General Fractional Integro-Differential Equation of Order $\varrho \in (2, 3]$ Involving Integral Boundary Conditions

Abdelati El Allaoui

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 1
Pages: 221-236

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.1990457.1264

Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 1 (2024), 221-236
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.1990457.1264

General Fractional Integro-Differential Equation of Order $\varrho \in (2,3]$ Involving Integral Boundary Conditions

Abdelati El Allaoui

Abstract. In this paper, we are interested in studying an integro-differential equation with two-point integral boundary conditions using the Caputo fractional derivative of order $2 < \varrho \leq 3$. The considered problem is transformed into an equivalent integral equation. To study existence and uniqueness results, our approaches used is based on two well-known fixed point theorems, Banach contraction and Krasnoselskii's theorems. To illustrate our obtained outcomes, an example is given at the end of this paper.

1. Introduction

Fractional calculus is widely employed for the mathematical modeling of various natural and engineering processes, as highlighted in [17]. Moreover, several works such as [12, 19, 20] delve into the application of fractional calculus in constructing models to explore theoretical physics problems. In particular, [18] presents a concrete real-world instance that demonstrates the physical interpretation of the Caputo fractional derivative. Numerous researchers have contributed to the study of fractional differential equations, including [2, 13–15].

Firstly, we present an overview of recent advancements in the analysis and solution of fractional differential equations with different types of boundary conditions. As a starting point, we will focus on three notable works by E. Shivanian.

The work [5] delves into the error estimation and stability analysis of a high-order nonlinear fractional differential equation featuring a Caputo derivative and integral boundary condition. Through rigorous analysis

2020 *Mathematics Subject Classification.* 26A33, 34A12, 47G20.
Key words and phrases. Caputo fractional derivative, Existence result, Integral boundary conditions, Integro-differential equation.
Received: 25 February 2023, Accepted: 16 July 2023.

and numerical investigations, the study provides valuable insights into the accuracy and stability of the equation's numerical solution.

In the work [6], E. Shivanian and A. Dinmohammadi tackle the challenging task of solving a nonlinear fractional integro-differential equation with a non-local boundary condition. The authors explore both analytical and numerical methods to obtain solutions and examine the impact of non-locality on the behavior of the equation.

Lastly, E. Shivanian and H. Fatahi's research focuses on the study of a specific class of three-point boundary fractional high-order problems subject to Robin conditions (see [7]). Their work investigates the existence of unique solutions and presents efficient numerical techniques for approximating the solution.

By reviewing these three works, we aim to highlight significant advancements in the understanding, analysis, and numerical solution of fractional differential equations with diverse boundary conditions. The insights gained from these studies contribute to the broader field of fractional calculus and offer valuable tools for modeling and solving complex mathematical problems arising in various scientific and engineering disciplines.

Ahmed et al. discussed the existence and uniqueness of solutions to the following fractional differential equation.

$$\begin{cases} {}^c\mathcal{D}^{\varrho}_{0+}u(t) = \phi(t, u(t)), & \text{for} \quad t \in (0,1), \quad n-1 < \varrho \leq n, \\ u^{(i)}(0) = 0, & \text{for} \quad i = 0, 1, \ldots, n-1, \\ u(\vartheta) = \delta_1 \displaystyle\int_0^{\gamma} u(s)ds + \delta_2 \int_{\sigma}^{1} u(s)ds, & 0 < \gamma < \vartheta < \sigma < 1, \end{cases}$$

where ${}^c\mathcal{D}^{\varrho}_{0+}$ represents the Caputo fractional derivative, for more details see [1].

Dong et al. in [8] investigated the solvability of the following fractional integro-differential equation

$$\begin{cases} {}^c\mathcal{D}^{\nu}_{0+}u(t) = \phi(t, u(t)) + \int_0^t \psi\,(t, s, u(s))\,ds, \quad t \in J = [0, T], \quad 0 < \nu < 1 \\ u(0) = \delta, \end{cases}$$

where the existence and uniqueness of solutions are established using the Banach contraction and the Schauder fixed point theorem.

Motivated by the aforementioned researches, the purpose of this paper is to study the following fractional integro-differential equation involving nonlocal double integral boundary conditions:

(1.1)
$$\begin{cases} {}^c\mathcal{D}^{\varrho}_{0+}u(t) + \lambda\mathcal{I}^{\vartheta}_{0+}\phi(t, u(t)) = \psi(t, u(t)), & t \in I = [0,1], \quad 2 < \varrho \leq 3, \quad \vartheta > 0, \\ u(0) = \displaystyle\int_0^{\gamma} u(\tau)d\tau, & \\ u(1) = \displaystyle\int_{\sigma}^{1} u(\tau)d\tau, & 0 < \gamma < \sigma < 1, \\ u'(0) = 0, & \end{cases}$$

where $\mathcal{D}^{\varrho}$ designates the Caputo fractional derivative [10], $\mathcal{I}_{0+}^{\vartheta}$ represents the Riemann-Liouville operators of fractional order ϑ (see [10]), ϕ, $\psi : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and $\lambda \in \mathbb{R}$.

To the best of our knowledge, no one has combined a fractional integro-differential equation of this type with these integral conditions and a higher order derivative. The innovation also lies in the construction of the equivalent integral equation for this problem.

2. Preliminaries

Firstly, we provide a concise overview of compact operators, including their definitions and key properties. The Krasnoselskii and Banach fixed point theorems, which are fundamental to this study, are also recalled.

Definition 2.1 ([4]).
(i) An operator $T : X \longrightarrow Y$ between two Banach spaces X and Y is called a compact operator if, for any bounded subset A in X, the closure of $T(A)$ be compact.
(ii) T is called a relatively compact operator if the closure of its image set $T(X)$ is compact in Y. This means that $\overline{T(X)}$ is both closed and compact.
(iii) T is said to be equicontinuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ with $\|\mathrm{x} - \mathrm{y}\| < \delta$, it holds that $\|T(x) - T(y)\| < \varepsilon$.

