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Common Solution for a Finite Family of Equilibrium Problems, Inclusion Problems and Fixed Points of a Finite Family of Nonexpansive Mappings in Hadamard Manifolds

Prashant Patel[1*] and Rahul Shukla[2]

Abstract. In this paper, we present an iterative algorithm and prove that the sequence generated by this algorithm converges strongly to a common solution of a finite family of equilibrium problems, the quasi-variational inclusion problem and the set of common fixed points of a countable family of nonexpansive mappings.

1. Introduction

In 1976, Rockafellar [19] studied the inclusion problem of finding

$$\eta^{\dagger} \in S^{-1}(0) \tag{1.1}$$

where S is a maximal monotone set-valued mapping defined on a Hilbert space M. To address the aforementioned inclusion problem (1.1), he created a method known as the proximal point method. Due to its applications in several fields of science, engineering, management and social sciences in the last many years, the inclusion problem has been broadened and generalized in numerous ways; see, for example, [6, 12, 18, 20, 22] and the references therein. In recent years, many authors extended the results obtained by the proximal point algorithm from classical spaces to Hadamard manifolds; see, for example, [1, 14] and the references therein.

In 2019, Al-Homidan et al. [1] considered the problem of finding

$$\eta^{\dagger} \in F(S) \bigcap (G+H)^{-1}(0),$$

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in Hadamard manifold, where S, H and G are nonexpansive, set-valued maximal monotone and single-valued continues and monotone mappings, respectively.

Let M be a Hilbert space, $\mathcal{E} \neq \emptyset$ be a subset of M and $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ a bifunction. Then a broad class of problems in optimization, such as variational inequality, convex minimization, fixed point and Nash equilibrium problems can be formulated as the equilibrium problem associated with the bifunction F and the set $\mathcal{E}$ [4, 17]

$$\text{find } \eta \in \mathcal{E} \text{ such that } F(\eta, \varsigma) \geq 0 \text{ for all } \varsigma \in \mathcal{E}.$$

A point $\eta \in \mathcal{E}$ solving this problem is called an equilibrium point. The set of equilibrium points is denoted by $EP(F)$. Many algorithms are available in the literature to analyze the existence and approximation of a solution to equilibrium problems in linear spaces. Recently, Colao et al. [8] and Khammahawong et al. [13] investigated equilibrium theory in Hadamard manifolds. Under suitable conditions, they proved the existence of equilibrium points for a bifunction and presented some applications to variational inequality, fixed point and Nash equilibrium problems.

Recently, Zhu et al. [23] presented an iterative algorithm for finding a common solution for a finite family of equilibrium problems, quasi-variational inclusion problems and fixed points of a nonexpansive mapping on Hadamard Manifolds and presented some strong convergence results. They considered the following problem of finding

$$\eta^{\dagger} \in \bigcap_{i=1}^{m} EP\left(F_i\right) \bigcap (G+H)^{-1}(0) \bigcap F(S), \tag{1.2}$$

in a Hadamard manifold, where G, H and S are the same as defined above.

In this paper, we consider the following common solution for a finite family of equilibrium problems, inclusion problems and fixed points of a finite family of nonexpansive mappings in Hadamard Manifolds, i.e., to obtain $\eta^{\dagger} \in \mathcal{E}$ such that

$$\eta^{\dagger} \in \bigcap_{i=1}^{m} EP(F_i) \bigcap (G+H)^{-1}(0) \bigcap_{i=1}^{m} F(S_i). \tag{1.3}$$

We present an algorithm and prove that the sequence generated by the algorithm converges strongly, which is the common solution to problem (1.3). In this way, we extend some results in the literature.

2. Preliminaries

Assume that Σ is a finite dimensional differentiable manifold, then for any $k \in \Sigma$ we denote by $T_k\Sigma$ the tangent space of Σ at k which is a vector space of the same dimension as Σ and we denote the tangent bundle of Σ by $T\Sigma = \bigcup_{k\in\Sigma} T_k\Sigma$. An inner product $\mathcal{R}_k(\cdot,\cdot)$ defined on the tangent space $T_k\Sigma$ is said to be a Riemannian metric on the tangent space $T_k\Sigma$. To become a Riemannian manifold, we assume that Σ can be endowed with a Riemannian metric $\mathcal{R}_k(\cdot,\cdot)$. We denote the corresponding norm to the inner product on $T_k\Sigma$ by $\|\cdot\|_k$. A manifold Σ is said to be a Riemannian manifold if it is differentiable endowed with a Riemannian metric $\mathcal{R}(\cdot,\cdot)$. We define the length of a piecewise smooth curve $\Lambda : [0,1] \to \Sigma$ joining k to l (i.e. $\Lambda(0) = k$ and $\Lambda(1) = l$) by $L(\Lambda) = \int_0^1 \|\Lambda'(t)\|dt$. The minimal length over the set of all such curves joining k to l, which includes the original topology on Σ is called the Riemannian distance $d(k,l)$. It is said that a Riemannian manifold Σ is complete if for all $k \in \Sigma$, all the geodesic emerging from k are defined $\forall\ t \in \mathbb{R}$. We say that a geodesics joining k to l in Σ is minimal if its length is equal to $d(k,l)$. A Riemannian manifold Σ with the Riemannian distance d is a metric space (Σ, d). According to the Hopf-Rinow Theorem [21], if the Riemannian manifold Σ be complete, then all the pairs of points in Σ can be joined by a minimal geodesic. Moreover, the metric space (Σ, d) is complete and its closed and bound subsets are compact.

Definition 2.1. Suppose Σ is a given complete Riemannian manifold. We define the exponential map $\exp_k : T_k\Sigma \to \Sigma$ at point $k \in \Sigma$ by $\exp_k v = \Lambda_v(1,k)\ \forall\ v \in T_k\Sigma$, where $\Lambda_v(\cdot,k)$ is the geodesic with the velocity v and starting from the point k i.e. $\Lambda'_v(0,k) = v$ and $\Lambda_v(0,k) = k$.

