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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 21
Number: 1
Pages: 237-254

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2023.1975918.1211

Sahand Communications in Mathematical Analysis (SCMA) Vol. 21 No. 1 (2024), 237-254
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2023.1975918.1211

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Abstract. This article concerns the existence of fast homoclinic solutions for the following damped vibration system

$$\frac{d}{dt}(P(t)\dot{u}(t)) + q(t)P(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0,$$

where $P, L \in C\left(\mathbb{R}, \mathbb{R}^{N^2}\right)$ are symmetric and positive definite matrices, $q \in C(\mathbb{R}, \mathbb{R})$ and $W \in C^1\left(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}\right)$. Applying the Fountain Theorem and Dual Fountain Theorem, we prove the above system possesses two different sequences of fast homoclinic solutions when L satisfies a new coercive condition and the potential $W(t,x)$ is combined nonlinearity.

1. Introduction

In this paper, we are interested in the following second-order differential system with damped term

$$\frac{d}{dt}(P(t)\dot{u}(t)) + q(t)P(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $P, L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ are symmetric and positive definite matrices for $t \in \mathbb{R}$, $q \in C(\mathbb{R}, \mathbb{R})$ and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$.

When P be the identity matrix of dimension N and $q = 0$, formally, system (1.1) reduces to the classical Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 34C37, 37J45, 58E05.

Key words and phrases. Damped vibration systems, Fast homoclinic solutions, Variational methods, Fountain Theorem, Dual Fountain Theorem.

Received: 02 December 2022, Accepted: 07 May 2023.

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As usual, we say that a solution u of (1.2) is homoclinic (to 0) if $u \neq 0$ and $u(t), \dot{u}(t) \to 0$ as $|t| \to \infty$. Over the past three decades, based on critical point theory and variational methods, for various conditions on L and the potential W, the existence and multiplicity of homoclinic solutions for system (1.2) have been investigated in the literature, see for example [2–4, 6–16, 19–30, 37, 40, 41].

When $P = I_N$ and $q \neq 0$, system (1.1) becomes damped vibration system

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0. \tag{1.3}$$

In the last ten years, the existence and multiplicity of fast homoclinic solutions (see Definition 2.6) of (1.3) have been studied by a few mathematicians via critical point theory and variational methods, see [1, 5, 17, 18, 31–35, 38, 39]. One of the difficulties in obtaining fast homoclinic solutions for (1.3) is the lack of compactness of embeddings. To solve this problem, various conditions on L were introduced in different papers and we mention some of them below (L_1) $l(t)$ is bounded from below and there exists a constant $\sigma < 0$ such that

$$\lim_{|t|\to\infty} l(t) \, |t|^{\sigma-1} = +\infty,$$

where $l(t) = \inf_{|\xi|=1} L(t)\xi \cdot \xi$,
(L_2) $l(t)$ is bounded from below and there exists a constant $r_0 > 0$ such that

$$\lim_{|s|\to\infty} meas_Q \left(\{t \in]s - r_0, s + r_0[/ L(t) < bI_N\}\right) = 0, \quad \forall b > 0,$$

where $meas_Q$ is the Lebesgue's measure with density $e^{Q(t)}$ where $Q(t) = \int_0^t q(s)ds$;
(L_3) $l(t)$ is bounded from below and there exists a constant $\sigma > 1$ such that

$$meas_Q \left(\left\{t \in \mathbb{R} / |t|^{-\sigma} L(t) < bI_N\right\}\right) < \infty, \quad \forall b > 0.$$

When $P \neq I_N$ and $q \neq 0$, there is no research about the existence of fast homoclinic solutions of (1.1). In this paper, we investigate the existence of two different sequences of fast homoclinic solutions of (1.1) via Fountain Theorem and Dual Fountain Theorem when the potential is of the form $W(t, x) = W_1(t, x) + W_2(t, x)$, where $W_1(t, x)$ is superquadratic as $|x| \to \infty$ and $W_2(t, x)$ is of subquadratic growth at infinity. To regain the compactness of embeddings, we consider a new coercive condition on L weaker than the above known conditions.

Now, we present the basic hypothesis on P, q, L and W in order to announce our main result.
(P) There exists a constant $p_0 > 0$ such that $P(t)\xi \cdot \xi \geq p_0 \, |\xi|^2$ for all

$\xi \in \mathbb{R}^N$;
(Q_σ) There exists a constant $\sigma > 1$ such that

$$Q(t) = \int_0^t q(s)ds \to +\infty \quad as \quad |t| \to \infty, \quad \int_{|t|\geq 1} \frac{e^{Q(t)}}{|t|\, ln^\sigma(|t|)} dt < \infty;$$

(L_σ) There exists a constant $l_0 > 0$ such that

$$L(t)\xi \cdot \xi \geq l_0|\xi|^2, \quad \forall (t,\xi) \in \mathbb{R} \times \mathbb{R}^N$$

and

$$meas_Q \left(\left\{ t \in \mathbb{R} / \frac{L(t)}{|t|\, ln^\sigma(|t|)} < bI_N \right\} \right) < +\infty, \quad \forall b > 0;$$

(W_1) There is a constant $\mu > 2$ such that

$$0 < \mu W_1(t,x) \leq \nabla W_1(t,x) \cdot x, \quad \forall (t,x) \in \mathbb{R} \times \left(\mathbb{R}^N \setminus \{0\}\right)$$

and $c_0 = \inf_{t\in\mathbb{R}, |x|=1} W_1(t,x) > 0$;
(W_2) There exists a constant $c > 0$ such that

$$|\nabla W_1(t,x)| \leq c \left(|x| + |x|^{\mu-1} \right), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_3) There exists a constant $1 < \nu < 2$ and a positive continuous function $a : \mathbb{R} \to \mathbb{R}$ such that

$$W_2(t,x) \geq a(t)\, |x|^\nu, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_4) $W_2(t,0) = 0$ and there exists a bounded positive continuous function $b : \mathbb{R} \to \mathbb{R}$ such that

$$|\nabla W_2(t,x)| \leq b(t)\, |x|^{\nu-1}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_5) There exists a bounded continuous function $d : \mathbb{R} \to \mathbb{R}$ such that

$$\mu W_2(t,x) - \nabla W_2(t,x) \cdot x \leq d(t)\, |x|^\nu, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W_6) $W_1(t,x)$ and $W_2(t,x)$ are even in x.