Theorem 2.2 (Arzelà-Ascoli theorem [9]). *Let X be a compact Hausdorff metric space. Then $K \subset \mathcal{C}(X)$ is relatively compact if and only if K is uniformly bounded and uniformly equicontinuous.*

Theorem 2.3 (Banach's Fixed Point Theorem [3]). *Let X be a Banach space and $\Lambda : X \longrightarrow X$ be a contraction on X. Then, Λ has a unique fixed point.*

Theorem 2.4 (Krasnoselskii's fixed point theorem [11]). *Let B be a closed bounded convex subset of a Banach space X. Assume that Λ_1 and Λ_2 are mappings from B into X such that:*

(i) $\Lambda_1(u) + \Lambda_2(v) \in B$ *for all* $u, v \in B$,
(ii) Λ_1 *is a contraction,*
(iii) Λ_2 *is continuous and compact.*

Then, $\Lambda_1 + \Lambda_2$ has a fixed point in B.

Now, we recall essential findings on fractional calculus, which form the foundation of our research. These fundamental results provide the necessary tools and concepts to investigate and analyze fractional differential equations.

Definition 2.5 ([10])**.** For $h \in L^1[0,1]$, we define the left fractional integral of order $\varrho > 0$ of Riemann-Liouville as follows:

$$\mathcal{I}_{0^+}^{\varrho} h(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t-\tau)^{\varrho-1} h(\tau) d\tau.$$

Let $n \in \mathbb{N}$ and $n-1 < \varrho \leq n.$ we recall the following definition:

Definition 2.6 ([10])**.** Let $AC^n[0,1]$ be the space of functions that have nth derivatives absolutely continuous.
The left Caputo fractional derivative for a function $h \in AC^n[0,1]$ of order ϱ is defined as

$${}^{C}\mathcal{D}_{0^+}^{\varrho} h(t) = \frac{1}{\Gamma(n-\varrho)} \int_t^b (t-\tau)^{n-\varrho-1} h^{(n)}(\tau) d\tau,$$

where, Γ represents the Gamma function of Euler [16].

Lemma 2.7 ([21])**.** *We have*

$${}^{C}\mathcal{I}_{0^+}^{\varrho} \left[{}^{C}\mathcal{D}_{0^+}^{\varrho} h(t)\right] = h(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1},$$

for $a_k \in \mathbb{R}$ *and* $k = 0, 1, 2, \ldots, n-1.$

3. Main Results

First, let us construct the solution to our problem. Consider the following problem

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\varrho} u(t) + \lambda \mathcal{I}_{0^+}^{\vartheta} \Phi(t) = \Psi(t), & t \in I = [0,1], \quad 2 < \varrho \leq 3, \quad \vartheta > 0, \\ u(0) = \displaystyle\int_0^{\gamma} u(\tau) d\tau, & \\ u(1) = \displaystyle\int_{\sigma}^{1} u(\tau) d\tau, & 0 < \gamma < \sigma < 1, \\ u'(0) = 0, & \end{cases} \tag{3.1}$$

with $\Phi,\ \Psi \in \mathcal{C}(I)$.

The solution of problem (3.1) is expressed as

$$\begin{aligned} u(t) = {} & \mathcal{I}_{0^+}^{\varrho} \Psi(t) - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta} \Phi(t) + \mu(t) \left[\mathcal{I}_{0^+}^{\varrho+1} \Psi(\gamma) - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta+1} \Phi(\gamma)\right] \\ & + \nu(t) \Big[\lambda \mathcal{I}_{0^+}^{\varrho+\vartheta} \Phi(1) + \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta+1} \Phi(\sigma) + \mathcal{I}_{0^+}^{\varrho+1} \Psi(1) - \mathcal{I}_{0^+}^{\varrho} \Psi(1) \\ & - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta+1} \Phi(1) - \mathcal{I}_{0^+}^{\varrho+1} \Psi(\sigma)\Big], \end{aligned}$$

with

$$\mu(t) = \frac{2 + \sigma^3 - 3\sigma t^2}{\omega}, \qquad \nu(t) = \frac{\gamma^3 + 3(1-\gamma) t^2}{\omega},$$

where $\omega = (1-\gamma)\left(2 + \sigma^3\right) + \sigma \gamma^3.$

Indeed:
Let us apply $\mathcal{I}_{0^+}^{\varrho}$ on both sides of the first equation of problem (3.1), we get
(3.2)

$$\begin{aligned} u(t) &= \mathcal{I}_{0^+}^{\varrho}\Psi(t) - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(t) + a_0 + a_1 t + a_2 t^2 \\ &= \frac{1}{\Gamma(\varrho)}\int_0^t (t-\tau)^{\varrho-1}\Psi(\tau)d\tau - \frac{\lambda}{\Gamma(\varrho+\vartheta)}\int_0^t (t-\tau)^{\varrho+\vartheta-1}\Phi(\tau)d\tau \\ &\quad + a_0 + a_1 t + a_2 t^2, \end{aligned}$$

where a_0, a_1, a_2 are unknowns to be determined.
From (3.2), we have

$$u(0) = a_0 = \int_0^{\gamma} u(\tau)d\tau. \tag{3.3}$$

By deriving both sides of equation (3.2), we have

$$u'(t) = \mathcal{I}_{0^+}^{\varrho-1}\Psi(t) - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta-1}\Phi(t) + a_1 + 2a_2 t,$$

which implies that

$$u'(0) = a_1 = 0. \tag{3.4}$$

From (3.2), we have

$$\begin{aligned} u(1) &= \mathcal{I}_{0^+}^{\varrho}\Psi(1) - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1) + a_0 + a_2 \\ &= \int_{\sigma}^{1} u(\tau)d\tau, \end{aligned}$$

which means that

$$a_2 = \int_{\sigma}^{1} u(\tau)d\tau + \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1) - \int_0^{\gamma} u(\tau)d\tau - \mathcal{I}_{0^+}^{\varrho}\Psi(1). \tag{3.5}$$

