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Abstract. This paper studies the concept of fuzzy generalized topologies, which are generalizations of smooth topologies and Chang's fuzzy topologies. A basis of fuzzy generalized topological space will be defined as functions from the family of all fuzzy subsets of a non-empty set X to $[0,1]$ and some basic properties of their structure will be obtained. Some characterizations of the basis of fuzzy generalized topology, fuzzy generalized cotopology and the product of fuzzy generalized topological spaces will also be shown.

1. Introduction

Chang [3] introduced the concept of fuzzy topology on a set X by axiomatizing a family T of fuzzy subsets of X. He referredd to each member of T as a T-open set. But, in his definition of fuzzy topology, the fuzziness in the concept of openness of a fuzzy subset is absent. To overcome this drawback in the fuzzification of topological spaces, R. N. Hazra et al [6] introduced the concept of gradation of openness (closedness) of fuzzy subsets of X and thereby gave a new definition of fuzzy topology on X. In this sense each fuzzy subset enjoys a definite grade of openness (closedness). Further to each fuzzy topology is an underlying Chang fuzzy topology. This generalisation of Chang's fuzzy topology has been developed in many directions [2, 4, 15]. Moreover, Srivastava [20] introduced the concept of a basis for a smooth fuzzy topological space in view of the definition of Hazra et al. [6], for which a basis is defined as a family of fuzzy sets of a set X satisfying a condition which depends on a fuzzy topology on X. According to these definitions, a

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basis or a sub-basis is defined only if a fuzzy topology is available on X. Likewise, Srivastava [20], Peeters [14], Park et al. [13], Kalaivani [7] and Roopkumar [8] studied the gradation of openness (closedness) of fuzzy subsets of a set X. Won Keun Min [10] introduced the concept of fuzzy generalized topologies which are generalizations of smooth topologies [15, 19] and Chang's fuzzy topologies [3]. He obtained some basic properties of their structure.

Molodtsov [11] studied the relationship between fuzzy sets and the soft set, he showed that Zadeh's fuzzy set may be considered as a special case of the soft set. In 2015, Mukherjee and Park [12] were first introduced the notion of fuzzy soft bitopological space and they presented the notions of $\tau_1\tau_2$-fuzzy soft open (closed) sets, $\tau_1\tau_2$-fuzzy soft interior (resp. closure). In [16, 17] Sayed characterized a new type of fuzzy soft sets and introduced some separation axioms in fuzzy soft bitopological spaces. He also investigated some characterizations of $(1,2)^{\star}$-fuzzy soft b-interior, $(1,2)^{\star}$-fuzzy soft b-closure fuzzy soft bitopological space mentioned in [18].

The purpose of this work is to define a the basis of fuzzy generalized topological space in view of the definition of fuzzy generalized topology [10] and to investigate some properties of basis. It is important to notice that our basis is different from the definition in classical theory, some functions from the family of all fuzzy subsets of a set X to $[0,1]$ will be used. Additionally, we will demonstrate some characterizations of basis of fuzzy generalized topology, fuzzy generalized cotopology and product of fuzzy generalized topological spaces. In this paper, all the notations and definitions are standard in fuzzy set theory.

2. Preliminaries

Let X be a set and $I=[0,1]$. Let I^X denote the set of all mappings $A: X \to I$. A member of I^X is called a fuzzy subset [3] of X. 1_X and 0_X will represent the characteristic functions of $\emptyset$ and X, respectively. Unions and intersections of fuzzy sets are denoted by $\vee$ and $\wedge$, respectively and defined by:

$$\underset{i\in J}{\vee} A_i = \sup\{A_i(x)/i\in J \text{ and } x\in X\},$$
$$\underset{i\in J}{\wedge} A_i = \inf\{A_i(x)/i\in J \text{ and } x\in X\},$$

in which J is an index set.

A Chang's fuzzy topological space [3] is an ordered pair (X,τ) where X is a non-empty set and $\tau \subset I^X$ satisfying the following conditions:

(i) $0_X, 1_X \in \tau$.
(ii) If $A, B \in \tau$, then $A \wedge B \in \tau$.
(iii) If $A_i \in \tau$, for all $i \in J$, then $\underset{i\in J}{\vee} A_i \in \tau$.

(X, τ) is a fuzzy topological space. Members of τ are called fuzzy open sets in (X, τ) and complement of this open set is called a fuzzy closed set. A smooth topological space [15] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(i) $\tau(0_X) = \tau(1_X) = 1$.
(ii) $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$ for $A_1, A_2 \in I^X$.
(iii) $\tau\left(\underset{i\in J}{\vee} A_i\right) \geq \underset{i\in J}{\wedge} \tau(A_i)$ for $A_i \in I^X$.