Our main result reads as follows

Theorem 1.1. *Assume that* $(P), (Q_\sigma), (L_\sigma)$ *and* $(W_1) - (W_6)$ *are satisfied. Then the damped vibration system* (1.1) *possesses two different sequences of fast homoclinic solutions* $(u_k)_{k\in\mathbb{N}}$ *and* $(v_k)_{k\in\mathbb{N}}$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{u}_k(t) \cdot \dot{u}_k(t) + L(t)u_k(t) \cdot u_k(t) \right] dt - \int_{\mathbb{R}} e^{Q(t)} W(t, u_k(t)) dt \to \infty, \quad as\ k \to \infty \tag{1.4}$$

and

$$\frac{1}{2}\int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{v}_k(t) \cdot \dot{v}_k(t) + L(t)v_k(t) \cdot v_k(t) \right] dt \tag{1.5}$$

$$-\int_{\mathbb{R}} e^{Q(t)} W(t, v_k(t))dt \to 0, \quad as\ k \to \infty.$$

2. Preliminaries

We will use $L^2_Q(\mathbb{R})$ to denote the Hilbert space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the inner product

$$\langle u, v\rangle_{L^2_Q} = \int_{\mathbb{R}} e^{Q(t)} u(t) \cdot v(t)dt,$$

and the induced norm

$$\|u\|_{L^2_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Similarly, $L^s_Q(\mathbb{R})$ $(1 \leq s < \infty)$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^s_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s dt\right)^{\frac{1}{s}},$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = ess\sup\left\{e^{\frac{Q(t)}{2}} |u(t)| / t \in \mathbb{R}\right\}.$$

Let E be the Banach space defined by

$$E = \left\{u \in L^2_Q(\mathbb{R}) / \int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{u}(t) \cdot \dot{u}(t) + L(t)u(t) \cdot u(t)\right] dt < \infty\right\},$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{u}(t) \cdot \dot{v}(t) + L(t)u(t) \cdot v(t)\right] dt,$$

and the associated norm

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

Lemma 2.1. *Suppose that (P), (Q_σ) and (L_σ) are satisfied. Then E is compactly embedded in $L^s_Q(\mathbb{R})$ for any $s \in [1, \infty[$. Moreover, for all $s \in [1, \infty]$, there exists a constant $\eta_s > 0$ such that*

$$\|u\|_{L^s_Q} \leq \eta_s \|u\|, \quad \forall u \in E. \tag{2.1}$$

Proof. For any $\epsilon > 0$, by conditions (P), (Q_σ), (L_σ) we can choose $r_\epsilon \geq e$ such that $meas_Q(B_\epsilon) \leq \epsilon$, where

$$B_\epsilon = \left\{t \in \mathbb{R} \setminus [-r_\epsilon, r_\epsilon] / \frac{L(t)}{|t|\, ln^\sigma |t|} < \frac{1}{\epsilon} I_N\right\},$$

and

$$\int_{|t|\geq r_\epsilon} \frac{e^{Q(t)}}{|t|\, ln^\sigma(|t|)} dt < \epsilon,$$

Let

$$D_\epsilon = \mathbb{R} \setminus (B_\epsilon \cup] - r_\epsilon, r_\epsilon[),$$

and

$$l_\epsilon = \inf_{t\in D_\epsilon, |x|=1} \frac{L(t)x \cdot x}{|t|\, ln^\sigma(|t|)}.$$

Then $\frac{1}{l_\epsilon} \leq \epsilon$. Let (u_k) be a sequence such that $u_k \rightharpoonup u$ weakly in E. The Banach-Steinhauss Theorem implies that

$$M = \sup_{k\in\mathbb{N}} \|u_k - u\| < \infty. \tag{2.2}$$

From (P) and (L_σ), we have $E \subset H^1_Q(\mathbb{R}) \subset L^s_Q(\mathbb{R})$ for all $s \in [2, \infty]$ with continuous embedding, there exists a constant $M_s > 0$ such that

$$\|u_k - u\|_{L^s_Q} \leq M_s, \ \forall k \in \mathbb{N}. \tag{2.3}$$

Since $P(t) \geq p_0 I_N$ and $L(t) \geq l_0 I_N$ on $I_\epsilon =] - r_\epsilon, r_\epsilon[$, the operator $E \to H^1_Q(I_\epsilon)$, $u \longmapsto u_{|I_\epsilon}$ is a continuous linear operator, where $H^1_Q(I_\epsilon)$ denotes the weighted Sobolev space

$$H^1_Q(I_\epsilon) = \left\{ u : I_\epsilon \to \mathbb{R}^N / \int_{I_\epsilon} e^{Q(t)} [|\dot{u}(t)|^2 + |u(t)|^2] dt < +\infty \right\}.$$