By substitution of (3.3), (3.4) and (3.5) in (3.2), we obtain
(3.6)

$$\begin{aligned} u(t) &= \mathcal{I}_{0^+}^{\varrho}\Psi(t) - \lambda \mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(t) + \lambda t^2 \mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1) - t^2 \mathcal{I}_{0^+}^{\varrho}\Psi(1) \\ &\quad + (1-t^2)\int_0^{\gamma} u(\tau)d\tau + t^2\int_{\sigma}^{1} u(\tau)d\tau \\ &= \frac{1}{\Gamma(\varrho)}\int_0^t (t-\tau)^{\varrho-1}\Psi(\tau)d\tau - \frac{\lambda}{\Gamma(\varrho+\vartheta)}\int_0^t (t-\tau)^{\varrho+\vartheta-1}\Phi(\tau)d\tau \\ &\quad + \lambda t^2 \mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1) - t^2 \mathcal{I}_{0^+}^{\varrho}\Psi(1) + (1-t^2)\int_0^{\gamma} u(\tau)d\tau \\ &\quad + t^2\int_{\sigma}^{1} u(\tau)d\tau, \end{aligned}$$

note that this solution is implicitly defined. In the next step, we will determine the two unknowns $\int_0^\gamma u(\tau)d\tau$ and $\int_\sigma^1 u(\tau)d\tau$.
Integrating both sides of equation (3.6) on $[0,\gamma]$, we have

$$\begin{aligned}\int_0^\gamma u(t)dt =& \frac{1}{\Gamma(\varrho+1)}\int_0^\gamma(\gamma-\tau)^\varrho\Psi(\tau)d\tau+\lambda\frac{\gamma^3}{3}\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)\\ &-\frac{\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^\gamma(\gamma-\tau)^{\varrho+\vartheta}\Phi(\tau)d\tau\\ &-\frac{\gamma^3}{3}\mathcal{I}_{0^+}^{\varrho}\Psi(1)+\left(\gamma-\frac{\gamma^3}{3}\right)\int_0^\gamma u(\tau)d\tau+\frac{\gamma^3}{3}\int_\sigma^1 u(\tau)d\tau,\end{aligned}$$

which is equivalent to

(3.7)

$$\begin{aligned}&\left(3+\gamma^3\ -3\gamma\right)\int_0^\gamma u(\tau)d\tau-\gamma^3\int_\sigma^1 u(\tau)d\tau\\ &\quad=\frac{3}{\Gamma(\varrho+1)}\int_0^\gamma(\gamma-\tau)^\varrho\Psi(\tau)d\tau+\lambda\gamma^3\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)\\ &\qquad-\frac{3\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^\gamma(\gamma-\tau)^{\varrho+\vartheta}\Phi(\tau)d\tau-\gamma^3\mathcal{I}_{0^+}^{\varrho}\Psi(1)\\ &\quad=3\mathcal{I}_{0^+}^{\varrho+1}\Psi(\gamma)-3\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\gamma)+\lambda\gamma^3\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)-\gamma^3\mathcal{I}_{0^+}^{\varrho}\Psi(1).\end{aligned}$$

Now, integrating both sides of equation (3.6) on $[\sigma,1]$, we obtain

$$\begin{aligned}&\int_\sigma^1 u(t)dt\\ &\quad=\frac{1}{\Gamma(\varrho+1)}\int_0^1(1-\tau)^\varrho\Psi(\tau)d\tau-\frac{1}{\Gamma(\varrho+1)}\int_0^\sigma(\sigma-\tau)^\varrho\Psi(\tau)d\tau\\ &\qquad-\frac{\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^1(1-\tau)^{\varrho+\vartheta}\Phi(\tau)d\tau+\frac{\lambda\left(1-\sigma^3\right)}{3}\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)\\ &\qquad+\frac{\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^\sigma(\sigma-\tau)^{\varrho+\vartheta}\Phi(\tau)d\tau-\frac{1-\sigma^3}{3}\mathcal{I}_{0^+}^{\varrho}\Psi(1)\\ &\qquad+\frac{2+\sigma^3-3\sigma}{3}\int_0^\gamma u(\tau)d\tau+\frac{1-\sigma^3}{3}\int_\sigma^1 u(\tau)d\tau,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\left(2+\sigma^3-3\sigma\right)\int_0^\gamma u(\tau)d\tau-\left(2+\sigma^3\right)\int_\sigma^1 u(\tau)d\tau\\ &\quad=\frac{3}{\Gamma(\varrho+1)}\int_0^\sigma(\sigma-\tau)^\varrho\Psi(\tau)d\tau\end{aligned}\tag{3.8}$$

$$
\begin{aligned}
&+\frac{3\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^1(1-\tau)^{\varrho+\vartheta}\Phi(\tau)d\tau\\
&-\frac{3}{\Gamma(\varrho+1)}\int_0^1(1-\tau)^{\varrho}\Psi(\tau)d\tau\\
&-\frac{3\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^\sigma(\sigma-\tau)^{\varrho+\vartheta}\Phi(\tau)d\tau\\
&-\lambda\left(1-\sigma^3\right)\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)+\left(1-\sigma^3\right)\mathcal{I}_{0^+}^{\varrho}\Psi(1)\\
=\,&\mathcal{I}_{0^+}^{\varrho+1}\Psi(\sigma)+3\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(1)-3\mathcal{I}_{0^+}^{\varrho+1}\Psi(1)\\
&-3\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\sigma)-\lambda\left(1-\sigma^3\right)\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)\\
&+\left(1-\sigma^3\right)\mathcal{I}_{0^+}^{\varrho}\Psi(1).
\end{aligned}
$$

We define

$$
S=\begin{pmatrix} 3+\gamma^3-3\gamma & -\gamma^3\\ 2+\sigma^3-3\sigma & -\left(2+\sigma^3\right)\end{pmatrix},\qquad U=\begin{pmatrix}\displaystyle\int_0^\gamma u(\tau)d\tau\\ \displaystyle\int_\sigma^1 u(\tau)d\tau\end{pmatrix},
$$

$$
B=\begin{pmatrix} b_1\\ b_2\end{pmatrix},
$$

where

$$
\begin{aligned}
b_1&=3\mathcal{I}_{0^+}^{\varrho+1}\Psi(\gamma)-3\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\gamma)+\lambda\gamma^3\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)-\gamma^3\mathcal{I}_{0^+}^{\varrho}\Psi(1),\\
b_2&=\mathcal{I}_{0^+}^{\varrho+1}\Psi(\sigma)+3\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(1)-3\mathcal{I}_{0^+}^{\varrho+1}\Psi(1)-3\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\sigma)\\
&\quad-\lambda\left(1-\sigma^3\right)\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)+\left(1-\sigma^3\right)\mathcal{I}_{0^+}^{\varrho}\Psi(1).
\end{aligned}
$$