It is also known that for any $t \in \mathbb{R}$ the exponential map $\exp_k tv = \Lambda_v(t,k)$. Hence one can easily see that for any zero tangent vector 0, the exponential map $\exp_k 0 = \Lambda_v(0,k) = k$. We also note that $\exp_k$ is differentiable on the tangent space $T_k\Sigma$ for all $k \in \Sigma$. Also, $d(k,l) = \left\|\exp_k^{-1} l\right\|$ for all $k,l \in \Sigma$.

Definition 2.2. The Hadamard Manifold is a Riemannian manifold with non-positive sectional curvature if it is simply connected and complete.

Proposition 2.3 ([21])**.** *Suppose Σ be any given Hadamard manifold. Then $\exp_k : T_k\Sigma \to \Sigma$ is a diffeomorphism for any $k \in \Sigma$ and for any pair of points $k,l \in \Sigma$, $\exists$ a unique normalized geodesic $\Lambda : [0,1] \to \Sigma$ joining points $k = \Lambda(0)$ to $l = \Lambda(1)$, in fact which is a minimal geodesic* *defined as*

$$\Lambda(t) = \exp_k t \exp_k^{-1} l \quad \text{for all } 0 \le t \le 1.$$

Lemma 2.4 ([5]). *Suppose Σ be any given finite dimensional Hadamard manifold.*

(1) *Suppose $\Lambda : [0,1] \to \Sigma$ be any geodesic joining points η to ς. Then*

$$d(\Lambda(t_1), \Lambda(t_2)) = |t_1 - t_2| d(\eta, \varsigma), \quad \forall t_1, t_2 \in [0,1].$$

(2) *The following hold true for any $z, u, \eta, \varsigma, w \in \Sigma$, $0 \le t \le 1$:*

$$d\left(\exp_\eta (1-t) \exp_\eta^{-1} \varsigma, z\right) \le t d(\eta, z) + (1-t) d(\varsigma, z);$$

$$d^2\left(\exp_\eta (1-t) \exp_\eta^{-1} \varsigma, z\right) \le t d^2(\eta, z) + (1-t) d^2(\varsigma, z) - t(1-t) d^2(\eta, \varsigma);$$

$$d\left(\exp_\eta (1-t) \exp_\eta^{-1} \varsigma, \exp_u (1-t) \exp_u^{-1} \zeta\right) \le t d(\eta, u) + (1-t) d(\varsigma, \zeta).$$

A subset $\mathcal{E}$ of a Hadamard manifold Σ is geodesic convex if for all $\eta, \varsigma \in \mathcal{E}$, the geodesic joining points η to ς is also contained in $\mathcal{E}$.

Now onwards we assume that the Hadamard manifold Σ is finite dimensional and $\mathcal{E}$ is a geodesic convex, bounded, nonempty and closed subset in Σ and $F(S)$ is the fixed point set of the mapping S.

Any function $h : \mathcal{E} \to (-\infty, \infty)$ is called geodesic convex if, $\forall\, \nu \in [0,1]$ the geodesic $\Lambda(\nu)$ joining points $\eta, \varsigma \in \mathcal{E}$, the function $h \circ \Lambda$ is convex, i.e.

$$\begin{aligned} h(\Lambda(\nu)) &\le \nu h(\Lambda(0)) + (1-\nu) h(\Lambda(1)) \\ &= \nu h(\eta) + (1-\nu) h(\varsigma). \end{aligned}$$

Definition 2.5. Suppose M be any given complete metric space and $\mathcal{E} \ne \emptyset$ a subset of M. The sequence $\{\eta_n\}$ is said to be Fejer monotone with respect to the subset $\mathcal{E}$ if $\forall\, \varsigma \in \mathcal{E}$, $0 \le n$,

$$d(\eta_n, \varsigma) \ge d(\eta_{n+1}, \varsigma).$$

Lemma 2.6 ([11]). *Suppose M be any given complete metric space and $\mathcal{E} \ne \emptyset$ a subset of M. If $\{\eta_n\} \subset M$ be a Fejer monotone with respect to $\mathcal{E}$, then $\{\eta_n\}$ is bounded. Furthermore, if a cluster point η of the sequence $\{\eta_n\}$ belongs to $\mathcal{E}$, then $\{\eta_n\}$ converges to η.*

Definition 2.7. A mapping $S : \mathcal{E} \to \mathcal{E}$ is said to be

(i) nonexpansive if

$$d(S(\eta), S(\varsigma)) \le d(\eta, \varsigma), \quad \text{for all } \eta, \varsigma \in \mathcal{E},$$

(ii) firmly nonexpansive, if $\forall\, \eta, \varsigma \in \mathcal{E}$, the function $\phi : [0,1] \to [0, \infty]$ defined as

$$\phi(t) = d\left(\exp_\eta t \exp_\eta^{-1} S(\eta), \exp_\varsigma t \exp_\varsigma^{-1} S(\varsigma)\right), \quad \text{for all } 0 \le t \le 1$$

is nonincreasing [21].

Proposition 2.8 ([15])**.** *Suppose $S : \mathcal{E} \to \mathcal{E}$ be any mapping, then these following are equivalent.*

(1) *S is a firmly nonexpansive mapping;*

(2) $\forall\ \eta, \varsigma \in \mathcal{E}$, $0 \leq t \leq 1$

$$d(S(\eta), S(\varsigma)) \leq d\left(\exp_\eta t \exp_\eta^{-1} S(\eta), \exp_\varsigma t \exp_\varsigma^{-1} S(\varsigma)\right);$$

(3) $\forall\ \eta, \varsigma \in \mathcal{E}$

$$\mathcal{R}\left(\exp_{S(\eta)}^{-1} S(\varsigma), \exp_{S(\eta)}^{-1} \eta\right) + \mathcal{R}\left(\exp_{S(\varsigma)}^{-1} S(\eta), \exp_{S(\varsigma)}^{-1} \varsigma\right) \leq 0.$$