Then $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^\star : I^X \to I$ is called a smooth cotopology [15] iff the following three conditions are satisfied:

(i) $\tau^\star(0_X) = \tau^\star(1_X) = 1$.
(ii) $\tau^\star(A_1 \vee A_2) \geq \tau^\star(A_1) \wedge \tau^\star(A_2)$ for $A_1, A_2 \in I^X$.
(iii) $\tau^\star\left(\underset{i\in J}{\wedge} A_i\right) \geq \underset{i\in J}{\wedge} \tau^\star(A_i)$ for $A_i \in I^X$.

Let f be a mapping from a set X into a set Y. Let A and B be respectively the fuzzy sets of X and Y. Then $f(A)$ is a fuzzy set in Y defined by [5]:

$$f(A)(y) = \begin{cases} \sup_{z\in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \neq 0, \\ 0, & \text{else,} \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x)), x \in X$.

3. Main Results

Definition 3.1 ([10, Definition 2.1]). Let $\mathfrak{J}_\mu = \left\{A \in I^X / A \leq \mu\right\}$ such that μ is a fuzzy subset of a non-empty set X. Let $\mathfrak{T} : \mathfrak{J}_\mu \to [0, 1]$ be a mapping satisfying the following conditions:

(i) $\mathfrak{T}(\mu) = 1$, $\mathfrak{T}(0_X) = 1$.
(ii) $\mathfrak{T}\left(\underset{\lambda\in\Lambda}{\vee} A_i\right) \geq \underset{\lambda\in\Lambda}{\wedge} \mathfrak{T}(A_\lambda)$ for any family $(A_\lambda)_{\lambda\in\Lambda}, A_\lambda \in \mathfrak{J}_\mu$.

$(\mu, \mathfrak{T})$ referred to as a fuzzy generalized topological space and $\mathfrak{T}$ is called a fuzzy generalized topology on μ. $\mathfrak{T}(A)$ represents the degree of openness of the fuzzy set A.

Chang's fuzzy topology is a smooth topology and a fuzzy generalized topology. But a smooth topology cannot be a fuzzy generalized topology in the following example.

Example 3.2. Let $X=[0,2]$, $\mu=1_X$. Define fuzzy sets $A_1, A_2 \in \mathcal{J}_\mu$ as follows

$$\begin{aligned} A_1(x) &= \frac{x^2}{4}, \quad \forall x \in [0,2], \\ A_2(x) &= \frac{x}{4}, \quad \forall x \in [0,2]. \end{aligned}$$

Consider a fuzzy generalized topology $\mathcal{T} : \mathcal{J}_\mu \to I$ defined as,

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A = 0_X \text{ or } 1_X, \\ \frac{1}{8}, & \text{if } A = A_1, \\ \frac{1}{4}, & \text{if } A = A_2, \\ \frac{1}{2}, & \text{if } A = A_1 \vee A_2, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{T}(A_1 \wedge A_2) \leq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2)$ then $\mathcal{T}$ is a fuzzy generalized topology which is not a smooth topology.

Definition 3.3 ([10, Definition 2.3])**.** Let $\mathcal{J}_\mu = \{A \in I^X / A \leq \mu\}$ such that μ is a fuzzy subset of a non-empty set X. A mapping $\mathcal{T}^* : I^X \to I$ is called a fuzzy generalized cotopology if the following conditions are satisfied:

(i) $\mathcal{T}^*(\mu) = 1$, $\mathcal{T}^*(0_X) = 1$.

(ii) $\mathcal{T}^*\left(\underset{\lambda\in\Lambda}{\wedge} A_\lambda\right) \geq \underset{\lambda\in\Lambda}{\vee} \mathcal{T}^*(A_\lambda)$, for any family $(A_\lambda)_{\lambda\in\Lambda}, A_\lambda \in \mathcal{J}_\mu$.

$\mathcal{T}^*(A)$ refers to the degree of generalized closedness of A.

Theorem 3.4 ([10, Theorem 2.4])**.** *If $\mathcal{T}$ is a fuzzy generalized topology on μ, then the mapping $\mathcal{T}^* : \mathcal{J}_\mu \to I$ defined by $\mathcal{T}^*(A) = \mathcal{T}(\mu - A)$, for all $A \in \mathcal{J}_\mu$, is a fuzzy generalized cotopology on μ.*

Proof. Let $\mathcal{J}_\mu = \{A \in I^X / A \leq \mu\}$ such that μ is a fuzzy subset of a non-empty set X. Let $\mathcal{T}$ be a fuzzy generalized topology on μ. It's obvious that $\mathcal{T}^*(\mu) = 1$, $\mathcal{T}^*(0_X) = 1$. For all $A \in \mathcal{J}_\mu$, we have:

$$\begin{aligned} \mathcal{T}^*\left(\underset{\lambda\in\Lambda}{\wedge} A_\lambda\right) &= \mathcal{T}\left(\mu - \underset{\lambda\in\Lambda}{\wedge} A_\lambda\right) \\ &= \mathcal{T}\left(\underset{\lambda\in\Lambda}{\vee} (\mu - A_\lambda)\right) \\ &\geq \underset{\lambda\in\Lambda}{\wedge} \mathcal{T}(\mu - A_\lambda) \\ &= \underset{\lambda\in\Lambda}{\wedge} \mathcal{T}^*(A_\lambda). \end{aligned}$$