From equations (3.7) and (3.8), we have the following system

$$
SU=B. \tag{3.9}
$$

Note that

$$
\det(S)=-3\left[(1-\gamma)\left(2+\sigma^3\right)+\sigma\gamma^3\right]<0.
$$

Then, the system (3.9) has a unique solution.
Using Cramer's rule, one can get

(3.10)

$$
\begin{aligned}
\int_0^\gamma u(\tau)d\tau=\,&\frac{\gamma^3}{\omega}\Big(\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)+\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\sigma)+\mathcal{I}_{0^+}^{\varrho+1}\Psi(1)-\mathcal{I}_{0^+}^{\varrho}\Psi(1)\\
&-\mathcal{I}_{0^+}^{\varrho+1}\Psi(\sigma)-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(1)\Big)\\
&+\frac{2+\sigma^3}{\omega}\left(\mathcal{I}_{0^+}^{\varrho+1}\Psi(\gamma)-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\gamma)\right),
\end{aligned}
$$

and

(3.11)

$$\int_\sigma^1 u(\tau)d\tau = \frac{3+\gamma^3-3\gamma}{\omega}\Big(\mathcal{I}_{0^+}^{\varrho+1}\Psi(1)+\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\sigma)-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(1)$$
$$-\mathcal{I}_{0^+}^{\varrho+1}\Psi(\sigma)\Big)+\frac{2+\sigma^3-3\sigma}{\omega}\Big(\mathcal{I}_{0^+}^{\varrho+1}\Psi(\gamma)-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\gamma)\Big)$$
$$+\frac{(1-\gamma)\left(1-\sigma^3\right)+\gamma^3(1-\sigma)}{\omega}\Big(\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)-\mathcal{I}_{0^+}^{\varrho}\Psi(1)\Big).$$

By substitution of (3.10) and (3.11) into (3.6), we get

(3.12)

$$u(t)=\mathcal{I}_{0^+}^{\varrho}\Psi(t)-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(t)+\mu(t)\Big[\mathcal{I}_{0^+}^{\varrho+1}\Psi(\gamma)-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\gamma)\Big]$$
$$+\nu(t)\Big[\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta}\Phi(1)+\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(\sigma)+\mathcal{I}_{0^+}^{\varrho+1}\Psi(1)-\mathcal{I}_{0^+}^{\varrho}\Psi(1)$$
$$-\lambda\mathcal{I}_{0^+}^{\varrho+\vartheta+1}\Phi(1)-\mathcal{I}_{0^+}^{\varrho+1}\Psi(\sigma)\Big],$$

Therefore, the problem (1.1) can be abstracted to the following fixed point problem

$$\Lambda u(t)=\frac{1}{\Gamma(\varrho)}\int_0^t(t-\tau)^{\varrho-1}\psi(\tau,u(\tau))d\tau \tag{3.13}$$
$$-\frac{\lambda}{\Gamma(\varrho+\vartheta)}\int_0^t(t-\tau)^{\varrho+\vartheta-1}\phi(\tau,u(\tau))d\tau$$
$$+\mu(t)\Big[\frac{1}{\Gamma(\varrho+1)}\int_0^\gamma(\gamma-\tau)^{\varrho}\psi(\tau,u(\tau))d\tau$$
$$-\frac{\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^\gamma(\gamma-\tau)^{\varrho+\vartheta}\phi(\tau,u(\tau))d\tau\Big]$$
$$+\nu(t)\Big[\frac{\lambda}{\Gamma(\varrho+\vartheta)}\int_0^1(1-\tau)^{\varrho+\vartheta-1}\phi(\tau,u(\tau))d\tau$$
$$+\frac{\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^\sigma(\sigma-\tau)^{\varrho+\vartheta}\phi(\tau,u(\tau))d\tau$$
$$+\frac{1}{\Gamma(\varrho+1)}\int_0^1(1-\tau)^{\varrho}\psi(\tau,u(\tau))d\tau$$
$$-\frac{1}{\Gamma(\varrho)}\int_0^1(1-\tau)^{\varrho-1}\psi(\tau,u(\tau))d\tau$$
$$-\frac{\lambda}{\Gamma(\varrho+\vartheta+1)}\int_0^1(1-\tau)^{\varrho+\vartheta}\phi(\tau,u(\tau))d\tau$$
$$-\frac{1}{\Gamma(\varrho+1)}\int_0^\sigma(\sigma-\tau)^{\varrho}\psi(\tau,u(\tau))d\tau\Big].$$

Let $\mathcal{X} = \mathcal{C}(I, \mathbb{R})$ the space formed by continuous functions on I.
The Banach space $\mathcal{X}$ is endowed with the norm $\|y\| = \sup_{t \in I} |y(t)|$.
Before citing our existence results, we consider the assumptions below:

($\mathcal{H}_1$) There exist k_ϕ, $k_\psi > 0$ such that, $\forall t \in I$, $\forall u, v \in \mathbb{R}$

$$|\phi(t, v) - \phi(t, u)| \leq k_\phi |v - u|,$$
$$|\psi(t, v) - \psi(t, u)| \leq k_\psi |v - u|$$

($\mathcal{H}_2$) There exist θ_1, $\theta_2 \in \mathcal{C}(I, \mathbb{R}^+)$ such that, $\forall t \in I$, $\forall u \in \mathbb{R}$

$$|\phi(t, u)| \leq \theta_1(t),$$
$$|\psi(t, u)| \leq \theta_2(t).$$

At this point, we are able to present our first existence result.

Theorem 3.1. *Suppose that assumptions* ($\mathcal{H}_1$) *and* ($\mathcal{H}_2$) *hold. Suppose in addition that*

$$\xi := \frac{|\lambda| k_\phi}{\Gamma(\varrho + \vartheta + 1)} + \frac{k_\psi}{\Gamma(\varrho + 1)} < 1. \tag{3.14}$$

Then, there exists at least one solution for problem (1.1).