Sobolev's embedding Theorem implies that

$$u_k \to u \text{ uniformly in } \overline{I}_\epsilon. \tag{2.4}$$

Step 1. E is compactly embedded in $L^2_Q(\mathbb{R})$. In fact, we have

(2.5)

$$\begin{aligned}
&\int_{|t|\geq r_\epsilon} e^{Q(t)} |u_k(t) - u(t)|^2 \, dt \\
&\quad = \int_{B_\epsilon} e^{Q(t)} |u_k(t) - u(t)|^2 \, dt + \int_{D_\epsilon} e^{Q(t)} |u_k(t) - u(t)|^2 \, dt \\
&\quad \leq meas_Q(B_\epsilon) \|u_k - u\|^2_\infty + \int_{D_\epsilon} e^{Q(t)} |t|\, ln^\sigma(|t|) \, |u_k(t) - u(t)|^2 \, dt \\
&\quad \leq meas_Q(B_\epsilon) \frac{M^2_\infty}{m_0} + \frac{1}{l_\epsilon} \int_{D_\epsilon} e^{Q(t)} L(t)(u_k(t) - u(t)) \cdot (u_k(t) - u(t)) dt \\
&\quad \leq \frac{M^2_\infty}{m_0} \epsilon + \epsilon \|u_k - u\|^2 \\
&\quad \leq \left(M^2 + \frac{M^2_\infty}{m_0} \right) \epsilon,
\end{aligned}$$

where $m_0 = \min_{t\in\mathbb{R}} e^{Q(t)}$. Combining (2.4) and (2.5) yields $\|u_k - u\|_{L^2_Q} \to 0$ as $k \to \infty$.

Step 2. $s \in]2,\infty[$. We claim that E is compactly embedded in $L^s_Q(\mathbb{R})$. We have

$$\begin{aligned}\|u_k - u\|^s_{L^s_Q} &= \int_{\mathbb{R}} e^{Q(t)} |u_k - u|^s \, dt \\ &\le m_0^{-\frac{s-2}{2}} \|u_k - u\|^{s-2}_{L^\infty_Q} \|u_k - u\|^2_{L^2_Q} .\end{aligned}$$

From step 1, we deduce that $u_k \to u$ in $L^s_Q(\mathbb{R})$.

Step 3. For $s \in [1,2[$, we claim that $u_k \to u$ in $L^s_Q(\mathbb{R})$. Let $\tau = \frac{\sigma}{2-s}$. Then $s > \frac{2}{1+\sigma}$ and $\tau s > 1$. For $v \in L^s_Q(\mathbb{R})$, Hölder's inequality implies

$$\begin{aligned}&\int_{|t|\ge r_\epsilon} e^{Q(t)} |v(t)|^s \, dt \\ &= \int_{B_\epsilon} e^{Q(t)} |v(t)|^s \, dt + \int_{D_\epsilon} e^{Q(t)} |v(t)|^s \, dt \\ &\le \left(\int_{B_\epsilon} e^{Q(t)} dt\right)^{\frac{1}{2}} \left(\int_{B_\epsilon} e^{Q(t)} |v(t)|^{2s} \, dt\right)^{\frac{1}{2}} \\ &\quad + \int_{\left\{t\in D_\epsilon / |t|^{\frac{1}{s}} ln^\tau(|t|)|v(t)|\le 1\right\}} \frac{e^{Q(t)}}{|t|\, ln^{\tau s}|t|} \left(|t|^{\frac{1}{s}} ln^\tau(|t|)\, |v(t)|\right)^s dt \\ &\quad + \int_{\left\{t\in D_\epsilon / |t|^{\frac{1}{s}} ln^\tau(|t|)|v(t)|\ge 1\right\}} \frac{e^{Q(t)}}{|t|\, ln^{\tau s}|t|} \left(|t|^{\frac{1}{s}} ln^\tau(|t|)\, |v(t)|\right)^s dt \\ &\le (meas_Q(B_\epsilon))^{\frac{1}{2}} \|v\|^s_{L^{2s}_Q} + \int_{|t|\ge r_\epsilon} \frac{e^{Q(t)}}{|t|\, ln^{\tau s}(|t|)} dt \\ &\quad + \int_{\left\{t\in D_\epsilon / |t|^{\frac{1}{s}} ln^\tau(|t|)|v(t)|\ge 1\right\}} \frac{e^{Q(t)}}{|t|\, ln^{\tau s}|t|} \left(|t|^{\frac{1}{s}} ln^\tau(|t|)\, |v(t)|\right)^2 dt \\ &\le (meas_Q(B_\epsilon))^{\frac{1}{2}} \|v\|^s_{L^{2s}_Q} + \int_{|t|\ge r_\epsilon} \frac{e^{Q(t)}}{|t|\, ln^{\sigma}(|t|)} dt \\ &\quad + \int_{|t|\ge r_\epsilon} e^{Q(t)} |t|^{\frac{2}{s}-1} ln^{(2-s)\tau}(|t|)\, |v(t)|^2 \, dt \\ &\le \epsilon^{\frac{1}{2}} \|v\|^s_{L^{2s}_Q} + \epsilon + \int_{|t|\ge r_\epsilon} e^{Q(t)} |t|\, ln^\sigma(|t|)\, |v(t)|^2 \, dt \\ &\le \epsilon^{\frac{1}{2}} \|v\|^s_{L^{2s}_Q} + \epsilon + \frac{1}{l_\epsilon} \int_{|t|\ge r_\epsilon} e^{Q(t)} L(t) v(t) \cdot v(t) dt \\ &\le \epsilon^{\frac{1}{2}} \|v\|^s_{L^{2s}_Q} + \epsilon + \frac{1}{l_\epsilon} \|v\|^2 .\end{aligned}$$

Hence, we have

$$\int_{|t|\geq r_\epsilon} e^{Q(t)}\left|u_k(t)-u(t)\right|^s dt \leq \epsilon^{\frac{1}{2}}\left\|u_k-u\right\|^p_{L^{2s}_Q}+\epsilon+\epsilon^{\frac{1}{2}}\left\|u_k-u\right\|^2.$$

Since $2s\geq 2$, we deduce that

$$\int_{|t|\geq r_\epsilon} e^{Q(t)}\left|u_k(t)-u(t)\right|^s dt \leq \epsilon^{\frac{1}{2}}\left(M_{2s}^{2s}+1+M^2\right).$$

