpha^k + c\,(\alpha-\beta)\,y^k$ in (2.13) we get

$$\begin{aligned} D_0\varphi &= y^i\frac{\partial\varphi}{\partial x^i} - 2G_\alpha^k\frac{\partial\varphi}{\partial y^k} - c\,(\alpha-\beta)\,y^k\frac{\partial\varphi}{\partial y^k} \\ &= \nabla_0\varphi - c\,(\alpha-\beta)\,y^k\frac{\partial\varphi}{\partial y^k}. \end{aligned} \tag{2.14}$$

Replacing (2.12) in (2.14) we have

$$\begin{aligned} D_0\varphi &= D_0D_0D_0f \\ &= \nabla_0\left(D_0D_0f\right) - c\,(\alpha-\beta)\,y^i\frac{\partial}{\partial y^i}\left(D_0D_0f\right) \\ &= \nabla_0\nabla_0\nabla_0 f - 2\nabla_0\left(cF\right)\nabla_0 f - 2cF\nabla_0\nabla_0 f \\ &\quad - c\,(\alpha-\beta)\,y^i\frac{\partial}{\partial y^i}\left(D_0D_0f\right). \end{aligned}$$

Again replacing (2.12) in the last term of the above equation we get

$$\begin{aligned} D_0D_0D_0f &= \nabla_0\nabla_0\nabla_0 f - 2\nabla_0\left(cF\right)\nabla_0 f - 2cF\nabla_0\nabla_0 f \\ &\quad - c\,(\alpha-\beta)\,y^i\frac{\partial}{\partial y^i}\left(\nabla_0\nabla_0 f - 2cF\nabla_0 f\right) \\ &= \nabla_0\nabla_0\nabla_0 f - \nabla_0\left(cF\right)\nabla_0 f - 2cF\nabla_0\nabla_0 f - c\,(\alpha-\beta)\,y^i\frac{\partial}{\partial y^i} \\ &\quad \left(y^i\nabla_i\nabla_0 f + y^i\nabla_0\nabla_i f - y^i 2cF\nabla_i f - y^i 2c\left(\frac{\partial}{\partial y^i}F\right)\nabla_0 f\right). \end{aligned} \tag{2.15}$$

By simplification, (2.15) becomes

$$\begin{aligned} D_0D_0D_0f &= \nabla_0\nabla_0\nabla_0 f - 2\nabla_0\left(cF\right)\nabla_0 f - 2cF\nabla_0\nabla_0 f \\ &\quad - 2c\,(\alpha-\beta)\,\nabla_0\nabla_0 f + 2c^2\left(\alpha^2-\beta^2\right)\nabla_0 f \\ &\quad + 2c^2\left(\alpha^2-\beta^2\right)\nabla_0 f. \end{aligned} \tag{2.16}$$

Replacing

$$\begin{aligned}\nabla_0(cF) &= \nabla_0(c)F + c\nabla_0(\alpha+\beta)\\ &= \nabla_0(c)(\alpha+\beta) + 2c^2\left(\alpha^2-\beta^2\right),\end{aligned}$$

in (2.16) we get

$$\begin{aligned}D_0D_0D_0f = {} & \nabla_0\nabla_0\nabla_0 f - 2(\nabla_0 c)(\alpha+\beta)\nabla_0 f\\ & - 4c^2\left(\alpha^2-\beta^2\right)\nabla_0 f - 2c(\alpha+\beta)\nabla_0\nabla_0 f\\ & + 4c^2\left(\alpha^2-\beta^2\right)\nabla_0 f - 2c(\alpha-\beta)\nabla_0\nabla_0 f.\end{aligned} \tag{2.17}$$