Proof. First, we consider the following closed ball

$$\mathcal{B}_r = \{u \in \mathcal{X} : \|u\| \leq r\},$$

with $r \geq \lambda_1 \|\theta_1\| + \lambda_2 \|\theta_2\|$, where

$$\lambda_1 = \frac{|\lambda| \left[\varrho + \vartheta + 1 + (\varrho + \vartheta + \sigma)\|\nu\| + \gamma \|\mu\|\right]}{\Gamma(\varrho + \vartheta + 2)},$$
$$\lambda_2 = \frac{\varrho + 1 + (\varrho + \sigma)\|\nu\| + \gamma \|\mu\|}{\Gamma(\varrho + 2)}.$$

The operator Λ defined by (3.13) can be written as $\Lambda_1 + \Lambda_2$, with

$$\Lambda_1 u(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t - \tau)^{\varrho - 1} \psi(\tau, u(\tau)) d\tau - \frac{\lambda}{\Gamma(\varrho + \vartheta)} \int_0^t (t - \tau)^{\varrho + \vartheta - 1} \phi(\tau, u(\tau)) d\tau,$$

and

$$\Lambda_2 u(t) = \mu(t) \left[\frac{1}{\Gamma(\varrho + 1)} \int_0^\gamma (\gamma - \tau)^\varrho \psi(\tau, u(\tau)) d\tau - \frac{\lambda}{\Gamma(\varrho + \vartheta + 1)} \int_0^\gamma (\gamma - \tau)^{\varrho + \vartheta} \phi(\tau, u(\tau)) d\tau\right]$$
$$+ \nu(t) \left[\frac{\lambda}{\Gamma(\varrho + \vartheta)} \int_0^1 (1 - \tau)^{\varrho + \vartheta - 1} \phi(\tau, u(\tau)) d\tau\right.$$

$$
\begin{aligned}
&+ \frac{\lambda}{\Gamma(\varrho+\vartheta+1)} \int_0^\sigma (\sigma-\tau)^{\varrho+\vartheta}\phi(\tau,u(\tau))d\tau \\
&+ \frac{1}{\Gamma(\varrho+1)} \int_0^1 (1-\tau)^{\varrho}\psi(\tau,u(\tau))d\tau \\
&- \frac{1}{\Gamma(\varrho)} \int_0^1 (1-\tau)^{\varrho-1}\psi(\tau,u(\tau))d\tau \\
&- \frac{\lambda}{\Gamma(\varrho+\vartheta+1)} \int_0^1 (1-\tau)^{\varrho+\vartheta}\phi(\tau,u(\tau))d\tau \\
&- \frac{1}{\Gamma(\varrho+1)} \int_0^\sigma (\sigma-\tau)^{\varrho}\psi(\tau,u(\tau))d\tau \Bigg].
\end{aligned}
$$

(*i*) For $u,\, v \in \mathcal{B}_r,\ \Lambda_1 u + \Lambda_2 v \in \mathcal{B}_r$. Indeed:

$$
\begin{aligned}
&\|\Lambda_1 u + \Lambda_2 v\| \\
&\quad \le \sup_{t\in I} \Bigg\{ \frac{1}{\Gamma(\varrho)} \int_0^t (t-\tau)^{\varrho-1} |\psi(\tau,u(\tau))| d\tau \\
&\qquad + \frac{|\lambda|}{\Gamma(\varrho+\vartheta)} \int_0^t (t-\tau)^{\varrho+\vartheta-1} |\phi(\tau,u(\tau))| d\tau \Bigg\} \\
&\qquad + \sup_{t\in I} |\mu(t)| \Bigg[\frac{1}{\Gamma(\varrho+1)} \int_0^\gamma (\gamma-\tau)^{\varrho} |\psi(\tau,v(\tau))| d\tau \\
&\qquad + \frac{|\lambda|}{\Gamma(\varrho+\vartheta+1)} \int_0^\gamma (\gamma-\tau)^{\varrho+\vartheta} |\phi(\tau,v(\tau))| d\tau \Bigg] \\
&\qquad + \sup_{t\in I} |\nu(t)| \Bigg[|\lambda| \int_0^1 \left| \frac{(1-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} - \frac{(1-\tau)^{\varrho+\vartheta-1}}{\Gamma(\varrho+\vartheta)} \right| |\phi(\tau,v(\tau))| d\tau \\
&\qquad + \int_0^1 \left| \frac{(1-\tau)^{\varrho}}{\Gamma(\varrho+1)} - \frac{(1-\tau)^{\varrho-1}}{\Gamma(\varrho)} \right| |\psi(\tau,v(\tau))| d\tau \\
&\qquad + |\lambda| \int_0^\sigma \frac{(\sigma-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} |\phi(\tau,v(\tau))| d\tau + \int_0^\sigma \frac{(\sigma-\tau)^{\varrho}}{\Gamma(\varrho+1)} |\psi(\tau,v(\tau))| d\tau \Bigg] \\
&\quad \le |\lambda| \|\theta_1\| \left[\frac{1}{\Gamma(\varrho+\vartheta+1)} + \frac{(\varrho+\vartheta+\sigma)\|\nu\| + \gamma\|\mu\|}{\Gamma(\varrho+\vartheta+2)} \right] \\
&\qquad + \|\theta_2\| \left[\frac{1}{\Gamma(\varrho+1)} + \frac{(\varrho+\sigma)\|\nu\| + \gamma\|\mu\|}{\Gamma(\varrho+2)} \right] \\
&\quad \le \frac{|\lambda| \left[\varrho+\vartheta+1+(\varrho+\vartheta+\sigma)\|\nu\| + \gamma\|\mu\|\right]}{\Gamma(\varrho+\vartheta+2)} \|\theta_1\| \\
&\qquad + \frac{\varrho+1+(\varrho+\sigma)\|\nu\| + \gamma\|\mu\|}{\Gamma(\varrho+2)} \|\theta_2\| \\
&\quad \le \lambda_1 \|\theta_1\| + \lambda_2 \|\theta_2\|
\end{aligned}
$$