As above $\int_{I_\epsilon} e^{Q(t)}\left|u_k(t)-u(t)\right|^s dt \to 0$ as $k\to\infty$. Hence $u_k\to u$ in $L^s_Q(\mathbb{R})$. □

In order to obtain two different sequences of fast homoclinic solutions of (1.1), we will apply the following two critical point theorems [42]. Let E be a reflexive and separable Banach space with the norm $\|.\|$, and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of E. For any $k\in\mathbb{N}$, denote by

$$X_k=span\left\{e_k\right\},\quad Y_k=\oplus_{j=0}^{k}X_j,\quad Z_k=\overline{\oplus_{j=k}^{\infty}X_j}.$$

Definition 2.2. Let $f\in C^1(E,\mathbb{R})$ and $c\in\mathbb{R}$, then

1) f satisfies the $(PS)_c-$condition if every sequence $(u_n)_{n\in\mathbb{N}}\subset E$ such that $f(u_n)\to c$ and $f'(u_n)\to 0$ possesses a convergent subsequence.
2) f satisfies the $(PS)^*_c-$condition with respect to $(Y_n)_{n\in\mathbb{N}}$ if any sequence $(u_{n_k})_{k\subset\mathbb{N}}\in E$ such that

$$n_k\to\infty,\quad u_{n_k}\in Y_{n_k},\quad f(u_{n_k})\to c,\quad f'_{|Y_{n_k}}(u_{n_k})\to 0,$$

has a subsequence converging to a critical point of f.

Lemma 2.3 (Fountain Theorem)**.** *Let E be a Banach space and $f\in C^1(E,\mathbb{R})$ be an even functional and suppose that for any $k\in\mathbb{N}$, there exist two constants $\rho_k>r_k>0$ such that*

$$a_k=\inf_{u\in Z_k,\|u\|=r_k} f(u)\to+\infty,\quad as\ k\to\infty \tag{2.6}$$

$$b_k=\max_{u\in Y_k,\|u\|=\rho_k} f(u)\leq 0, \tag{2.7}$$

and f satisfies the $(PS)_c-$condition for every $c>0$. Then f has an unbounded sequence of critical values.

Lemma 2.4 (Dual Fountain Theorem)**.** *Let $f\in C^1(\mathbb{R},\mathbb{R})$ be an even functional. If there exists a constant $k_0\in\mathbb{N}$ such that for every $k\geq k_0$, there exists $\rho_k>r_k$ such that*

$$a_k=\inf_{u\in Z_k,\|u\|=\rho_k} f(u)\geq 0, \tag{2.8}$$

$$b_k=\max_{u\in Y_k,\|u\|=r_k} f(u)<0, \tag{2.9}$$

$$d_k = \inf_{u\in Z_k, \|u\|\le \rho_k} f(u) \to 0, \quad as \quad k\to\infty \tag{2.10}$$

and f satisfies the $(PS)_c^*-$condition for every $c\in[d_{k_0},0]$. Then f has a sequence of negative critical values converging to zero.

Remark 2.5. Since the $(PS)_c^*-$condition implies the $(PS)_c-$condition, then Lemma 2.3 also holds if in (A_3) we replace the $(PS)_c-$condition by the $(PS)_c^*-$condition.

Definition 2.6. A solution u of (1.1) is called a fast homoclinic solution if u is in E.

3. Proof of Theorem 1.1

Let us consider the variational functional $f : E\to\mathbb{R}$ associated to the system (1.1)

$$f(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\left[P(t)\dot{u}(t)\cdot\dot{u}(t) + L(t)u(t)\cdot u(t)\right]dt - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt.$$

Lemma 3.1. *Assume that $(P),(Q_\sigma),(L_\sigma),(W_2)$ and (W_4) are satisfied. If $u_n \rightharpoonup u$, then $\nabla W(t,u_n)\to\nabla W(t,u)$ in $L_Q^{\frac{\nu}{\nu-1}}(\mathbb{R})$ as $n\to\infty$.*

Proof. Let $u_n \rightharpoonup u$. Arguing indirectly, by Lemma 2.1 we may assume that there exists a subsequence (u_{n_k}) such that as $k\to\infty$

$$\begin{aligned} &u_{n_k}\to u \quad \text{both in } L_Q^{\nu}(\mathbb{R}) \text{ and in } L_Q^{\frac{\nu(\mu-1)}{\nu-1}}(\mathbb{R}) \\ &\text{and } u_{n_k}\to u \quad \text{a.e. in } \mathbb{R}\end{aligned} \tag{3.1}$$

and

$$\int_{\mathbb{R}} e^{Q(t)}\left|\nabla W(t,u_{n_k}(t)) - \nabla W(t,u(t))\right|^{\frac{\nu}{\nu-1}} dt \ge \epsilon_0, \quad \forall k\in\mathbb{N}, \tag{3.2}$$

for a positive constant ϵ_0. Using (3.1) and up to a subsequence if necessary, we may assume that

$$\sum_{k=1}^{\infty}\|u_{n_k}-u\|_{L_Q^{\frac{\nu}{\nu-1}}} < \infty, \qquad \sum_{k=1}^{\infty}\|u_{n_k}-u\|_{L_Q^{\frac{\nu(\mu-1)}{\nu-1}}} < \infty.$$

Let $w(t) = \sum_{k=1}^{\infty}|u_{n_k}(t)-u(t)|$ for $t\in\mathbb{R}$. Then w belongs to $L_Q^{\frac{\nu}{\nu-1}}(\mathbb{R})$ and $L_Q^{\frac{\nu(\mu-1)}{\nu-1}}(\mathbb{R})$. Combining (W_2) and (W_4), there exists a positive constant $c_1>0$ such that

$$|\nabla W(t,x)| \le c_1\left(|x|^{\nu-1} + |x|^{\mu-1}\right), \quad \forall (t,x)\in\mathbb{R}\times\mathbb{R}^{\mathbb{N}}. \tag{3.3}$$