Lemma 2.9 ([7])**.** *Suppose $S : \mathcal{E} \to \mathcal{E}$ be any given firmly nonexpansive mapping with $F(S) \neq \emptyset$, then for all $\eta \in \mathcal{E}$, $k \in F(S)$ the following condition holds true,*

$$d^2(S(\eta), k) \leq d^2(\eta, k) - d^2(S(\eta), \eta).$$

In the continuation, suppose $G : \Sigma \to T\Sigma$ be a single-valued vector field in such a way that $G(\eta) \in T_\eta\Sigma$, $\forall\ \eta \in \Sigma$, we denote the set of all single-valued vector fields by $\Theta(\Sigma)$. Suppose that the domain $D(G)$ of vector field G is defined as

$$D(G) = \{\eta \in \Sigma : G(\eta) \in T_\eta\Sigma\}.$$

Definition 2.10 ([16])**.** A single-valued vector field $G : \Sigma \to T\Sigma$ is monotone if

$$\left\langle G(\varsigma), -\exp_\varsigma^{-1} \eta\right\rangle \geq \left\langle G(\eta), \exp_\eta^{-1} \varsigma\right\rangle \text{ for all } \eta, \varsigma \in \Sigma.$$

Suppose $H : \Sigma \to 2^{T\Sigma}$ is a set-valued vector field in such a way that $H(\eta) \subset T_\eta\Sigma$, $\forall\ \eta \in \Sigma$ and we denote the set of all set-valued vector fields by $\mathcal{X}(\Sigma)$. Suppose that domain $D(H)$ of set-valued vector field H is defined as

$$D(H) = \{\eta \in \Sigma : \emptyset \neq B(\eta)\}.$$

Definition 2.11 ([10])**.** A set-valued vector field $H : \Sigma \to 2^{T\Sigma}$ is called

(i) monotone if $\forall\ \eta, \varsigma \in D(H)$

$$\mathcal{R}\left(v, -\exp_\varsigma^{-1} \eta\right) \geq \mathcal{R}\left(u, \exp_\eta^{-1} \varsigma\right), \quad \text{for all } v \in H(\varsigma), u \in H(\eta);$$

(ii) maximal monotone if the mapping is monotone and $\forall\ \eta \in D(H)$, $u \in T_\eta\Sigma$, following assumption

$$\mathcal{R}\left(v, -\exp_\varsigma^{-1} \eta\right) \geq \mathcal{R}\left(u, \exp_\eta^{-1} \varsigma\right), \quad \text{for all } v \in H(\varsigma), \varsigma \in D(H),$$

implies that $u \in H(\eta)$.

(iii) For any given positive ν, the resolvent of set-valued vector field H of the order ν is also a set-valued mapping $J_\nu^H : \Sigma \to 2^{T\Sigma}$ defined as

$$J_\nu^H(\eta) = \{z \in \Sigma : \eta \in \exp_z \nu H(z)\}, \quad \text{for all } \eta \in \Sigma.$$

Theorem 2.12 ([15])**.** *Suppose $H \in \mathcal{X}(\Sigma)$. The following statements hold for a given positive ν*

1. *the given set-valued vector field H is monotone iff J_ν^H is a single-valued and firmly nonexpansive mapping;*
2. *if $D(H) = \Sigma$, the set-valued vector field H is maximal monotone iff J_ν^H is a single-valued firmly nonexpansive mapping and domain $D(J_\nu^H) = \Sigma$.*

Proposition 2.13 ([15])**.** *Suppose $\mathcal{E} \neq \emptyset$ be a subset of Σ, $S : \mathcal{E} \to \Sigma$ a firmly nonexpansive mapping. Then*

$$\mathcal{R}\left(\exp_{S(\varsigma)}^{-1} \eta, \exp_{S(\varsigma)}^{-1} \varsigma\right) \leq 0,$$

holds $\forall\ \eta \in F(S)$, $\varsigma \in \mathcal{E}$.

Lemma 2.14 ([1])**.** *Suppose $\mathcal{E} \neq \emptyset$ be a closed subset of Σ, $H : \Sigma \to 2^{T\Sigma}$ a maximal monotone set-valued vector field. Suppose $\{\nu_n\}$ be a sequence of positive real numbers along with $\lim_{n\to\infty} \nu_n = \nu > 0$, a sequence $\{\eta_n\} \subset \mathcal{E}$ along with $\lim_{n\to\infty} \eta_n = \eta \in \mathcal{E}$ in such a way that $\lim_{n\to\infty} J_{\nu_n}^H(\eta_n) = \varsigma$. Then, we get $\varsigma = J_\nu^H(\eta)$.*

Proposition 2.15 ([2])**.** *Suppose $G : \Sigma \to T\Sigma$ be a given single-valued monotone, $H : \Sigma \to 2^{T\Sigma}$ a given set-valued maximal monotone vector field. Then $\forall\ \eta \in \mathcal{E}$, following conditions are equivalent*

1. *$\eta \in (G + H)^{-1}(0)$;*
2. *$\eta = J_\nu^H(\exp_\eta(-\nu G(\eta)))$ for all $\nu > 0$.*

Suppose $\mathcal{E} \neq \emptyset$ be a geodesic convex and closed set in Σ, $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ a bifunction satisfying given following suppositions:

- (A_1) $\forall\ \eta \in \mathcal{E}$, $0 \leq F(\eta, \eta)$;
- (A_2) F is monotone, i.e. $\forall\ \eta, \varsigma \in \mathcal{E}$, $F(\varsigma, \eta) + F(\eta, \varsigma) \leq 0$;
- (A_3) $\forall\ \varsigma \in \mathcal{E}$, $\eta \mapsto F(\eta, \varsigma)$ is upper semicontinuous;
- (A_4) $\forall\ \eta \in \mathcal{E}$, $\varsigma \mapsto F(\eta, \varsigma)$ is lower semicontinuous and geodesic convex;
- (A_5) $\eta \mapsto F(\eta, \eta)$ is lower semicontinuous;
- (A_6) $\exists$ a compact set $L \subseteq \Sigma$ in such a way that $\eta \in \mathcal{E}/L \quad \Rightarrow \quad \exists\, \varsigma \in \mathcal{E} \cap L$ in such a way that $F(\eta, \varsigma) < 0$.