$\leq r.$

(ii) Λ_1 is a contraction. Indeed:
For $u,\, v \in \mathcal{B}_r$, we have

$$\begin{aligned}\|\Lambda_1 v - \Lambda_1 u\| &\leq \sup_{t\in I}\left\{\frac{1}{\Gamma(\varrho)}\int_0^t (t-\tau)^{\varrho-1}|\psi(\tau, v(\tau)) - \psi(\tau, u(\tau))|d\tau\right.\\ &\quad \left. + \frac{|\lambda|}{\Gamma(\varrho+\vartheta)}\int_0^t (t-\tau)^{\varrho+\vartheta-1}|\phi(\tau, v(\tau)) - \phi(\tau, u(\tau))|d\tau\right\}\\ &\leq \left(\frac{\|\lambda\| k_\phi}{\Gamma(\varrho+\vartheta+1)} + \frac{k_\psi}{\Gamma(\varrho+1)}\right)\|v-u\|\\ &\leq \xi\|v-u\|.\end{aligned}$$

(iii) Note that Λ_2 is continuous, and for $u \in \mathcal{B}_r$, we have

$$\begin{aligned}\|\Lambda_2 u\| \leq \|\mu\| &\left[\frac{1}{\Gamma(\varrho+1)}\int_0^\gamma (\gamma-\tau)^{\varrho}|\psi(\tau,u(\tau))|d\tau\right.\\ &\left. + \frac{|\lambda|}{\Gamma(\varrho+\vartheta+1)}\int_0^\gamma (\gamma-\tau)^{\varrho+\vartheta}|\phi(\tau,u(\tau))|d\tau\right]\\ &+ \|\nu\|\left[|\lambda|\int_0^1\left|\frac{(1-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)} - \frac{(1-\tau)^{\varrho+\vartheta-1}}{\Gamma(\varrho+\vartheta)}\right||\phi(\tau,u(\tau))|d\tau\right.\\ &+ \int_0^1\left|\frac{(1-\tau)^{\varrho}}{\Gamma(\varrho+1)} - \frac{(1-\tau)^{\varrho-1}}{\Gamma(\varrho)}\right||\psi(\tau,u(\tau))|d\tau\\ &+ |\lambda|\int_0^\sigma \frac{(\sigma-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}|\phi(\tau,u(\tau))|d\tau\\ &\left.+ \int_0^\sigma \frac{(\sigma-\tau)^{\varrho}}{\Gamma(\varrho+1)}|\psi(\tau,u(\tau))|d\tau\right]\\ &\leq \frac{|\lambda|\left[(\varrho+\vartheta+\sigma)\|\nu\| + \gamma\|\mu\|\right]}{\Gamma(\varrho+\vartheta+2)}\|\theta_1\| + \frac{(\varrho+\sigma)\|\nu\| + \gamma\|\mu\|}{\Gamma(\varrho+2)}\|\theta_2\|,\end{aligned}$$

which prove that Λ_2 is uniformly bounded on this ball.
Now, let us prove that Λ_2 is compact:
Denote

$$\xi_\phi = \sup_{(t,u)\in I\times\mathcal{B}_r}|\phi(t,u)| \quad \text{and} \quad \xi_\psi = \sup_{(t,u)\in I\times\mathcal{B}_r}|\psi(t,u)|.$$

For $u \in \mathcal{B}_r$ and $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}&\|(\Lambda_2 u)\,(t_2) - (\Lambda_2 u)\,(t_1)\|\\ &\quad\leq \left[\frac{1}{\Gamma(\varrho+1)}\int_0^\gamma (\gamma-\tau)^{\varrho}|\psi(\tau,v(\tau))|d\tau\right.\\ &\qquad\left. + \frac{|\lambda|}{\Gamma(\varrho+\vartheta+1)}\int_0^\gamma (\gamma-\tau)^{\varrho+\vartheta}|\phi(\tau,v(\tau))|d\tau\right]|\mu(t_2) - \mu(t_1)|\end{aligned}$$

$$
\begin{aligned}
&+\left[|\lambda|\int_0^1\left|\frac{(1-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}-\frac{(1-\tau)^{\varrho+\vartheta-1}}{\Gamma(\varrho+\vartheta)}\right||\phi(\tau,v(\tau))|d\tau\right.\\
&+\int_0^1\left|\frac{(1-\tau)^{\varrho}}{\Gamma(\varrho+1)}-\frac{(1-\tau)^{\varrho-1}}{\Gamma(\varrho)}\right||\psi(\tau,v(\tau))|d\tau\\
&+|\lambda|\int_0^\sigma\frac{(\sigma-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}|\phi(\tau,v(\tau))|d\tau\\
&\left.+\int_0^\sigma\frac{(\sigma-\tau)^{\varrho}}{\Gamma(\varrho+1)}|\psi(\tau,v(\tau))|d\tau\right]|\nu(t_2)-\nu(t_1)|\\
\leq&\left[\frac{|\lambda|\gamma\xi_\phi}{\Gamma(\varrho+\vartheta+2)}+\frac{\gamma\xi_\psi}{\Gamma(\varrho+2)}\right]|\mu(t_2)-\mu(t_1)|\\
&+\left[\frac{|\lambda|(\varrho+\vartheta+\sigma)\xi_\phi}{\Gamma(\varrho+\vartheta+2)}+\frac{(\varrho+\sigma)\xi_\psi}{\Gamma(\varrho+2)}\right]|\nu(t_2)-\nu(t_1)|,
\end{aligned}
$$

since

$$
\begin{aligned}
|\mu(t_2)-\mu(t_1)|&\leq\frac{3(1-\gamma)}{\omega}|t_2^2-t_1^2|\\
&\leq\frac{6(1-\gamma)}{\omega}|t_2-t_1|,
\end{aligned}
$$

and

$$
|\nu(t_2)-\nu(t_1)|\leq\frac{3\sigma}{\omega}|t_2^2-t_1^2|\leq\frac{6\sigma}{\omega}|t_2-t_1|.
$$

Therefore,

$$
\begin{aligned}
&\|(\Lambda_2u)(t_2)-(\Lambda_2u)(t_1)\|\\
&\quad\leq\left[\frac{6(1-\gamma)}{\omega}\left(\frac{|\lambda|\gamma\xi_\phi}{\Gamma(\varrho+\vartheta+2)}+\frac{\gamma\xi_\psi}{\Gamma(\varrho+2)}\right)\right.\\
&\qquad\left.+\frac{6\sigma}{\omega}\left(\frac{|\lambda|(\varrho+\vartheta+\sigma)\xi_\phi}{\Gamma(\varrho+\vartheta+2)}+\frac{(\varrho+\sigma)\xi_\psi}{\Gamma(\varrho+2)}\right)\right]|t_2-t_1|.
\end{aligned}
$$

Hence, Λ_2 is equicontinuous. Then, it is relatively compact on $\mathcal{B}_r$.