Therefore, $\mathcal{T}^*\left(\underset{\lambda\in\Lambda}{\wedge} A_\lambda\right) \geq \underset{\lambda\in\Lambda}{\vee} \mathcal{T}^*(A_\lambda)$. Thus, $\mathcal{T}^*$ is a fuzzy generalized cotopology on μ. □

Similarly, we have the next theorem.

Theorem 3.5 ([10, Theorem 2.5])**.** *If $\mathcal{T}^*$ is a fuzzy generalized cotopology on μ, then the mapping $\mathcal{T} : \mathcal{J}_\mu \to I$ defined by $\mathcal{T}(A) = \mathcal{T}^*(\mu - A)$, for all $A \in \mathcal{J}_\mu$, is a fuzzy generalized topology on μ.*

Definition 3.6 ([10, Definition 2.6])**.** Given a non-empty set X and fuzzy generalized topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X. We say that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ or $\mathcal{T}_2$ if coarser than $\mathcal{T}_1$ (denoted by $\mathcal{T}_1 \geq \mathcal{T}_2$) if $\mathcal{T}_1(A) \geq \mathcal{T}_2(A)$ for every $A \in \mathcal{J}_\mu$.

Definition 3.7. Let μ be a fuzzy subset of a non-empty set X. Let $\mathcal{B} : \mathcal{J}_\mu \to [0,1]$ be a mapping. $\mathcal{B}$ is called a basis for a fuzzy generalized topology on μ if for every $x \in X$ and $\epsilon > 0$ there exists $A \leq \mu$ such that $A(x) = \mu(x)$ and $\mathcal{B}(A) \geq 1 - \epsilon$.

Definition 3.8. Let μ be fuzzy subset of a non-empty set X and let $\mathcal{B}$ be a basis for a fuzzy generalized topology on μ. Let $\mathcal{T}(A) : \mathcal{J}_\mu \to [0,1]$ be the mapping defined by $\mathcal{B}$ as

$$\mathcal{T}(A) = \begin{cases} 1, & \text{if } A = 0_X, \\ \sup\limits_{\Lambda\in\Gamma}\left\{\sup\limits_{A_\lambda\in C_\Lambda}\{\mathcal{B}(A_\lambda)\}\right\}, & \text{if } A \neq 0_X, \end{cases}$$

where $(C_\Lambda)_{\Lambda\in\Gamma}$ is the family of all inner covers $C_\Lambda = (A_\lambda)_{\lambda\in\Lambda}$ of A.

Theorem 3.9. *Let μ be a fuzzy subset of a non-empty set X and $\mathcal{B}$ be a basis. If $\mathcal{T}$ is as defined in the previous Definition, then $\mathcal{T}$ is a fuzzy generalized topology on μ.*

Proof. Since $\mathcal{B}$ takes the value in $[0,1]$, $\mathcal{T}$ is well defined. From the definition of $\mathcal{T}$ we have, $\mathcal{T}(0_X) = 1$. Now, we prove that $\mathcal{T}(\mu) = 1$. For each $x \in X$ and $\epsilon > 0$ let $A_{x,\epsilon} \leq \mu$ such that $A_{x,\epsilon}(x) = \mu(x)$ and $\mathcal{B}(A_{x,\epsilon}) \geq 1 - \epsilon$. Then, $\sup\limits_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\} \geq 1 - \epsilon$. The family $\mathcal{C}_\epsilon = (A_{x,\epsilon})_{x\in X}$ is an inner cover for μ. Thus, for each $\epsilon > 0$ there exists an inner cover $\mathcal{C}_\epsilon = (A_{x,\epsilon})_{x\in X}$ of μ such that $\sup\limits_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\} \geq 1 - \epsilon$. Therefore, $\mathcal{T}(\mu) \geq \sup\left\{\sup\limits_{x\in X}\{\mathcal{B}(A_{x,\epsilon})\}\right\} \geq 1$ and hence, $\mathcal{T}(\mu) = 1$. Now, we prove that for any family $(A_\lambda)_{\lambda\in\Lambda}, A_\lambda \in \mathcal{J}_\mu$, $\mathcal{T}\left(\underset{\lambda\in\Lambda}{\vee} A_\lambda\right) \geq \underset{\lambda\in\Lambda}{\wedge} T(A_\lambda)$. For each $\epsilon > 0$ and for each A_λ, let $(A_{\lambda,\gamma})_{\gamma\in\Gamma_\lambda}$ be an inner cover for A_λ, then by hypothesis $\sup\limits_{\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\} \geq \mathcal{T}(A_\lambda)-\epsilon$. Since $(A_{\lambda,\gamma})_{\gamma\in\Gamma_\lambda}$ is an inner cover for A_λ, we have $(A_{\lambda,\gamma})_{\lambda\in\Lambda,\gamma\in\Gamma_\lambda}$ is an inner cover for $\bigvee\limits_{\lambda\in\Lambda} A_\lambda$. Now,