Using the above equation, since F is weak Einstein we have

$$D_0D_0D_0f + (n-1)\left(\frac{3\theta}{F}+\sigma\right)F^2\nabla_0 f = 0.$$

Hence

$$\begin{aligned}& D_0D_0D_0f + (n-1)\left(\frac{3\theta}{F}+\sigma\right)F^2\nabla_0 f\\ &= \nabla_0\nabla_0\nabla_0 f - 2\alpha(\nabla_0 c)\nabla_0 f - 2(\nabla_0 c)\nabla_0 f\beta\\ &\quad - 4c^2\alpha^2\nabla_0 f + 4c^2\beta^2\nabla_0 f - 2c\alpha\nabla_0\nabla_0 f\\ &\quad - 2c\beta\nabla_0\nabla_0 f - 2c\alpha\nabla_0\nabla_0 f + 2c\beta\nabla_0\nabla_0 f\\ &\quad + 4c^2\alpha^2\nabla_0 f - 4c^2\beta^2\nabla_0 f + (n-1)\nabla_0 f 3\theta\alpha\\ &\quad + \nabla_0 f(n-1)3\theta\beta + (n-1)\nabla_0 f\sigma\alpha^2\\ &\quad + (n-1)\nabla_0 f\sigma\beta^2 + 2(n-1)\nabla_0 f\sigma\alpha\beta\\ &= 0.\end{aligned}$$

Let us rewrite the above equation using the Rat and Irrat

$$\begin{aligned}Rat = {} & \nabla_0\nabla_0\nabla_0 f - 2\beta\nabla_0 c\nabla_0 f + 3(n-1)\theta\beta\nabla_0 f\\ & + (n-1)\nabla_0 f\sigma\left(\alpha^2+\beta^2\right),\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}Irrat = {} & -2\nabla_0 c\nabla_0 f - 4c\nabla_0\nabla_0 f + 3(n-1)\nabla_0 f\theta\\ & + 2(n-1)\nabla_0 f\sigma\beta.\end{aligned} \tag{2.19}$$

This complete the proof of Lemma 2.4. □

We are now in a position to prove the theorem 1.1.

Proof. To prove Theorem 1.1, by means of Lemma 2.4 we compute the term $Rat - \beta Irrat$ as follows.

$$Rat - \beta Irrat = \nabla_0\nabla_0\nabla_0 f + 4c\beta\nabla_0\nabla_0 f + (n-1)\nabla_0 f\sigma\left(\alpha^2-\beta^2\right) = 0.$$

The above equation along a geodesic $\gamma(t), t \in R$ has the following form

$$f''' + 4c(t)\,u(t)\,f'' + \gamma(t)(n-1)\,f' = 0, \tag{2.20}$$

where, $f(t) = f(x(t))$, $u(t) = \beta(\dot{\gamma}(t))$, and $\gamma(t) = \gamma(x(t))$. To solve this differential equation, we assume first $y = f'$, therefore we have the following second-order differential equation

$$y'' + 4c(t)\,u(t)\,y' + (n-1)\,\gamma(t)\,y = 0, \tag{2.21}$$

where we suppose $l \le \gamma(t) \le l_1$, $m \le u(t) \le m_1$ and $b \le c(t) \le b_1$ are bounded in (2.21). Therefore, we have the following equation

$$\frac{d^2}{dt^2}y(t) + 4bm\left(\frac{d}{dt}y(t)\right) + (n-1)\,ly(t) = 0. \tag{2.22}$$

To solve the equation above, we make the assumption that the solution can be represented as a series: $y(t) = \sum_{k=0}^{\infty} a_k t^k$, and rewrite the above ODE with the series expansion.

Convert $(n-1)\,ly(t)$ to the series expansion

$$(n-1)\,ly(t) = \sum_{k=0}^{\infty} l(n-1)\,a_k t^k. \tag{2.23}$$

Convert $4\text{bm}\frac{d}{dt}y(t)$ to the series expansion

$$4bm\frac{d}{dt}y(t) = \sum_{k=0}^{\infty} 4bma_{k+1}(k+1)\,t^k. \tag{2.24}$$

Convert $\frac{d^2}{dt^2}y(t)$ to the series expansion

$$\frac{d^2}{dt^2}y(t) = \sum_{k=0}^{\infty} a_{k+2}(k+1)(k+2)\,t^k. \tag{2.25}$$

Replacing (2.23) and (2.24), (2.25) in (2.22) we get

$$\sum_{k=0}^{\infty} a_{k+2}(k+1)(k+2)\,t^k + \sum_{k=0}^{\infty} 4bma_{k+1}(k+1)\,t^k + \sum_{k=0}^{\infty} l(n-1)\,a_k t^k = 0. \tag{2.26}$$

Evaluating each term at $t = 0$ gives the recursion relation,

$$\left(k^2 + 3k + 2\right) a_{k+2} + 4bma_{k+1}k + 4bma_{k+1} + l(n-1)\,a_k = 0. \tag{2.27}$$