Hence, for any $k\in\mathbb{N}$ and $t\in\mathbb{R}$, for a positive constant $c_2>0$, we have

$$\left|\nabla W(t,u_{n_k}(t)) - \nabla W(t,u(t))\right|^{\frac{\nu}{\nu-1}}$$

$$
\begin{aligned}
&\le \left[|\nabla W(t,u_{n_k}(t))| + |\nabla W(t,u(t))|\right]^{\frac{\nu}{\nu-1}} \\
&\le 2^{\frac{1}{\nu-1}}\left[|\nabla W(t,u_{n_k}(t))|^{\frac{\nu}{\nu-1}} + |\nabla W(t,u(t))|^{\frac{\nu}{\nu-1}}\right] \\
&\le 2^{\frac{1}{\nu-1}} c_1^{\frac{\nu}{\nu-1}}\left[\left(|u_{n_k}|^{\nu-1} + |u_{n_k}|^{\mu-1}\right)^{\frac{\nu}{\nu-1}} + \left(|u|^{\nu-1} + |u|^{\mu-1}\right)^{\frac{\nu}{\nu-1}}\right] \\
&\le 2^{\frac{2}{\nu-1}} c_1^{\frac{\nu}{\nu-1}}\left[|u_{n_k}|^{\nu} + |u_{n_k}|^{\frac{\nu(\mu-1)}{\nu-1}} + |u|^{\nu} + |u|^{\frac{\nu(\mu-1)}{\nu-1}}\right] \\
&\le 2^{\frac{2}{\nu-1}} c_1^{\frac{\nu}{\nu-1}}\left[(|u_{n_k}-u| + |u|)^{\nu} + (|u_{n_k}-u| + |u|)^{\frac{\nu(\mu-1)}{\nu-1}}\right] \\
&\quad + 2^{\frac{2}{\nu-1}} c_1^{\frac{\nu}{\nu-1}}\left[|u|^{\nu} + |u|^{\frac{\nu(\mu-1)}{\nu-1}}\right] \\
&\le 2^{\frac{2}{\nu-1}} c_1^{\frac{\nu}{\nu-1}}\left[(|w| + |u|)^{\nu} + (|w| + |u|)^{\frac{\nu(\mu-1)}{\nu-1}} + |u|^{\nu} + |u|^{\frac{\nu(\mu-1)}{\nu-1}}\right] \\
&\le c_2\left[w^{\nu} + |u|^{\nu} + w^{\frac{\nu(\mu-1)}{\nu-1}} + |u|^{\frac{\nu(\mu-1)}{\nu-1}}\right],
\end{aligned}
$$

which with the Dominated Convergence Theorem implies

$$\lim_{k\to\infty}\int_{\mathbb{R}} e^{Q(t)}\,|\nabla W(t,u_{n_k}(t)) - \nabla W(t,u(t))|^{\frac{\nu}{\nu-1}}\,dt = 0,$$

and contradicts (3.2). Hence the above claim holds. □

Lemma 3.2. *Assume that* $(P), (Q_\sigma), (L_\sigma)$ *and* (W_4) *are satisfied. Then* $f \in C^1(E,\mathbb{R})$ *and for all* $u, v \in E$

$$
\begin{aligned}
f'(u)v = &\int_{\mathbb{R}} e^{Q(t)}\left[P(t)\dot u(t)\cdot \dot v(t) + L(t)u(t)\cdot v(t)\right]dt \\
&- \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt.
\end{aligned}
$$

Moreover, any critical point u *of* f *on* E *is a fast homoclinic solution of* (1.1).

Proof. Let $g : E \to \mathbb{R}$ defined by

$$g(u) = \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt, \quad u \in E.$$

By the Mean Value Theorem, Hölder's inequality and Lemma 2.1, we have for all $u, v \in E$

$$
\begin{aligned}
&\left|g(u+v) - g(u) - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt\right| \\
&\quad = \left|\int_{\mathbb{R}} e^{Q(t)}\left[[W(t,u(t)+v(t)) - W(t,u(t))\right]dt - \int_{\mathbb{R}} e^{Q(t)}\nabla W(t,u(t))\cdot v(t)dt\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}} e^{Q(t)} \int_0^1 \left[\nabla W(t,u(t)+sv(t)) - \nabla W(t,u(t)) \right] \cdot v(t) ds dt \right| \\
&\le \int_0^1 \int_{\mathbb{R}} e^{Q(t)} \left| \nabla W(t,u(t)+sv(t)) - \nabla W(t,u(t)) \right| |v(t)| \, dt ds \\
&\le \int_0^1 \left(\int_{\mathbb{R}} e^{Q(t)} \left| \nabla W(t,u(t)+sv(t)) - \nabla W(t,u(t)) \right|^{\frac{\nu}{\nu-1}} dt \right)^{\frac{\nu-1}{\nu}} \\
&\quad \times \left(\int_{\mathbb{R}} e^{Q(t)} |v(t)|^{\nu} \, dt \right)^{\frac{1}{\nu}} ds \\
&\le \eta_\nu \int_0^1 \left(\int_{\mathbb{R}} e^{Q(t)} \left| \nabla W(t,u(t)+sv(t)) - \nabla W(t,u(t)) \right|^{\frac{\nu}{\nu-1}} dt \right)^{\frac{\nu-1}{\nu}} \|v\| \, ds.
\end{aligned}
$$