$$\begin{aligned}\mathcal{T}\left(\bigvee_{\lambda\in\Lambda} A_\lambda\right) &\geq \sup_{\gamma\in\Gamma_\lambda,\lambda\in\Lambda}\{\mathcal{B}(A_{\lambda,\gamma})\}\\ &= \sup_{\lambda\in\Lambda}\left\{\sup_{\gamma\in\Gamma_\lambda}\{\mathcal{B}(A_{\lambda,\gamma})\}\right\}\\ &\geq \sup_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)-\epsilon\}\\ &\geq \sup_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\}-\epsilon.\end{aligned}$$

Since this is true for every $\epsilon>0$, we have $\mathcal{T}\left(\bigvee\limits_{\lambda\in\Lambda} A_\lambda\right) \geq \sup\limits_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\}$ and $\sup\limits_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\} \geq \inf\limits_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\}$. Therefore, $\mathcal{T}\left(\bigvee\limits_{\lambda\in\Lambda} A_\lambda\right) \geq \inf\limits_{\lambda\in\Lambda}\{\mathcal{T}(A_\lambda)\}$. Hence, $\mathcal{T}\left(\bigvee\limits_{\lambda\in\Lambda} A_\lambda\right) \geq \bigwedge\limits_{\lambda\in\Lambda} T(A_\lambda)$. Thus, $\mathcal{T}$ is a fuzzy generalized topology on μ. □

Theorem 3.10. *Let $(\mu,\mathcal{T})$ be a fuzzy generalized topological space, then $\mathcal{T}$ is a basis for the fuzzy generalized topology $\mathcal{T}$.*

Proof. We prove that $\mathcal{T}$ is a basis for the fuzzy generalized topology $\mathcal{T}$ on μ. For any $x\in X$ and $\epsilon>0$, we choose $A=\mu$. Then we have, $A(x)=\mu(x)$, $A\leq\mu$ and $\mathcal{T}(A)=\mathcal{T}(\mu)\geq 1-\epsilon$. Thus, $\mathcal{T}$ is a basis for a fuzzy generalized topology on μ.

We prove now that the fuzzy generalized topology generated by $\mathcal{T}$ is $\mathcal{T}$. Let $\mathcal{T}'$ be another fuzzy generalized topology generated by $\mathcal{T}$. Let $D\in\mathcal{J}_\mu$, since $\mathcal{T}'$ is another fuzzy generalized topology generated by $\mathcal{T}$, we have

$$\mathcal{T}'(D)=\sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda}\{\mathcal{T}(D_\lambda)\}\right\},$$

such that $(C_\Lambda)_{\Lambda\in\Gamma}$ is the family of all possible inner covers $(D_\lambda)_{\lambda\in\Lambda}$ of D. Since D itself is an inner cover for D, we have $\mathcal{T}'(D)\geq\mathcal{T}(D)$. Therefore, $\mathcal{T}'\geq\mathcal{T}$. By hypothesis, $\mathcal{T}$ is also a fuzzy generalized topology on μ. Hence, for any inner cover $(D_\lambda)_{\lambda\in\Lambda}$ of D, we have

$$\mathcal{T}(D)=\sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda}\{\mathcal{T}(D_\lambda)\}\right\}$$

$$\leq \sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda}\left\{\mathcal{T}'\left(D_\lambda\right)\right\}\right\} \quad \left(\text{By } \mathcal{T}' \geq \mathcal{T}\right)$$
$$\leq \mathcal{T}'\left(D\right)$$

Therefore, $\mathcal{T}' \geq \mathcal{T}$ and Thus, $\mathcal{T} = \mathcal{T}'$. □

Theorem 3.11. *Let $(\mu, \mathcal{T})$ be a fuzzy generalized topological space. Let $\mathcal{B} : \mathcal{J}_\mu \to [0, 1]$ be a mapping satisfying:*

(i) $\mathcal{T}(A) \geq \mathcal{B}(A)$ *for all* $A \in \mathcal{J}_\mu$,

(ii) if $A \in \mathcal{J}_\mu, x \in X$ *and* $\epsilon > 0$*, then there exists* $B \in \mathcal{J}_\mu$ *such that* $B(x) = A(x)$, $B \leq A$ *and* $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Then, $\mathcal{B}$ *is a basis for the fuzzy generalized topology* $\mathcal{T}$ *on* μ.