Definition 2.16 ([9])**.** Suppose $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be a given bifunction. Then the resolvent of bifunction F is a multivalued mapping $T_r^F : \Sigma \to 2^{\mathcal{E}}$ defined as

$$T_r^F(\eta) = \left\{ z \in \mathcal{E} : 0 \leq F(z, \varsigma) - \frac{1}{r} \left\langle \exp_z^{-1} \eta, \exp_z^{-1} \varsigma \right\rangle \text{ for all } \varsigma \in \mathcal{E} \right\}.$$

Theorem 2.17 ([9]). *Suppose $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be a given bifunction satisfying following assertions:*

(1) *the given bifunction F is monotone;*
(2) *$\forall\ r > 0$, T_r^F is properly defined, i.e. the domain $D\left(T_r^F\right) \neq \emptyset$. Then $\forall\ r > 0$,*
  (a) *T_r^F is a single-valued mapping;*
  (b) *T_r^F is a firmly nonexpansive mapping;*
  (c) *the set of fixed points of mapping T_r^F is the set of equilibrium points of bifunction F i.e.,*

$$EP(F) = F\left(T_r^F\right).$$

Theorem 2.18 ([15]). *Suppose $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be a given bifunction satisfying the above suppositions (A_1)-(A_3). Then $D\left(T_r^F\right) = \Sigma$.*

Theorem 2.19 ([15]). *Suppose $F : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ be a given bifunction satisfying the above suppositions (A_1),(A_3),(A_4),(A_5) and (A_6). Then $\exists\ z \in \mathcal{E}$ in such a way that*

$$0 \le F(z,\varsigma) - \frac{1}{r}\left\langle \exp_z^{-1}\eta, \exp_z^{-1}\varsigma\right\rangle, \quad \text{for all } \varsigma \in \mathcal{E}$$

$\forall\ r > 0$, $\eta \in \Sigma$.

3. Main Results

In the sequel, we always assume that

(i) $\mathcal{E} \neq \emptyset$ is a closed bounded geodesic convex subset of a Hadamard manifold Σ;
(ii) $H : \mathcal{E} \to 2^{T\Sigma}$ is maximal monotone setvalued vector field;
(iii) $G : \mathcal{E} \to T\Sigma$ is monotone and continuous single-valued vector field satisfying following condition $\forall \eta, \varsigma \in \mathcal{E}, 0 < \nu, 0 \le \rho \le 1$

$$d\left(\exp_\eta(-\nu G(\eta)), \exp_\varsigma(-\nu G(\varsigma))\right) \le (1-\rho) d(\eta,\varsigma). \tag{3.1}$$

(iv) $S_i : \mathcal{E} \to \mathcal{E},\ i = 1, 2, \dots, m$ is a finite family of nonexpansive mappings;
(v) $F_i : \mathcal{E} \times \mathcal{E} \to \mathbb{R},\ i = 1, 2, \dots, m$ is a finite family of bifunctions satisfying the above suppositions (A_1) - (A_6), for any given $0 < r$, the resolvent of family of bifunctions F_i is multivalued mapping $T_r^{F_i} : \Sigma \to 2^{\mathcal{E}}$ in such a way that $\forall\ \eta \in \Sigma$, $i = 1, 2, \dots, m$,

$$T_r^{F_i}(\eta) = \left\{ z \in \mathcal{E} : 0 \le F_i(z,\varsigma) - \frac{1}{r}\left\langle \exp_z^{-1}\eta, \exp_z^{-1}\varsigma\right\rangle, \forall\ \varsigma \in \mathcal{E} \right\}.$$

(vi) Denote by

$$S_r^j = T_r^{F_j} \circ T_r^{F_{j-1}} \circ \cdots \circ T_r^{F_2} \circ T_r^{F_1}, \quad j = 1, 2, \dots, m.$$

Theorem 3.1. *Suppose $\mathcal{E}, \Sigma, G, H, \{F_i\}_{i=1}^m, \{S_r^j\}_{r=1}^m$ and S_i be the same as defined above. Suppose $\{\eta_n\}, \{u_n\}, \{\varsigma_n\}$ and $\{z_n\}$ are the sequences generated by $\eta_0 \in \mathcal{E}$*

$$\begin{cases} u_n = J_{\nu_n}^H \left(\exp_{\eta_n}(-\nu_n G(\eta_n))\right), \\ \varsigma_n^i = \exp_{\eta_n} \vartheta_n \exp_{\eta_n}^{-1} S_i(u_n), \\ w_n \in \{\varsigma_n^i, \ i = 1, 2, \dots m\}, \\ \textit{such that } d(w_n, \eta_n) = \max\limits_{1 \le i \le m} d(\varsigma_n^i, \eta_n), \\ z_n = S_r^m(w_n), \\ \eta_{n+1} = \exp_{\eta_n} \Upsilon_n \exp_{\eta_n}^{-1}(z_n), \ \forall n \ge 0, \end{cases} \tag{3.2}$$

where $\forall\ n \in \mathbb{N}$, $\{\vartheta_n\}$, $\{\Upsilon_n\}$, $\{\nu_n\}$ are the given sequences satisfying these following conditions:

(a) $0 < a \le \vartheta_n, \Upsilon_n \le b < 1$,
(b) $0 < \hat{\nu} \le \nu_n \le \tilde{\nu} < \infty$,
(c) $\sum\limits_{n=1}^{\infty} \vartheta_n \Upsilon_n = \infty$.

If $\Theta = \bigcap\limits_{i=1}^m EP(F_i) \bigcap (G+H)^{-1}(0) \bigcap\limits_{i=1}^m F(S_i) \neq \emptyset$, then the sequence $\{\eta_n\}$ converges strongly to a solution of problem (1.3).

Proof. Suppose $\Lambda_n : [0,1] \to \Sigma$ be the geodesic joining $\Lambda_n(0) = \eta_n$ to $\Lambda_n(1) = z_n$, and $\hat{\Lambda}_n : [0,1] \to \Sigma$ be the geodesic joining $\hat{\Lambda}_n(0) = \eta_n$ to $\hat{\Lambda}_n(1) = S_i(u_n)$ then we can write $\{\eta_{n+1}\}$ as $\eta_{n+1} = \Lambda_n(\Upsilon_n)$ and $\varsigma_n = \hat{\Lambda}_n(\vartheta_n)$.

First, we prove that Θ is geodesic convex and closed.

Since all the nonexpansive mappings are continuous, hence $F(S_i)$ is closed. Now, we show $F(S_i)$ is geodesic convex.