By Lemma 3.1, the map

$$
h : v \longmapsto \left(\int_{\mathbb{R}} e^{Q(t)} \left| \nabla W(t,u(t)+v(t)) - \nabla W(t,u(t)) \right|^{\frac{\nu}{\nu-1}} dt \right)^{\frac{\nu-1}{\nu}}
$$

is continuous on zero, so for all $\epsilon > 0$, there exists a constant $\alpha > 0$ such that $h(v) \le \epsilon$ for all $\|v\| \le \alpha$. Therefore, for all $v \in E$, $\|v\| \le \alpha$, we have

$$
\left| g(u+v) - g(u) - \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u(t)) \cdot v(t) dt \right| \le \epsilon \eta_\nu \|v\|,
$$

which implies g is differentiable and

$$
g'(u)v = \int_{\mathbb{R}} e^{Q(t)} \nabla W(t,u(t)) \cdot v(t) dt, \quad \forall u, v \in E.
$$

Now, let $u_n \to u$. By Hölder's inequality and Lemma 3.1, as $n \to \infty$, we have

$$
\begin{aligned}
\|g'(u_n) - g'(u)\| &= \sup_{\|v\|=1} \left| \int_{\mathbb{R}} e^{Q(t)} \left(\nabla W(t,u_n(t)) - \nabla W(t,u(t)) \right) \cdot v(t) dt \right| \\
&\le \sup_{\|v\|=1} \left(\int_{\mathbb{R}} e^{Q(t)} \left| \nabla W(t,u_n(t)) - \nabla W(t,u(t)) \right|^{\frac{\nu}{\nu-1}} dt \right)^{\frac{\nu-1}{\nu}} \|v\|_{L_Q^\nu} \\
&\le \eta_\nu \left(\int_{\mathbb{R}} e^{Q(t)} \left| \nabla W(t,u_n(t)) - \nabla W(t,u(t)) \right|^{\frac{\nu}{\nu-1}} dt \right)^{\frac{\nu-1}{\nu}} \to 0.
\end{aligned}
$$

So, $g \in C^1(E, \mathbb{R})$. It is clear that the quadratic form

$$
\varphi : u \longmapsto \int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{u}(t) \cdot \dot{u}(t) + L(t)u(t) \cdot u(t) \right] dt,
$$

is continuously differentiable on E and

$$
\varphi'(u)v = \int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{u}(t) \cdot \dot{v}(t) + L(t)u(t) \cdot v(t) \right] dt, \quad \forall u, v \in E.
$$

Consequently $f \in C^1(E, \mathbb{R})$ and

$$f'(u)v = \int_{\mathbb{R}} e^{Q(t)} \left[P(t)\dot{u}(t) \cdot \dot{v}(t) + L(t)u(t) \cdot v(t)\right] dt$$
$$- \int_{\mathbb{R}} e^{Q(t)} \nabla W(t, u(t)) \cdot v(t) dt, \quad \forall u, v \in E.$$

Let $u \in E$ be a critical point of f, we have for $v \in E$

$$\int_{\mathbb{R}} e^{Q(t)} P(t)\dot{u}(t) \cdot \dot{v}(t) = -\int_{\mathbb{R}} e^{Q(t)} \left[L(t)u(t) - \nabla W(t, u(t))\right] \cdot v(t) dt,$$

which implies that

$$\frac{d}{dt}\left(e^{Q(t)} P(t)\dot{u}(t)\right) = e^{Q(t)} \left(L(t)u(t) - \nabla W(t, u(t))\right),$$

and

$$\frac{d}{dt}\left(P(t)\dot{u}(t)\right) + q(t)P(t)\dot{u}(t) = L(t)u(t) - \nabla W(t, u(t)).$$

Hence u is a fast homoclinic solution of (1.1). □

Next, we consider an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of E and we denote

$$X_k = \mathbb{R} e_k, \qquad Y_k = \oplus_{j=0}^{k} X_j, \qquad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

Lemma 3.3. *Assume that* $(P), (Q_\sigma), (L_\sigma)$, (W_1), (W_2), (W_4) *and* (W_5) *are satisfied. Then* f *satisfies the* $(PS)_c^*-$*condition for any* $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ and $(u_{n_k}) \subset E$ be a sequence such that

$$n_k \to \infty, \quad u_{n_k} \in Y_{n_k}, \quad f(u_{n_k}) \to c \quad \text{and} \quad f'_{|Y_{n_k}}(u_{n_k}) \to 0 \tag{3.4}$$

as $k \to \infty$. Let c_3 be a positive constant such that

$$|f(u_{n_k})| \le c_3 \text{ and } \left\|f'(u_{n_k})\right\| \le c_3, \quad \forall k \in \mathbb{N}. \tag{3.5}$$

By (W_1), (W_5) and Lemma 2.1, we have

$$\begin{aligned}
c_3 + \frac{1}{\mu} c_3 \|u_{n_k}\| &\ge f(u_{n_k}) - \frac{1}{\mu} f'(u_{n_k}) u_{n_k} \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_{n_k}\|^2 \\
&\quad - \int_{\mathbb{R}} e^{Q(t)} \left[W_1(t, u_{n_k}(t)) - \frac{1}{\mu} \nabla W_1(t, u_{n_k}(t)) \cdot u_{n_k}(t)\right] dt \\
&\quad - \int_{\mathbb{R}} e^{Q(t)} \left[W_2(t, u_{n_k}(t)) - \frac{1}{\mu} \nabla W_2(t, u_{n_k}(t)) \cdot u_{n_k}(t)\right] dt \\
&\ge \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_{n_k}\|^2 - \int_{\mathbb{R}} e^{Q(t)} d(t) \left|u_{n_k}(t)\right|^{\nu} dt
\end{aligned}$$

$$\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_{n_k}\|^2 - \frac{1}{\mu} \|d\|_\infty \eta_\nu^\nu \|u_{n_k}\|^\nu ,$$

which implies that (u_{n_k}) is bounded in E since $1 < \nu < 2$. A classical computation shows that

$$\begin{aligned}
&(f'(u_{n_k}) - f'(u))(u_{n_k} - u) \\
&\quad = \|u_{n_k} - u\|^2 - \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_{n_k}(t)) - \nabla W(t, u(t))) \cdot (u_{n_k}(t) - u(t))dt.
\end{aligned}$$