Proof. We prove that $\mathcal{B}$ is a basis for the fuzzy generalized topology $\mathcal{T}$ on μ. By hypothesis, for any $x \in X$ and $\epsilon > 0$, we choose $A = \mu$. Then, we have $A(x) = \mu(x)$, $A \leq \mu$ and $\mathcal{B}(A) \geq \mathcal{T}(\mu) - \epsilon$. Thus, $\mathcal{T}$ is a basis for a fuzzy generalized topology on μ. We prove now that the fuzzy generalized topology generated by $\mathcal{B}$ is $\mathcal{T}$. Let $\mathcal{T}'$ be another fuzzy generalized topology generated by $\mathcal{B}$. Let $D \in \mathcal{J}_\mu$ and let $(D_\lambda)_{\lambda\in\Lambda}$ be an inner cover for D. Then, by hypothesis $(ii.)$, for all $x \in X$ and for each D_λ, there exists $D_{\lambda,x} \in \mathcal{J}_\mu$ such that $D_{\lambda,x}(x) = D_\lambda(x), D_{\lambda,x} \leq D_\lambda$ and $\mathcal{B}(D_{\lambda,x}) \geq \mathcal{T}(D_\lambda) - \epsilon$. Then, the family $(D_{\lambda,x})_{x\in X}$ is an inner cover for D_λ and therefore, the family $(D_{\lambda,x})_{\lambda\in\Lambda,x\in X}$ is an inner cover for D. Therefore, for any given inner cover $(D_\lambda)_{\lambda\in\Lambda}$ of D there exists an inner cover $(D_{\lambda,x})_{\lambda\in\Lambda,x\in X}$ of D such that $\mathcal{B}(D_{\lambda,x}) \geq \mathcal{T}(D_\lambda) - \epsilon$, for all $\lambda \in \lambda$, $x \in X$. Hence,

$$\sup_{\lambda\in\Lambda,x\in X}\left\{\mathcal{B}\left(D_{\lambda,x}\right)\right\} \geq \sup_{\lambda\in\Lambda}\left\{\mathcal{T}\left(D_\lambda\right) - \epsilon\right\} = \sup_{\lambda\in\Lambda}\left\{\mathcal{T}\left(D_\lambda\right)\right\} - \epsilon. \tag{3.1}$$

Since this is true for every inner cover $C_\Lambda = (D_\lambda)_{\lambda\in\Lambda}$, then

$$\sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda,x\in X}\left\{\mathcal{B}\left(D_\lambda, x\right)\right\}\right\} \geq \sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda}\left\{\mathcal{T}\left(D_\lambda\right)\right\}\right\} - \epsilon,$$

where $(C_\Lambda)_{\Lambda\in\Gamma}$ is the family of all possible inner covers $(D_\lambda)_{\lambda\in\Lambda}$ of D.

By Definition 3.8 of $\mathcal{T}'$, we have, $\mathcal{T}'(D) = \sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda,x\in X}\left\{\mathcal{B}\left(D_\lambda, x\right)\right\}\right\}$.

Therefore, by (3.1) and Theorem 3.10, for every $\epsilon > 0$:

$$\mathcal{T}'\left(D\right) \geq \sup_{\Lambda\in\Gamma}\left\{\sup_{D_\lambda\in C_\Lambda}\left\{\mathcal{T}\left(D_\lambda\right)\right\}\right\} - \epsilon$$
$$= \mathcal{T}\left(D\right) - \epsilon.$$

Hence, $\mathcal{T}' \geq \mathcal{T}$. Conversely, let $D \in \mathfrak{I}_\mu$ and $C_\Lambda = (D_\lambda)_{\lambda \in \Lambda}$ be an inner cover for D. By (i), for all $\lambda \in \Lambda$, $\mathcal{T}(D_\lambda) \geq \mathcal{B}(D_\lambda)$. Then, $\sup\limits_{D_\lambda \in C_\Lambda} \{\mathcal{T}(D_\lambda)\} \geq \sup\limits_{D_\lambda \in C_\Lambda} \{\mathcal{B}(D_\lambda)\}$. Therefore,

$$\sup_{\Lambda \in \Gamma} \left\{ \sup_{D_\lambda \in C_\Lambda} \{\mathcal{T}(D_\lambda)\} \right\} \geq \sup_{\Lambda \in \Gamma} \left\{ \sup_{D_\lambda \in C_\Lambda} \{\mathcal{B}(D_\lambda)\} \right\}.$$

Hence, by Definition 3.8 and Theorem 3.10 once again, $\mathcal{T} \geq \mathcal{T}'$. Thus, $\mathcal{T} = \mathcal{T}'$. □

Theorem 3.12. *If $\mathcal{B}$ is a basis for the fuzzy generalized topological space $(\mu, \mathcal{T})$, then*

(i) $\mathcal{T}(A) \geq \mathcal{B}(A)$ *for all* $A \in \mathfrak{I}_\mu$.