Thus, according to Arzelà-Ascoli theorem, Λ_2 is compact. Consequently, all conditions of Krasnoselskii's fixed-point theorem hold.

The result is then proved. □

Based on Banach contraction, the result of existence and uniqueness will be the subject of the next theorem.

Theorem 3.2. *Suppose that* $(\mathcal{H}_1)$ *is satisfied. Suppose also that*

$$
k:=\lambda_1k_\phi+\lambda_2k_\psi<1 \tag{3.15}
$$

Then, there exists a unique solution for problem (1.1).

Proof. Denote

$$\eta_\phi = \sup_{t\in I} |\phi(t,0)|, \qquad \eta_\psi = \sup_{t\in I} |\psi(t,0)|,$$

let

$$\mathcal{B}_\rho = \{u \in \mathcal{X} : \|u\| \le \rho\},$$

the closed ball of radius ρ defined by

$$\rho \ge \frac{\lambda_1 \eta_\phi + \lambda_2 \eta_\psi}{1-k}.$$

Firstly, show that Λ is defined on $\mathcal{B}_\rho$ into itself.
For $t \in I$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}|\phi(t,x)| &\le |\phi(t,x) - \phi(t,0)| + |\phi(t,0)| \\ &\le k_\phi |x| + \eta_\phi.\end{aligned}$$

Then,

$$\begin{aligned}|\phi(t,u(t))| &\le k_\phi \|u\| + \eta_\phi \\ &\le k_\phi \rho + \eta_\phi, \quad \forall (t,u) \in I \times \mathcal{B}_\rho.\end{aligned}$$

Similarly, we have

$$|\psi(t,u(t))| \le k_\psi \rho + \eta_\psi, \quad \forall (t,u) \in I \times \mathcal{B}_\rho,$$

let $u \in \mathcal{B}_\rho$. In a similar way to the proof given in (i), we get

$$\begin{aligned}\|\Lambda u\| &\le \lambda_1 \sup_{t\in I} |\phi(t,u(t))| + \lambda_2 \sup_{t\in I} |\psi(t,u(t))| \\ &\le \lambda_1 (k_\phi \rho + \eta_\phi) + \lambda_2 (k_\psi \rho + \eta_\psi) \\ &\le k\rho + \lambda_1 \eta_\phi + \lambda_2 \eta_\psi \\ &\le \rho.\end{aligned}$$

Now, let us prove that Λ is a contraction.
For u, $v \in \mathcal{B}_\rho$, we have

$$\begin{aligned}&\|\Lambda v - \Lambda u\| \\ &\le \sup_{t\in I} \left\{ \frac{1}{\Gamma(\varrho)} \int_0^t (t-\tau)^{\varrho-1} |\psi(\tau, v(\tau)) - \psi(\tau, u(\tau))| d\tau \right. \\ &\quad \left. + \frac{|\lambda|}{\Gamma(\varrho+\vartheta)} \int_0^t (t-\tau)^{\varrho+\vartheta-1} |\phi(\tau, v(\tau)) - \phi(\tau, u(\tau))| d\tau \right\} \\ &\quad + \|\mu\| \left[\frac{1}{\Gamma(\varrho+1)} \int_0^\gamma (\gamma-\tau)^{\varrho} |\psi(\tau, v(\tau)) - \psi(\tau, u(\tau))| d\tau \right. \\ &\quad \left. + \frac{|\lambda|}{\Gamma(\varrho+\vartheta+1)} \int_0^\gamma (\gamma-\tau)^{\varrho+\vartheta} |\phi(\tau, v(\tau)) - \phi(\tau, u(\tau))| d\tau \right]\end{aligned}$$

$$
\begin{aligned}
&+\|\nu\|\left[\int_0^1\left|\frac{(1-\tau)^{\varrho}}{\Gamma(\varrho+1)}-\frac{(1-\tau)^{\varrho-1}}{\Gamma(\varrho)}\right||\psi(\tau,v(\tau))-\psi(\tau,u(\tau))|d\tau\right.\\
&+|\lambda|\int_0^1\left|\frac{(1-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}-\frac{(1-\tau)^{\varrho+\vartheta-1}}{\Gamma(\varrho+\vartheta)}\right||\phi(\tau,v(\tau))-\phi(\tau,u(\tau))|d\tau\\
&+|\lambda|\int_0^\sigma\frac{(\sigma-\tau)^{\varrho+\vartheta}}{\Gamma(\varrho+\vartheta+1)}|\phi(\tau,v(\tau))-\phi(\tau,u(\tau))|d\tau\\
&\left.+\int_0^\sigma\frac{(\sigma-\tau)^{\varrho}}{\Gamma(\varrho+1)}|\psi(\tau,v(\tau))-\psi(\tau,u(\tau))|d\tau\right]\\
\leq&\left[k_\phi|\lambda|\left(\frac{1}{\Gamma(\varrho+\vartheta+1)}+\frac{(\varrho+\vartheta+\sigma)\|\nu\|+\gamma\|\mu\|}{\Gamma(\varrho+\vartheta+2)}\right)\right.\\
&\left.+k_\psi\left(\frac{1}{\Gamma(\varrho+1)}+\frac{(\varrho+\sigma)\|\nu\|+\gamma\|\mu\|}{\Gamma(\varrho+2)}\right)\right]\|v-u\|\\
\leq&\left(\lambda_1k_\phi+\lambda_2k_\psi\right)\|v-u\|\\
\leq&\, k\|v-u\|,
\end{aligned}
$$

hence, the existence and uniqueness of the solution due to Banach's theorem □

Remark 3.3. We observe that,

$$
k=\xi+\left(\frac{|\lambda|\left[(\varrho+\vartheta+\sigma)\|\nu\|+\gamma\|\mu\|\right]}{\Gamma(\varrho+\vartheta+2)}k_\phi+\frac{(\varrho+\sigma)\|\nu\|+\gamma\|\mu\|}{\Gamma(\varrho+2)}k_\psi\right).
$$

Which means that, if the condition (3.15) of Theorem 3.2 holds, then the one given by (3.14) in Theorem 3.1 also holds.