The above recursion relation, defines the series solution of the ODE

$$a_{k+2} = -\frac{4bma_{k+1}(k+1) + l(n-1)\,a_k}{(k+2)(k+1)}. \tag{2.28}$$

Using the recursion equation, we have the following solution

$$y = a_0 \left(1 - \left(\frac{n-1}{2}\right) lt^2 + \cdots \right) + a_1 \left(t - 2bmt^2 + \cdots\right). \tag{2.29}$$

The limit at infinity is

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} a_0 \left(1 - \left(\frac{n-1}{2}\right) lt^2 + \cdots \right) + a_1 \left(t - 2bmt^2 + \cdots\right) = \infty, \tag{2.30}$$

which is a contradiction with the assumption. Since y is assumed to be bounded, the coefficients $a_0, a_1, \ldots$ must be zero, hence $y = 0$, and f is constant. Since f is constant, we have $\nabla_0 f = \Psi = 0$. Therefore every special projective vector field is Killing on (M, F). This completes the proof of Theorem 1.1. □

Recall that a vector field W on M is conformal with respect to the Riemannian metric h if the Lie derivative of the metric h with respect to W is proportional to h, namely, there is a positive function $\sigma = \sigma(x)$ such that $\mathcal{L}_W h = 2\sigma(x) h$. In [8], it is proved that given any Randers metric $F = \alpha + \beta$ on M expressed in terms of the navigation data (h, W), F has isotropic S-curvature $S = (n+1) c(x) F(x, y)$ if and only if W is a conformal vector field satisfying $\mathcal{L}_W h = -4c(x) h$. Compact Riemannian manifolds that admit a non-trivial conformal vector field exhibit intriguing properties. For instance, the following result summarize information on the interaction between some Riemannian manifolds and conformal vector fields, see [10, 19].

Theorem 2.5 ([10, p.243])**.** *Let (M, h) be a connected n-dimensional $(n \geq 3)$ Riemannian Einstein manifold with Ric $= (n-1)\mu h^2$ and the conformal vector field W on M.*

(a) If $\mu \leq 0$ then, W is a Killing vector field for (M, h).
(b) If $\mu > 0$ then, (M, h) is isometric to the sphere $\mathbb{S}^n\left(\sqrt{\mu}\right)$.

Theorem 2.6 ([7])**.** *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n expressed by (2.5) with the navigation data (h, W). Assume that F is of the isotropic S-curvature $S = (n+1) cF$. Then F is a weak Einstein metric with*

$$\mathbf{Ric} = (n-1)\left(\frac{3\theta}{F} + \sigma\right) F^2,$$

if and only if h is an Einstein metric with

$$\mathbf{Ric}_h = (n-1)\mu h^2,$$

where $\mu = \sigma(x) + c^2 + 2c_{x^m} W^m$, $c = c(x)$, $\sigma = \sigma(x)$ are scalar functions on M, and $\mathbf{Ric}_h$ is the Ricci curvature of the Riemannian metric h.

We are now in a position to prove the theorem 1.2.

Proof. To prove Theorem1.2, let us consider a weak Einstein-Randers metric $F = \alpha + \beta$ with the navigation data (h, W). It is well-known that the S-curvature of F is isotropic, see [3]. By Theorem (2.6) h is an Einstein metric whose Ricci scalar $\mathbf{Ric}_h$ satisfies $\mathbf{Ric}_h = (n-1)\,\mu\,(x)\,h^2$, where $\mu\,(x)$ is a constant say μ since $n \geq 3$. Now, W is a conformal vector field on the Einstein manifold (M, h). By Theorem 2.5, either W is a Killing vector field for h if $\mu \leq 0$ or, (M, h) is isometric to the Euclidean sphere $\mathbb{S}^n\left(\sqrt{\mu}\right)$ if $\mu > 0$. In the former case, S-curvature vanishes since W is Killing. This completes the proof of Theorem 1.2. □

In conclusion, Theorems 1.1 and 1.2 make significant contributions to the study of Randers metrics and weak Einstein-Finsler metrics. Theorem 1.1 demonstrates that under certain conditions, every special projective vector field on a Randers manifold is a Killing vector field. This finding not only illuminates the intricate connection between Randers metrics and special projective vector fields but also carries significant implications for the investigation of geometry and physics within Randers spaces.