(ii) if $x \in X$, $A \in \mathfrak{I}_\mu$ *and* $\epsilon > 0$, *then there exists* $B \in \mathfrak{I}_\mu$ *such that* $B(x) = A(x)$, $B \leq A$ *and* $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Proof. Let $D \in \mathfrak{I}_\mu$. By Definition 3.8 of $\mathcal{T}$ and since (D) is an inner cover for D, hence $(i.)$ is true. For $(ii.)$, let $x \in X, D \in \mathfrak{I}_\mu$ and $\epsilon > 0$. By Definition 3.8 of $\mathcal{T}$ there exists an inner cover $C_\Lambda = (D_\lambda)_{\lambda \in \Lambda}$ of D such that $\sup\limits_{D_\lambda \in C_\Lambda} \{\mathcal{B}(D_\lambda)\} \geq \mathcal{T}(D) - \epsilon$ and there exists $D_{\lambda_0} \in (D_\lambda)_{\lambda \in \Lambda}$ such that $D_{\lambda_0}(x) = D(x)$. Since $D_{\lambda_0} \in (D_\lambda)_{\lambda \in \Lambda}$, then $D_{\lambda_0} \leq D$ and hence, $\mathcal{B}(D_{\lambda_0}) \geq \mathcal{T}(D) - \epsilon$. □

Theorem 3.13. *Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for the fuzzy generalized topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on μ. Then, the following conditions are equivalent.*

(i) $\mathcal{T}'$ *is finer than* $\mathcal{T}$.

(ii) If $B \in \mathfrak{I}_\mu$, $x \in X$ *and* $\epsilon > 0$ *there exists* $B' \in \mathfrak{I}_\mu$ *such that* $B' \leq B$, $B'(x) = B(x)$ *and* $\mathcal{B}'(B') \geq \mathcal{B}(B) - \epsilon$.

Proof. Suppose that $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $B \in \mathfrak{I}_\mu, x \in X$ and $\epsilon > 0$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, by Theorem 3.12, there exists $B' \in \mathfrak{I}_\mu$ such that

$$B'(x) = B(x), B' \leq B \text{ and } \mathcal{B}'\left(B'\right) \geq \mathcal{T}'(B) - \epsilon.$$

By hypothesis, $\mathcal{T}'(B) \geq \mathcal{T}(B)$. Therefore,

$$\mathcal{B}'\left(B'\right) \geq \mathcal{T}'(B) - \epsilon \geq \mathcal{T}(B) - \epsilon \geq \mathcal{B}(B) - \epsilon.$$

Conversely, let $D \in \mathfrak{I}_\mu$ and $\epsilon > 0$. Let $(C_\Lambda)_{\Lambda \in \Gamma}$ be the family of all possible inner covers $(D_\lambda)_{\lambda \in \Lambda}$ of D. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, there exists an inner cover $C_\Lambda = (D_\lambda)_{\lambda \in \Lambda}$ such that

$$\sup_{D_\lambda \in C_\Lambda} \{\mathcal{B}(D_\lambda)\} \geq \mathcal{T}(D) - \epsilon.$$

By hypothesis (*ii.*), for each D_λ and for each $x \in X$ there exists $D_{\lambda,x} \in \mathcal{J}_\mu$ such that $D_{\lambda,x}(x) = D_\lambda(x), D_{\lambda,x} \leq D_\lambda$ and $\mathcal{B}'(D_{\lambda,x}) \geq \mathcal{B}(D_\lambda) - \epsilon$. Then, $(D_{\lambda,x})_{x\in X}$ is an inner cover for D_λ and hence, $(D_{\lambda,x})_{\lambda\in\Lambda,x\in X}$ is an inner cover for D. Therefore, for any given inner cover $C_\Lambda = (D_\lambda)_{\lambda\in\Lambda}$, there exists an inner cover $(D_{\lambda,x})_{\lambda\in\Lambda,x\in X}$ of D such that, for all $\lambda \in \Lambda$, $x \in X$, $\mathcal{B}'(D_{\lambda,x}) \geq \mathcal{B}(D_\lambda) - \epsilon$. Then,

$$\sup_{D_\lambda \in C_\Lambda} \left\{\mathcal{B}'(D_{\lambda,x})\right\} \geq \sup_{D_\lambda \in C_\Lambda} \left\{\mathcal{B}(D_\lambda) - \epsilon\right\} \geq \sup_{D_\lambda \in C_\Lambda} \left\{\mathcal{B}(D_\lambda)\right\} - \epsilon.$$