Suppose $k, l \in F(S_i)$, to prove $F(S_i)$ is geodesic convex, we have to show that geodesic $\Lambda : [0,1] \to \Sigma$ joining points k and l is also contained in $F(S_i)$. We know that in a given Hadamard manifold Σ, $\forall\ k, l \in \Sigma$, $0 \le t \le 1$, $\exists$ a unique point $\Lambda(t) = \exp_k t \exp_k^{-1} l = \zeta_t$ such that

$$d(k, l) = d(k, \zeta_t) + d(\zeta_t, l).$$

Using the geodesic convexity of Riemannian distance, the nonexpansiveness of S_i, we get

$$d(k, S_i(\zeta_t)) = d(S_i(k), \qquad S_i(\zeta_t)) \le d(k, \zeta_t) = d(k, \Lambda(t)) \le t d(k, l).$$

Similarly, we can also get

$$d(S_i(\zeta_t), l) \le (1-t) d(k, l).$$

Using above equations we get

$$d(k, l) \le d(k, S_i(\zeta_t)) + d(S_i(\zeta_t), l) \le d(k, l).$$

Hence

$$d(k,l) = d(k, S_i(\zeta_t)) + d(S_i(\zeta_t), l).$$

Since ζ_t is unique, hence we have $\zeta_t = S_i(\zeta_t)$. Hence, $\Lambda(t) = \zeta_t \in F(S_i)$. Therefore $F(S_i)$ is geodesic convex.

Now, using Proposition 2.15 we can say

$$(G+H)^{-1}(0) = F\left(J_\nu^H\left(\exp(-\nu G)\right)\right).$$

Since J_ν^H is nonexpansive mapping using this together assumption (3) we can easily get $J_\nu^H\left(\exp(-\nu G)\right)$ is also a nonexpansive mapping. Hence $(G+H)^{-1}(0)$ is also geodesic convex and closed in Σ.

Now, using Theorem 2.17, we say $T_r^{F_i}$ is a firmly nonexpansive mapping and $F\left(T_r^{F_i}\right) = EP(F_i)$. Therefore, $EP(F_i)$ is also geodesic convex and closed in Σ and hence Θ is geodesic convex and closed.

Now, we show that the sequence $\{\eta_n\}$ is Fejer monotone with respect to Θ.

Suppose $\zeta \in \Theta$

$$\begin{aligned}
d(u_n, \zeta) &= d\left(J_{\nu_n}^H\left(\exp_{\eta_n}\left(-\nu_n G(\eta_n)\right)\right), \zeta\right) \\
&= d\left(J_{\nu_n}^H\left(\exp_{\eta_n}\left(-\nu_n G(\eta_n)\right)\right), J_{\nu_n}^H\left(\exp_\zeta(-\nu_n G(\zeta))\right)\right) \\
&\le d\left(\exp_{\eta_n}\left(-\nu_n G(\eta_n)\right), \exp_\zeta(-\nu_n G(\zeta))\right) \\
&\le (1-\rho) d(\eta_n, \zeta) \le d(\eta_n, \zeta).
\end{aligned}$$

Since $\zeta \in \Theta$ using Theorem 2.17, we say $T_r^{F_i}$ is firmly nonexpansive and hence $T_r^{F_i}$ is nonexpansive, therefore S_r^m is also nonexpansive and $\zeta \in F(S_r^m)$, we get

$$d(z_n, \zeta) = d(S_r^m(w_n), S_r^m(\zeta)) \le d(w_n, \zeta) \le d(\varsigma_n^i, \zeta).$$

Now,

$$\begin{aligned}
d^2(\varsigma_n^i, \zeta) &= d^2\left(\exp_{\eta_n} \vartheta_n \exp_{\eta_n}^{-1} S_i(u_n), \zeta\right) \\
&\le (1-\vartheta_n) d^2(\eta_n, \zeta) + \vartheta_n d^2(S_i(u_n), \zeta) - \vartheta_n(1-\vartheta_n) d^2(\eta_n, S_i(u_n)) \\
&\le (1-\vartheta_n) d^2(\eta_n, \zeta) + \vartheta_n d^2(u_n, \zeta) - \vartheta_n(1-\vartheta_n) d^2(\eta_n, S_i(u_n)) \\
&\le (1-\vartheta_n) d^2(\eta_n, \zeta) + \vartheta_n d^2(u_n, \zeta) \\
&\le (1-\vartheta_n) d^2(\eta_n, \zeta) + \vartheta_n d^2(\eta_n, \zeta) \\
&= d^2(\eta_n, \zeta).
\end{aligned}$$

Thus

$$d(\varsigma_n^i, \zeta) \le d(\eta_n, \zeta).$$

And

$$\begin{aligned}
d^2(\eta_{n+1}, \zeta) &= d^2\left(\exp_{\eta_n} \Upsilon_n \exp_{\eta_n}^{-1} z_n, \zeta\right) \\
&\le (1-\Upsilon_n) d^2(\eta_n, \zeta) + \Upsilon_n d^2(z_n, \zeta) - \Upsilon_n(1-\Upsilon_n) d^2(\eta_n, z_n)
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\Upsilon_n)d^2(\eta_n,\zeta)+\Upsilon_n d^2(\varsigma_n^i,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\leq (1-\Upsilon_n)d^2(\eta_n,\zeta)+\Upsilon_n d^2(\eta_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\leq d^2(\eta_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\leq d^2(\eta_n,\zeta).
\end{aligned}$$

Thus

$$d(\eta_{n+1},\zeta)\leq d(\eta_n,\zeta), \qquad \text{for all } n\geq 0, \zeta\in\Theta,$$

and hence the sequence $\{\eta_n\}$ is Fejer monotone with respect to Θ. Using Lemma 2.6 implies that sequence $\{\eta_n\}$ is bounded alongwith $\{u_n\}$, $\{\varsigma_n^i\}$, $\{z_n\}$ and $\lim_{n\to\infty} d(\eta_n,\zeta)$ exists for any $\zeta\in\Theta$.