To illustrate our results, we consider the following example.

Example 3.4. we consider the problem defined by

$$
\begin{cases}
{}^{c}\mathcal{D}_{0^+}^{\frac{5}{2}}u(t)+\dfrac{1}{3}\mathcal{I}_{0^+}^{\frac{1}{2}}\left[\dfrac{\cos(u(t))+\sin(t)}{30+t^2}\right]\\
\qquad=\dfrac{1}{\sqrt{2500+t^2}}\left(\dfrac{|u(t)|}{1+|u(t)|}+e^{-t}\right), \quad t\in I,\\
u(0)=\int_0^{\frac{1}{4}}u(\tau)d\tau,\\
u(1)=\int_{\frac{1}{2}}^{1}u(\tau)d\tau,\\
u'(0)=0.
\end{cases}
\tag{3.16}
$$

The problem (3.16) can be abstracted into (1.1), with

$$
\phi(t,u)=\frac{\cos(u(t))+\sin(t)}{30+t^2},\quad \psi(t,u)=\frac{1}{\sqrt{2500+t^2}}\left(\frac{|u|}{1+|u|}+e^{-t}\right),
$$

$$
\vartheta=\sigma=\frac{1}{2},\ \gamma=\frac{1}{4},\ \varrho=\frac{5}{2}\ \text{and}\ \lambda=\frac{1}{3}.
$$

Note that assumption $(\mathcal{H}_1)$ holds, where

$$k_\phi = \frac{1}{30}, \qquad k_\psi = \frac{1}{50}.$$

We have also

$$\mu(t) = \frac{272}{205} - \frac{192}{205}t^2, \qquad \nu(t) = \frac{2}{205} + \frac{288}{205}t^2.$$

Then, we get $k = 0.0182 < 1$. According to Theorem 3.2, problem (3.16) has a unique solution.

Note that, according to the Remark 3.3, the existence of the solution is guaranteed.

References

1. B. Ahmad and R.P. Agarwal, *Some new versions of fractional boundary value problems with slitstrips conditions*, Bound. Value Probl., 175 (2014), 12 pages.
2. Y. Allaoui, S. Melliani and A. EL Allaoui, & K. Hilal, *Coupled System of Mixed Hybrid Fractional Differential Equations: Linear Perturbations of First and Second Type*, In Recent Advances in Fuzzy Sets Theory, Fractional Calculus, Dynamic Systems and Optimization. Cham: Springer International Publishing, (2022), pp. 446-457
3. S. Banach, *On operations in abstract sets and their application to integral equations*, Fundamenta Mathematicae, 3 (1922), pp. 133-181.
4. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, Chapter 9, Section 1, 2010.
5. E. Shivanian, *Error estimate and stability analysis on the study of a high-order nonlinear fractional differential equation with Caputo-derivative and integral boundary condition*, Comp. Appl. Math., 41 395 (2022).
6. E. Shivanian and A. Dinmohammadi, *On the solution of a nonlinear fractional integro-differential equation with non-local boundary condition*, Int. J. Nonlinear Anal. Appl., In Press, (2023), pp. 1-12.
7. E. Shivanian and H. Fatahi, *To study existence of unique solution and numerically solving for a kind of three-point boundary fractional high-order problem subject to Robin condition*, Comput. Methods Differ. Equ., 11 (2) (2023), pp. 332-344.
8. L. Dong, N. Hoa and H. Vu, *Existence and Ulam stability for random fractional integro-differential equation*, Afr. Mat., (2020), pp. 1-12.
9. D. Hadwin, *Noncommutative Arzela-Ascoli theorems*, Journal of Operator Theory, 38 (1) (1997), pp. 43-66.

10. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science B. V., Amsterdam, 204 (2006).
11. M.A. Krasnosel'skii, *Some Problems of Nonlinear Analysis*, Am. Math. Soc. Transl. Ser., 2 (10) (1958), pp. 345-409.
12. D. Kumar and D. Baleanu, *Fractional calculus and its applications in physics*, Frontiers in Physics, 7 (2019).
13. S. Muthaiah, S. Muthu, M. Murugesan and N.G. Thangaraj, *On generalized Caputo fractional differential equations and inclusions with non-local generalized fractional integral boundary conditions*, Malaya J. Mat., 8 (2020), pp. 1099-1109.
14. M. Rahaman, S.P. Mondal, A. El Allaoui, S. Alam, A. Ahmadian, and S. Salahshour, *Solution strategy for fuzzy fractional order linear homogeneous differential equation by Caputo-H differentiability and its application in fuzzy EOQ model*, Advances in Fuzzy Integral and Differential Equations, (2022), pp. 143-157.
15. E. Raniyah Alsulaiman, A. Mohamed Abdou, M. Eslam Youssef and Mai Taha, *Solvability of a nonlinear integro-differential equation with fractional order using the Bernoulli matrix approach[J]*, AIMS Mathematics, 8 (3) 2023, pp. 7515-7534.
16. D. Riddhi, *Beta Function and Its Applications*, The University of Tennesse: Knoxville, TN, USA, 2008.
17. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, (Yverdon: Gordon and Breach), 1993.
18. H. Sun, A. Chang, Y. Zhang and W. Chen, *A review on variable-order fractional differential equations:mathematical foundations, physical models, numerical methods and applications*, Fract. Calc. Appl. Anal., 22 (1) (2019), pp. 27-59.
19. J.A. Tenreiro Machado, *Handbook of Fractional Calculus with Applications*, (Berlin-Boston: Walter de Gruyter GmbH), 2019.
20. S. Ullah , M.A. Khan, M. Farooq, Z. Hammouch and D. Baleanu, *A fractional model for the dynamics of tuberculosis infection using Caputo-Fabrizio derivative*, Discrete Contin. Dyn. Syst., 13 (3) (2020), pp. 975-993.
21. J. Wang, Y. Zhou and W. Wei1, *Study in fractional differential equations by means of topological degree methods*, Numer. Funct. Anal. Optim., 33 (2) (2012), pp. 216-238.

MISCOM, National School of Applied Sciences, Cadi Ayyad University, Safi, Morocco.

Email address: a.elallaoui@uca.ma