Therefore,

$$\sup_{\Lambda\in\Gamma} \left\{ \sup_{D_\lambda \in C_\Lambda, x\in X} \left\{\mathcal{B}'(D_{\lambda,x})\right\} \right\} \geq \sup_{\Lambda\in\Gamma} \left\{ \sup_{D_\lambda \in C_\Lambda} \left\{\mathcal{B}(D_\lambda)\right\} \right\} - \epsilon.$$

By Definition 3.8 of $\mathcal{T}'$ and $\mathcal{T}$ we have, for every $\epsilon > 0$,

$$\begin{aligned}
\mathcal{T}'(D) &= \sup_{\Lambda\in\Gamma} \left\{ \sup_{D_\lambda \in C_\Lambda, x\in X} \left\{\mathcal{B}'(D_{\lambda,x})\right\} \right\} \\
&\geq \sup_{\Lambda\in\Gamma} \left\{ \sup_{D_\lambda \in C_\Lambda} \left\{\mathcal{B}(D_\lambda)\right\} \right\} - \epsilon \\
&= \mathcal{T}(D) - \epsilon.
\end{aligned}$$

Hence, $\mathcal{T}'(D) \geq \mathcal{T}(D)$ and thus, $\mathcal{T}' \geq \mathcal{T}$. □

Theorem 3.14. *If $(\mathcal{T}_\delta)_{\delta\in\Delta}$ is a family of fuzzy generalized topologies on μ, then $\inf_{\delta\in\Delta} \{\mathcal{T}_\delta\}$ is also a fuzzy generalized topology on μ.*

Proof. Let $\mathcal{T} = (\mathcal{T}_\delta)_{\delta\in\Delta}$. It's obvious that $\mathcal{T}(\mu) = 1$ and $\mathcal{T}(0_X) = 1$. Let $(D_\lambda)_{\lambda\in\Lambda}$, $D_\lambda \in \mathcal{J}_\mu$ be any family of fuzzy subsets of X. Then,

$$\begin{aligned}
\mathcal{T}\left(\underset{\lambda\in\Lambda}{\vee} D_\lambda\right) &= \inf_{\delta\in\Delta} \left\{ \mathcal{T}_\delta\left(\underset{\lambda\in\Lambda}{\vee} D_\lambda\right)\right\} \\
&\geq \inf_{\delta\in\Delta} \left\{ \underset{\lambda\in\Lambda}{\wedge} \left\{\mathcal{T}_\delta(D_\lambda)\right\}\right\} \\
&= \underset{\lambda\in\Lambda}{\wedge} \left\{ \inf_{\delta\in\Delta} \left\{\mathcal{T}_\delta(D)\right\}\right\} \\
&= \underset{\lambda\in\Lambda}{\wedge} \mathcal{T}(D_\lambda).
\end{aligned}$$

Thus, $\mathcal{T}$ is a fuzzy generalized topology on μ. □

Definition 3.15 ([10, Definition 2.11])**.** Let μ and ν be fuzzy subsets of X and Y, respectively. Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two fuzzy generalized topological spaces. Let $f : (\mu, \mathcal{T}_\mu) \to (\nu, \mathcal{T}_\nu)$ be a mapping on fuzzy generalized topological spaces. Then,

(i) f is said to be fuzzy generalized continuous if for every $A \in \mathcal{J}_\nu$, $\mathcal{T}_\mu\left(f^{-1}(A)\right) \geq \mathcal{T}_\nu(A)$.

(ii) f is said to be weakly fuzzy generalized continuous if for every $A \in \mathcal{J}_\nu$, we have $\mathcal{T}_\nu(A) > 0 \quad \Rightarrow \quad \mathcal{T}_\mu\left(f^{-1}(A)\right) > 0$.

Theorem 3.16. *Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two fuzzy generalized topological spaces. Let $\mathcal{B}_\mu, \mathcal{B}_\nu$ be basis for the fuzzy generalized topologies $\mathcal{T}_\mu, \mathcal{T}_\nu$, respectively. Then,*

(i) f is fuzzy generalized continuous if for every $A \in \mathcal{J}_\nu$,

$$\mathcal{B}_\mu\left(f^{-1}(A)\right) \geq \mathcal{B}_\nu(A).$$

(ii) f is said to be weakly fuzzy generalized continuous if for every $A \in \mathcal{J}_\nu$,

$$\mathcal{B}_\nu(A) > 0 \quad \Rightarrow \quad \mathcal{B}_\mu\left(f^{-1}(A)\right) > 0.$$