Now, we show that $\lim_{n\to\infty} d(\eta_{n+1},\eta_n)=0$.

$$\begin{aligned}
&d^2(\eta_{n+1},\zeta)\leq d^2(\eta_n,\zeta)-\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\\
&\Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\leq d^2(\eta_n,\zeta)-d^2(\eta_{n+1},\zeta)\\
&a(1-b)d^2(\eta_n,z_n)\leq \Upsilon_n(1-\Upsilon_n)d^2(\eta_n,z_n)\leq d^2(\eta_n,\zeta)-d^2(\eta_{n+1},\zeta)\to 0.
\end{aligned}$$

Since $a(1-b)>0$, it implies

$$d^2(\eta_n,z_n)\to 0 \quad \Rightarrow \quad d(\eta_n,z_n)\to 0.$$

Since $\eta_{n+1}=\Lambda_n(\Upsilon_n)$, we have

$$\begin{aligned}
d(\eta_{n+1},\eta_n)&=d(\Lambda_n(\Upsilon_n),\eta_n)\\
&\leq (1-\Upsilon_n)d(\Lambda_n(0),\eta_n)+\Upsilon_n d(\Lambda_n(1),\eta_n)\\
&=(1-\Upsilon_n)d(\eta_n,\eta_n)+\Upsilon_n d(z_n,\eta_n)\\
&=\Upsilon_n d(z_n,\eta_n)\\
&\leq b d(z_n,\eta_n).
\end{aligned}$$

Applying limit we get $\lim_{n\to\infty} d(\eta_{n+1},\eta_n)=0$.

Now, we prove that $\lim_{n\to\infty} d(S_i(u_n),\eta_n)=0$, $\lim_{n\to\infty} d(\varsigma_n^i,u_n)=0$, $\lim_{n\to\infty} d(u_n,\eta_n)=0$ and $\lim_{n\to\infty} d(S_r^m(\eta_n),\eta_n)=0$.

Now,

$$\begin{aligned}
d(\eta_{n+1},\zeta)&=d(\Lambda_n(\Upsilon_n),\zeta)\\
&\leq (1-\Upsilon_n)d(\Lambda_n(0),\zeta)+\Upsilon_n d(\Lambda_n(1),\zeta)\\
&\leq (1-\Upsilon_n)d(\eta_n,\zeta)+\Upsilon_n d(z_n,\zeta)\\
&\leq (1-\Upsilon_n)d(\eta_n,\zeta)+\Upsilon_n d(\varsigma_n^i,\zeta).
\end{aligned}$$

And

$$\begin{aligned}
d(\varsigma_n^i,\zeta)&=d(\hat{\Lambda}_n(\vartheta_n),\zeta)\\
&\leq (1-\vartheta_n)d(\hat{\Lambda}_n(0),\zeta)+\vartheta_n d(\hat{\Lambda}_n(1),\zeta)
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\vartheta_n)d(\eta_n,\zeta)+\vartheta_n d(S_i(u_n),\zeta)\\
&\leq (1-\vartheta_n)d(\eta_n,\zeta)+\vartheta_n d(S_i(u_n),S_i(\zeta))\\
&\leq (1-\vartheta_n)d(\eta_n,\zeta)+\vartheta_n d(u_n,\zeta).
\end{aligned}
$$

Similarly, using above two equations, we have

$$
\begin{aligned}
d(\eta_n,\zeta) &\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)+\Upsilon_{n-1}d(\varsigma^i_{n-1},\zeta)\\
&\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)\\
&\quad +\Upsilon_{n-1}\{(1-\vartheta_{n-1})d(\eta_{n-1},\zeta)+\vartheta_{n-1}d(u_{n-1},\zeta)\}\\
&\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)\\
&\quad +\Upsilon_{n-1}\{(1-\vartheta_{n-1})d(\eta_{n-1},\zeta)+\vartheta_{n-1}(1-\rho)d(\eta_{n-1},\zeta)\}\\
&\leq (1-\Upsilon_{n-1})d(\eta_{n-1},\zeta)+\Upsilon_{n-1}(1-\rho\vartheta_{n-1})d(\eta_{n-1},\zeta)\\
&= (1-\rho\vartheta_{n-1}\Upsilon_{n-1})d(\eta_{n-1},\zeta).
\end{aligned}
$$

Since, $\{\eta_n\}$ is a bounded sequence, so $\exists$ a constant Q in such a way that $d(\eta_n,\zeta)\leq Q\ \forall\ 0\leq n$.

$$d(\eta_n,\zeta)\leq (1-\rho\vartheta_{n-1}\Upsilon_{n-1})Q.$$

Suppose $0\leq m\leq n$, we get

$$d(\eta_n,\zeta)\leq Q\prod_{j=m}^{n-1}(1-\rho\vartheta_j\Upsilon_j).$$

Using condition (c), we can get

$$\lim_{n\to\infty}\prod_{j=m}^{n-1}(1-\rho\vartheta_j\Upsilon_j)=0,$$

and hence

$$\lim_{n\to\infty}d(\eta_n,\zeta)=0.$$

Now

$$
\begin{aligned}
d(\eta_n,\varsigma^i_n) &\leq d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\zeta)+d(\varsigma^i_n,\zeta)\\
&\leq d(\eta_n,\eta_{n+1})+d(\eta_n,\zeta)+d(\eta_n,\zeta),
\end{aligned}
$$

applying $\lim n\to\infty$ we get $\lim_{n\to\infty}d(\eta_n,\varsigma^i_n)=0$, and

$$
\begin{aligned}
d(\eta_n,S_i(u_n)) &\leq d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\zeta)+d(S_i(u_n),\zeta)\\
&\leq d(\eta_n,\eta_{n+1})+d(\eta_{n+1},\zeta)+d(u_n,\zeta)\\
&\leq d(\eta_n,\eta_{n+1})+2d(\eta_n,\zeta),
\end{aligned}
$$

applying $\lim n\to\infty$ we get $\lim_{n\to\infty}d(\eta_n,S_i(u_n))=0$ and

$$d(\eta_n,u_n)\leq d(\eta_n,\zeta)+d(\zeta,u_n)$$

$$\leq 2d(\eta_n, \zeta),$$

applying $\lim n \to \infty$ we get $\lim_{n\to\infty} d(\eta_n, u_n) = 0$ and

$$\begin{aligned} d(S_r^m(\eta_n), \eta_n) &\leq d(S_r^m(\eta_n), S_r^m(w_n)) + d(S_r^m(w_n), \eta_n) \\ &\leq d(\eta_n, w_n) + d(z_n, \eta_n) \\ &\leq d(\varsigma_n^i, \eta_n) + d(\eta_n, z_n), \end{aligned}$$

applying $\lim n \to \infty$ we get $\lim_{n\to\infty} d(S_r^m(\eta_n), \eta_n) = 0$.