On the other hand, we have $f'(u)(u_{n_k} - u) \to 0$ as $k \to \infty$. Since $E = \overline{\bigcup_{k=0}^{\infty} Y_{n_k}}$, there exists a sequence $(v_{n_k}) \subset E$ with $v_{n_k} \in Y_{n_k}$ for all $k \in \mathbb{N}$ such that $v_{n_k} \to u$ as $k \to \infty$, which implies

$$\begin{aligned}
\lim_{k\to\infty} f'(u_{n_k})(u_{n_k} - u) &= \lim_{k\to\infty} f'(u_{n_k})(u_{n_k} - v_{n_k}) + \lim_{k\to\infty} f'(u_{n_k})(v_{n_k} - u) \\
&= \lim_{k\to\infty} f'_{|Y_{n_k}}(u_{n_k})(u_{n_k} - v_{n_k}) \\
&= 0.
\end{aligned}$$

On the other hand, by Hölder's inequality, Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} e^{Q(t)} (\nabla W(t, u_{n_k}(t)) - \nabla W(t, u(t))) \cdot (u_{n_k}(t) - u(t))dt \right| \\
&\quad \leq \eta_\nu \left(\int_{\mathbb{R}} e^{Q(t)} |W(t, u_{n_k}(t)) - \nabla W(t, u(t))|^{\frac{\nu}{\nu-1}} dt \right)^{\frac{\nu-1}{\nu}} \|u_{n_k} - u\| \to 0.
\end{aligned}$$

Therefore $u_{n_k} \to u$ in E. It remains to prove that $f'(u) = 0$. Let $m \in \mathbb{N}$ and $v \in Y_m$, then for all $k \in \mathbb{N}$ with $n_k \geq m$, we have $v \in Y_{n_k}$. Hence

$$f'(u)v = \left(f'(u) - f'(u_{n_k})\right) v + f'_{|Y_{n_k}}(u_{n_k})v \to 0, \quad \text{as} \quad k \to \infty.$$

Therefore $f'(u)v = 0$ for all $m \in \mathbb{N}$ and $v \in Y_m$, so that $f'(u) = 0$. Hence for all $c \in \mathbb{R}$, the functional f satisfies the $(PS)_c^*$−condition. ☐

3.1. **Proof of Theorem 1.1** (1.4)**.**

Lemma 3.4. *Assume that* $(P), (Q_\sigma), (L_\sigma)$, (W_2) *and* (W_4) *are satisfied. Then there exists a constant* $r_k > 0$ *such that*

$$b_k = \inf_{u\in Z_k, \|u\|=r_k} f(u) \to +\infty, \quad \textit{as} \quad k \to \infty.$$

Proof. For any $k \in \mathbb{N}$, define

$$\zeta_k = \sup_{v\in Z_k, \|u\|=1} \int_{\mathbb{R}} e^{Q(t)} |v(t)|^\nu dt,$$

and

$$\xi_k = \sup_{v\in Z_k, \|u\|=1} \int_{\mathbb{R}} e^{Q(t)} |v(t)|^\mu dt.$$

Then it is well known that $\zeta_k > 0$, $\xi_k > 0$ for any $k \in \mathbb{N}$ and $\zeta_k \to 0$, $\xi_k \to 0$ as $k \to \infty$. Let $u \in Z_k \setminus \{0\}$ and define $v = \frac{u}{\|u\|}$, then $v \in Z_k$ and $\|v\| = 1$. Since $W(t,0) = 0$ for all $t \in \mathbb{R}$, then (3.3) implies

$$\begin{aligned}(3.6)\qquad \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt &\le c_1 \int_{\mathbb{R}} e^{Q(t)} \left[|u(t)|^\nu + |u(t)|^\mu\right] dt \\ &= c_1 \int_{\mathbb{R}} e^{Q(t)} \left[\|u\|^\nu |v(t)|^\nu + \|u\|^\mu |v(t)|^\mu\right] d \\ &\le c_1 \left[\zeta_k \|u\|^\nu + \xi_k \|u\|^\mu\right].\end{aligned}$$

Hence, we obtain

$$f(u) \ge \frac{1}{2}\|u\|^2 - c_1\zeta_k \|u\|^\nu - c_1\xi_k \|u\|^\mu.$$

Since $\zeta_k \to 0$ as $k \to \infty$, then for k large enough, $c_1\zeta_k < \frac{1}{4}$. So, one gets

$$f(u) \ge \frac{1}{4}\|u\|^2 - c_1\xi_k \|u\|^\mu, \quad \forall u \in Z_k, \quad \|u\| \ge 1.$$

Let $r_k = \left(\frac{1}{2\mu c_1 \xi_k}\right)^{\frac{1}{\mu-2}}$, then for any $u \in Z_k$ with $\|u\| = r_k$, we have

$$f(u) \ge \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\mu}\right) r_k^2.$$

Since $\xi_k \to 0$ as $k \to \infty$ we deduce

$$\inf_{\{u \in Z_k, \|u\| = r_k\}} f(u) \to +\infty, \quad \text{as } k \to \infty. \qquad \square$$

Lemma 3.5. *Assume that $(P), (Q_\sigma), (L_\sigma)$ and (W_1) are satisfied. Then there exists a constant $\rho_k > r_k$ such that*