Theorem 1.2 provides a complete characterization of the geometry of compact weak Einstein-Randers spaces in terms of the Zermelo navigation data. It reveals that such spaces either have vanishing S-curvature or are isometric to Euclidean spheres, providing a deep understanding of the geometric structure of these spaces. These results are likely to be of great interest to researchers in differential geometry and Finsler geometry, and may have applications in a variety of other areas, including physics, engineering, and computer science.

Acknowledgment. The second author would like to thank the Institut de Mathématique de Toulouse, Université Paul Sabatier, France, (IMT) where parts of this work were carried out.

References

1. H. Akbar-Zadeh, *Initiation to Global Finslerian Geometry*, North-Holland Math. Libr., 2006.
2. H. Akbar-Zadeh, *Champ de vecteurs projectifs sur le fibré unitary*, J. Math. Pures Appl., 65(1986), pp. 47-79.
3. D. Bao and C. Robles, *On Randers spaces of constant flag curvature*, Rep. Math. Phys., 53 (2003).
4. B. Bidabad, B. Lajmiri, M. Rafie-Rad and Y. Keshavarzi, *On projective symmetries on Finsler Spaces*, Diff. Geom. and its App., 129 (2021), Pages 21.

5. S. Bácsó and X. Cheng, *Finsler conformal transformations and the curvature invariances.*, Publ. Mathematicae, 70 (1) (2007).
6. B. Bidabad and A. Asanjarani, *Classification of complete Finsler manifolds through a second order differential equation*, Diff. Geom. and its App., 26 (2008), pp. 434–444.
7. X. Cheng, Q. Qu and S. Xu, *The navigation problems on a class of conic Finsler manifolds*, Diff. Geom. and its App., 74 (2021).
8. X. Cheng and Z. Shen, *Finsler Geometry– An Approach Via Randers Spaces*, Springer, 2012.
9. S. Ishihara and Y. Tashiro, *On Riemannian manifolds admitting a concircular transformation*, Math. J. Okayama Univ., 9 (1959), pp. 19-47.
10. W. Kuhnel and H. Radmacher, *Conformal vector fields on pseudo-Riemannian spaces*,J. Math. Soc. Japan, 14 (3) 3, 1962.
11. M. Molaei, *Hyperbolic dynamics of discrete dynamical systems on pseudo-Riemannian manifolds*, Elect. Res. Ann., in Math. Sci., 25 (2018), pp. 8–15.
12. M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan., 14 (3) (1962).
13. M. Rafie-Rad, *Some new characterizations of projective Randers metrics with constant S-curvature*, J. Geom. Phys., 62 (2) (2012), pp. 272–278.
14. M. Rafie-Rad, *Special projective Lichnérowicz–Obata theorem for Randers spaces* , 351 (2013), pp. 927-930.
15. M. Rafie-Rad, *On Obata theorem in Randers spaces*, J. Geom. Phys., 49 (2016), pp. 380-387.
16. M. Rafie-Rad and B. Rezaei, *On the projective algebra of Randers metrics of constant flag curvature*, SIGMA, 7, 085, (2011), 12 pages.
17. S. Tanno, *Some differential equations on Riemannian manifolds*, J. Math. Soc. Japan., 30(3) (1978).
18. H. Yamabe,*On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J., 12 (1960), pp. 21-37.
19. K. Yano and T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformations*, Ann. of Math., (2) 69 (1959), pp. 451-461.
20. X. Zhai, C. Huang and G. Ren *Extended harmonic mapping connects the equations in classical, statistical, fluid, quantum physics and general relativity*, Sci. Rep., 10 (18281), (2020).

[1] Department of Mathematics and Computer Science Amirkabir University of Technology (Tehran Polytechnic) 424 Hafez Ave. 15914 Tehran, Iran.

Email address: bidabad@aut.ac.ir

[2] Department of Mathematics and Computer Science Amirkabir University of Technology (Tehran Polytechnic) 424 Hafez Ave. 15914 Tehran, Iran.

Email address: behnaz.lajmiri@aut.ac.ir

[3] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.

Email address: rafie-rad@umz.ac.ir