Now, we show that the cluster point $\eta^\dagger$ of sequence $\{\eta_n\}$ belongs to Θ. Since we already proved that sequence $\{\eta_n\}$ is bounded. Therefore $\exists$ a subsequence $\{\eta_{n_j}\}$ of sequence $\{\eta_n\}$ which converges to the cluster point $\eta^\dagger$ of sequence $\{\eta_n\}$. Since $\lim_{n\to\infty} d(\eta_n, u_n) = 0$ it implies $\lim_{j\to\infty} d(u_{n_j}, \eta^\dagger) = 0$. Using nonexpansiveness of S_i, we get

$$\begin{aligned} d(\eta^\dagger, S_i(\eta^\dagger)) &\leq d(\eta^\dagger, \eta_{n_j}) + d(\eta_{n_j}, S_i(u_{n_j})) + d(S_i(u_{n_j}), S_i(\eta^\dagger)) \\ &\leq d(\eta^\dagger, \eta_{n_j}) + d(\eta_{n_j}, S_i(u_{n_j})) + d(u_{n_j}, \eta^\dagger). \end{aligned}$$

Applying $\lim_{j\to\infty}$, we get

$$d(\eta^\dagger, S_i(\eta^\dagger)) = 0 \quad \Rightarrow \quad \eta^\dagger \in F(S_i).$$

Now, we show that $\eta^\dagger \in \bigcap_{i=1}^{m} EP(F_i)$. We also have for any subsequence $\{\eta_{n_j}\}$ of $\{\eta_n\}$, $\lim_{j\to\infty} d(S_r^m(\eta_{n_j}), \eta_{n_j}) = 0$. We know that the mapping S_r^m is nonexpansive, it is demiclosed at 0 and hence $\eta^\dagger \in F(S_r^m)$. To prove $\eta^\dagger \in \bigcap_{i=1}^{m} EP(F_i)$ we have to prove that $F(S_r^m) = \bigcap_{i=1}^{m} F\left(T_r^{F_i}\right)$. It is obvious that $\bigcap_{i=1}^{m} F\left(T_r^{F_i}\right) \subseteq F(S_r^m)$, we only have to prove that $F(S_r^m) \subseteq \bigcap_{i=1}^{m} F\left(T_r^{F_i}\right)$.

Let $l \in F(S_r^m)$ and $k \in \bigcap_{i=1}^{m} F\left(T_r^{F_i}\right)$, we have

$$\begin{aligned} d(l, k) &= d(S_r^m(l), k) \\ &= d(T_r^{F_m} S_r^{m-1}(l), k) \\ &\leq d(S_r^{m-1}(l), k) \\ &\leq d(S_r^{m-2}(l), k) \\ &\ \ \vdots \\ &\leq d(S_r^1(l), k) \end{aligned}$$

$$\begin{aligned} &= d(T_r^{F_1}(l), k) \\ &\leq d(l, k). \end{aligned}$$

It implies that

$$\begin{aligned} d(l,k) &= d(S_r^m(l), k) \\ &= d(S_r^{m-1}(l), k) \\ &= d(S_r^{m-2}(l), k) \\ &\ \ \vdots \\ &= d(S_r^1(l), k) \\ &= d(T_r^{F_1}(l), k). \end{aligned}$$

Applying Lemma 2.9, we get

$$d^2\left(S_r^i(l), k\right) = d^2\left(S_r S_r^{i-1}(l), k\right) \leq d^2\left(S_r^{i-1}(l), k\right) - d^2\left(S_r^i(l), S_r^{i-1}(l)\right),$$

$$d^2\left(S_r^i(l), k\right) + d^2\left(S_r^i(l), S_r^{i-1}(l)\right) \leq d^2\left(S_r^{i-1}(l), k\right) = d^2(l, k).$$

Since $d(S_r^i(l), k) = d(l, k)$, from the above equation $\forall\ i = 1, 2, \ldots, m$, we have

$$\begin{aligned} &d\left(S_r^i(l), S_r^{i-1}(l)\right) = 0 = d\left(T_r^{F_i}(l), S_r^{i-1}(l)\right) \\ \Rightarrow \quad &S_r^{i-1}(l) \in F\left(T_r^{F_i}\right). \end{aligned} \tag{3.3}$$

Now, if we take $i = 1$ in (3.3), we get $l \in F(T_r^{F_1}) \Rightarrow l = T_r^{F_1}(l)$ again taking $i = 2$ in (3.3), we get $l = S_r^1(l) \in F(T_r^{F_2}) \Rightarrow l = T_r^{F_2}(l)$. Similarly taking $i = 2, 3, \ldots, m$ in (3.3), we get

$$l = T_r^{F_1}(l) = T_r^{F_2}(l) = \cdots = T_r^{F_{m-1}}(l) = T_r^{F_m}(l).$$

It implies that

$$l \in \bigcap_{i=1}^{m} F\left(T_r^{F_i}\right).$$

That is

$$F(S_r^m) = \bigcap_{i=1}^{m} F\left(T_r^{F_i}\right).$$

Now, finally we prove that $\eta^\dagger \in (G + H)^{-1}(0)$. Since $\hat{\nu} \leq \nu_n \leq \tilde{\nu}$, we can choose a $\nu > 0$ in such a way that the subsequence $\{\nu_{n_j}\}$ of $\{\nu_n\}$ converges to ν. Since $u_n = J_{\nu_n}^H(\exp_{\eta_n}(-\nu_n G(\eta_n)))$. Using Lemma 2.14 and $\lim_{n\to\infty} d(\eta_n, u_n) = 0$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} d(\eta_n, u_n) \\ &= \lim_{j\to\infty} d(\eta_{n_j}, u_{n_j}) \end{aligned}$$

$$= \lim_{j\to\infty} d\left(\eta_{n_j}, J^H_{\nu_{n_j}}\left(\exp_{\eta_{n_j}}\left(-\nu_{n_j} G\left(\eta_{n_j}\right)\right)\right)\right)$$
$$= d\left(\eta^{\dagger}, J^H_{\nu}\left(\exp_{\eta_{n_j}}\left(-\nu_{n_j} G\left(\eta_{n_j}\right)\right)\right)\right).$$