$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} f(u) < 0.$$

Proof. Firstly, let us remark that according to (W_1), it is easy to verify that the function $\varphi : \mathbb{R}_+^* \to \mathbb{R}$, $\xi \longmapsto W_1(t, \xi^{-1}x)\xi^\mu$ is non-increasing. Consequently, for all $t \in \mathbb{R}$ and $|x| \ge 1$,

$$W_1(t,x) \ge W_1\left(t, \frac{x}{|x|}\right)|x|^\mu \ge c_0 |x|^\mu.$$

As a result, we deduce

$$W_1(t,x) \ge c_0\left(|x|^\mu - |x|^2\right), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

which with Lemma 2.1 implies

$$\begin{aligned}(3.7)\qquad f(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt \\ &\le \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(t)} W_1(t,u(t))dt\end{aligned}$$

$$\leq \frac{1}{2}\|u\|^2 - c_0 \int_{\mathbb{R}} e^{Q(t)} |u(t)|^\mu \, dt + c_0 \int_{\mathbb{R}} e^{Q(t)} |u(t)|^2 \, dt$$
$$\leq \left(\frac{1}{2} + c_0\eta_2^2\right) \|u\|^2 - c_0 \int_{\mathbb{R}} e^{Q(t)} |u(t)|^\mu \, dt.$$

Since $Y_k \subset E$ is a finite dimensional subspace, all the norms on Y_k are equivalent, so there exists a constant $d_k > 0$ such that

$$d_k \|u\|^\mu \leq \|u\|^\mu_{L^\mu_Q}, \quad \forall u \in Y_k. \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$f(u) \leq \left(\frac{1}{2} + c_0\eta_2^2\right) \|u\|^2 - c_0 d_k \|u\|^\mu, \quad \forall u \in Y_k.$$

Since $\mu > 2$, there exists a constant $\rho_k > r_k$ such that

$$\max_{u \in Y_k, \|u\| = \rho_k} f(u) < 0. \qquad \square$$

According to (W_6), f is even. Lemmas 3.2-3.5 imply that all Lemma 2.3 conditions are satisfied. Therefore, f has an unbounded sequence of critical values, i.e., (1.1) possesses infinitely many fast homoclinic solutions satisfying condition (1.4) of Theorem 1.1.

3.2. **Proof of Theorem 1.1** (1.5).

Lemma 3.6. *Assume that $(P), (Q_\sigma), (L_\sigma), (W_2)$ and (W_4) are satisfied. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, there exists a constant $\rho_k > 0$ such that*

$$a_k = \inf_{u \in Z_k, \|u\| = \rho_k} f(u) \geq 0.$$

Proof. For every $v \in Z_k$ with $\|v\| = 1$ and $0 < s < 1$, from (3.6) we have

$$f(sv) \geq \frac{s^2}{2} - c_1\zeta_k s^\nu - c_1\xi_k s^\mu$$

where ζ_k and ξ_k are defined in the proof of Lemma 3.4. Since $\xi_k \to 0$ as $k \to \infty$, then there exists $k_1 \in \mathbb{N}$ such that $c_1\xi_k < \frac{1}{4}$ for all $k \geq k_1$. Moreover, since $\mu > 2$ and $0 < s < 1$, one gets

$$f(sv) \geq \frac{s^2}{4} - c_1\zeta_k s^\nu. \tag{3.9}$$

Taking $\rho_k = (8\nu c_1\zeta_k)^{\frac{\nu}{2-\nu}}$, we can find $k_0 \geq k_1$ such that $0 < \rho_k < 1$. For $v \in Z_k$ with $\|v\| = 1$, we have $u = \rho_k v \in Z_k$ with $\|u\| = \rho_k$ and $f(u) \geq (2\nu - 1)\rho_k^\nu c_1\zeta_k > 0$. Therefore $a_k \geq 0$ for all $k \geq k_0$. $\square$

Lemma 3.7. *Assume that* $(P), (Q_\sigma), (L_\sigma), (W_1)$ *and* (W_3) *are satisfied. Then there exist* $0 < r_k < \rho_k$ *such that*

$$b_k = \max_{u \in Y_k, \|u\| = r_k} f(u) < 0.$$

Proof. For $k \geq k_0$, put

$$\delta_k = \inf_{v \in Y_k, \|v\|=1} \int_{\mathbb{R}} a(t) \, |v(t)|^\nu \, dt.$$

It is clear that $\delta_k > 0$. Let $0 < s < 1$ and $v \in Y_k$ with $\|v\| = 1$, then in view of (W_1) and (W_3), one has

$$f(sv) \leq \frac{s^2}{2} - \delta_k s^\nu.$$

Choose $r_k \in \left]0, \inf\left\{\rho_k, (\nu\delta_k)^{\frac{1}{2-\nu}}\right\}\right[$, we obtain

$$f(r_k v) \leq \frac{r_k^2}{2} - \delta_k r_k^\nu < 0, \quad \forall v \in Y_k \quad \text{with} \quad \|v\| = 1.$$

As a result, we deduce $b_k \leq \frac{r_k^2}{2} - \delta_k r_k^\nu < 0$. □

Lemma 3.8. *Assume that* $(P), (Q_\sigma), (L_\sigma)$ *and* $(W_1)-(W_4)$ *are satisfied. Then*

$$d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} f(u) \to 0, \quad as \quad k \to \infty.$$

Proof. Since $Y_k \bigcap Z_k \neq \phi$ and $r_k < \rho_k$, it is easy to see that

$$d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} f(u) \leq \max_{u \in Y_k, \|u\| = r_k} f(u) < 0.$$

On the other hand, for all $u \in Z_k$ with $0 < \|u\| \leq \rho_k < 1$, (3.9) implies

$$f(u) = f\left(\|u\| \frac{u}{\|u\|}\right) \geq \frac{1}{4}\|u\|^2 - c_1\zeta_k \|u\|^2 \geq -c_1\zeta_k.$$

Therefore $d_k \to 0$ as $k \to \infty$. □

Finally, f is even and Lemmas 3.2, 3.3 and Lammas 3.6-3.8 imply that the functional f satisfies all the conditions of Lemma 2.4. Consequently, f has a sequence of negative critical values converging to zero, i.e., (1.1) admits a sequence of fast homoclinic solutions satisfying condition (1.5) of Theorem 1.1.

Acknowledgment. The author would like to thank the anonymous referees and the Executive Director for their carefully reading this paper and their useful comments and suggestions.

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