Using Proposition 2.15 we get $\eta^{\dagger} \in (G+H)^{-1}(0)$ and hence $\eta^{\dagger} \in \Theta$. This completes the proof. □

Corollary 3.2. *Suppose* $\mathcal{E}, \Sigma, G, H, \{F_i\}_{i=1}^m, \left\{S_r^j\right\}_{j=1}^m$ *and* S *the same as above. Suppose* $\{\eta_n\}, \{u_n\}, \{\varsigma_n\}$ *and* $\{z_n\}$ *are the sequences generated by* $\eta_0 \in \mathcal{E}$, $\forall n \geq 0$,

$$\begin{cases} u_n = J^H_{\nu_n}\left(\exp_{\eta_n}\left(-\nu_n G\left(\eta_n\right)\right)\right), \\ \varsigma_n = \exp_{\eta_n} \vartheta_n \exp^{-1}_{\eta_n} S u_n, \\ z_n = S^m_r\left(\varsigma_n\right), \\ \eta_{n+1} = \exp_{\eta_n} \Upsilon_n \exp^{-1}_{\eta_n} z_n, \end{cases}$$

where $\forall n \in \mathbb{N}$, $\{\vartheta_n\}, \{\Upsilon_n\}$ *and* $\{\nu_n\}$ *are the sequences of positive real numbers satisfying given following assumptions:*

(i) $0 < a \leq \vartheta_n, \Upsilon_n \leq b < 1$;
(ii) $0 < \hat{\nu} \leq \nu_n \leq \tilde{\nu} < \infty$;
(iii) $\sum_{n=1}^{\infty} \vartheta_n \Upsilon_n = \infty$.

If $\Theta = \bigcap_{i=1}^m EP\left(F_i\right) \bigcap (G+H)^{-1}(0) \bigcap F(S)$ *is nonempty, therefore the sequence* $\{\eta_n\}$ *converges strongly to solution of the problem* (1.2).

4. Example

In the sequel, we first recall hyperbolic space. We equip $\mathbb{R}^{n+1}$ with the inner product.

$$\langle \eta, \varsigma \rangle = -\eta_0 \varsigma_0 + \sum_{i=1}^n \eta_i \varsigma_i,$$

for $\eta = (\eta_0, \eta_1, \ldots, \eta_n)$ and $\varsigma = (\varsigma_0, \varsigma_1, \ldots, \varsigma_n)$. Define

$$\mathbb{H}^n = \left\{\eta = (\eta_0, \eta_1, \ldots, \eta_n) \in \mathbb{R}^n : \langle \eta, \eta \rangle = -1, \eta_0 > 0\right\}.$$

Then $\langle \cdot, \cdot \rangle$ induces the Riemannian metric d, on the tangent space $T_p\mathbb{H}^n \subset T_p\mathbb{R}^{n+1}$ as

$$d(\eta, \varsigma) = \operatorname{arccosh}(-\langle \eta, \varsigma \rangle), \quad \forall \eta, \varsigma \in \mathbb{H}^n,$$

for $p \in \mathbb{H}^n$. Then $(\mathbb{H}^n, d)$ is Hadamard manifold with sectional curvature -1 at every point [3].

Example 4.1. Let $\Sigma = \mathbb{H}^n$ be the hyperbolic space and $S_i : \Sigma \to \Sigma$ is the family of nonexpansive mappings for $i = 1, 2$ defined by

$$S_1(\eta) = (\eta_0, -\eta_1, -\eta_2, \ldots, \eta_n),$$

$$S_2(\eta) = (\eta_0, 0, 0 \cdots, 0).$$

Here $\bigcap_{i=1}^{2} F(S_i) = (1, 0, \ldots, 0)$. Now, we define $F_i : \Sigma \times \Sigma \to \mathbb{R}$ for all $i = 1, 2$, as $F_i(\eta, \varsigma) = \langle \eta - S_i(\eta), \varsigma - \eta \rangle$. If η is an equilibrium point of $EP(F_i)$, then $F_i(\eta, \varsigma) \geq 0$ for all $\varsigma \in \Sigma$. Taking $\varsigma = S_i(\eta)$, we get

$$\langle \eta - S_i(\eta), S_i(\eta) - \eta \rangle \geq 0.$$

For $i = 1$, $F_1(\eta, S_1(\eta)) = \langle \eta - S_1(\eta), S_1(\eta) - \eta \rangle \geq 0$. Which implies $-4\eta_1^2 - 4\eta_2^2 - \cdots - 4\eta_n^2 \geq 0$ that is $\eta_1 = \eta_2 = \cdots = \eta_n = 0$.

For $i = 2$, $F_2(\eta, S_2(\eta)) = \langle \eta - S_2(\eta), S_2(\eta) - \eta \rangle \geq 0$. Which implies $-\eta_1^2 - \eta_2^2 - \cdots - \eta_n^2 \geq 0$ that is $\eta_1 = \eta_2 = \cdots = \eta_n = 0$. On the other hand we have $\langle \eta, \eta \rangle = -1$. Therefore we can conclude that $\eta_0 = 1$. Hence $(1, 0, \ldots, 0)$ is an equilibrium point of $\bigcap_{i=1}^{2} EP(F_i)$ for $i = 1, 2$.

Now, define

$$H(\eta_0, \eta_1, \ldots, \eta_n) = (-\eta_0, \eta_1, \eta_2, \ldots, \eta_n),$$

and

$$G(\eta_0, \eta_1, \ldots, \eta_n) = (\eta_0(1 + \ln(\eta_0)), \eta_1, \eta_2, \ldots, \eta_n).$$

Here H is a maximal monotone, setvalued vector field and G is a continuous and monotone vector field and $(G+H)^{-1}(0) = (1, 0, \ldots, 0)$. If $\{\eta_n\}$ is the sequence generated by (3.2), then the sequence $\{\eta_n\}$ converges to

$$\bigcap_{i=1}^{2} EP(F_i) \bigcap (G + H)^{-1}(0) \bigcap_{i=1}^{2} F(S_i) = (1, 0, \ldots, 0).$$

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