Proof. (i) Let $(C_\Lambda)_{\Lambda \in \Gamma}$ be the family of all inner covers $C_\Lambda = (A_\lambda)_{\lambda \in \Lambda}$ of A. By hypthesis for all $A_\lambda \in C_\Lambda$ and for all $\Lambda \in \Gamma$,

$$\begin{aligned} \mathcal{B}_\mu\left(f^{-1}(A_\lambda)\right) \geq \mathcal{B}_\nu(A_\lambda) \quad &\Rightarrow \quad \sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\mu\left(f^{-1}(A_\lambda)\right)\right\} \geq \\ &\qquad \sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\nu(A_\lambda)\right\} \\ &\Rightarrow \quad \sup_{\Lambda \in \Gamma}\left\{\sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\mu\left(f^{-1}(A_\lambda)\right)\right\}\right\} \geq \\ &\qquad \sup_{\Lambda \in \Gamma}\left\{\sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\nu\left(f^{-1}(A_\lambda)\right)\right\}\right\} \\ &\Rightarrow \quad \mathcal{T}_\mu\left(f^{-1}(A)\right) \geq \mathcal{T}_\nu(A). \end{aligned}$$

Thus, f is fuzzy generalized continuous.

(ii) Let $(C_\Lambda)_{\Lambda \in \Gamma}$ be the family of all inner covers $C_\Lambda = (A_\lambda)_{\lambda \in \Lambda}$ of A. By hypthesis, for all $A_\lambda \in C_\Lambda$ and for all $\Lambda \in \Gamma$:

$$\mathcal{B}_\nu(A_\lambda) > 0 \quad \Rightarrow \quad \mathcal{B}_\mu\left(f^{-1}(A_\lambda)\right) > 0.$$

Then,

$$\sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\nu\left(f^{-1}(A_\lambda)\right)\right\} > 0 \quad \Rightarrow \quad \sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\mu\left(f^{-1}(A_\lambda)\right)\right\} > 0.$$

Therefore,

$$\sup_{\Lambda \in \Gamma}\left\{\sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\nu\left(f^{-1}(A_\lambda)\right)\right\}\right\} > 0 \Rightarrow \sup_{\Lambda \in \Gamma}\left\{\sup_{A_\lambda \in C_\Lambda} \left\{\mathcal{B}_\mu\left(f^{-1}(A_\lambda)\right)\right\}\right\} > 0.$$

Hence, $\mathcal{T}_\nu(A) > 0 \quad \Rightarrow \quad \mathcal{T}_\mu\left(f^{-1}(A)\right) > 0$. Thus, f is weakly fuzzy generalized continuous.

□

Definition 3.17 ([1])**.** Let μ and ν be fuzzy subsets of X and Y respectively. The product $\mu \times \nu$ of μ and ν is defined as a fuzzy subset of $X \times Y$ by

$$(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y), \quad \text{for each } (x, y) \in X \times Y.$$

Definition 3.18. Let μ and ν be fuzzy subsets of X and Y, respectively. Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two fuzzy generalized topological spaces. Let $D \in \mathcal{J}_{\mu\times\nu}$. Define the mapping $\mathcal{B} : \mathcal{J}_{\mu\times\nu} \to [0, 1]$ as

- If D cannot be written as $A \times B$ for any $A \in \mathcal{J}_\mu$ and $B \in \mathcal{J}_\nu$, then $\mathcal{B}(D) = 0$.
- Otherwise, $\mathcal{B}(D) = \sup\limits_{\lambda\in\Lambda} \{\mathcal{T}_\mu(A_\lambda) \wedge \mathcal{T}_\nu(B_\lambda)\}$, such that $(A_\lambda \times B_\Lambda)_{\lambda\in\Lambda}$ is the family of all possible ways of writing E as $D = A_\lambda \times B_\lambda$, with $A_\lambda \in \mathcal{J}_\mu, B_\lambda \in \mathcal{J}_\nu$.

Theorem 3.19. *Let $(\mu, \mathcal{T}_\mu)$ and (ν, T_ν) be two fuzzy generalized topological spaces. Let $\mathcal{B} : \mathcal{J}_{\mu\times\nu} \to [0, 1]$ be the mapping defined as in the previous Definition. Then, $\mathcal{B}$ is a basis for a fuzzy generalized topology on $\mu \times \nu$ (in this case, this topology is called the product topology on $\mu \times \nu$).*

Proof. For all $(x, y) \in X \times Y$, $\epsilon > 0$:

- If D cannot be written as $A \times B$ for any $A \in \mathcal{J}_\mu$ and $B \in \mathcal{J}_\nu$ then, $\mathcal{B}(D) = 0 > 1 - \epsilon$ and hence Definition 3.1 follows.
- Otherwise we choose $D = \mu \times \nu$ such that $\mathcal{T}_\mu(\mu) = \mathcal{T}_\nu(\nu) = 1$. By definition 3.18 of $\mathcal{B}$, $\mathcal{B}(D) = 1$ and hence, $\mathcal{B}(D) > 1 - \epsilon$.

Thus, $\mathcal{B}$ is a basis for a fuzzy generalized topology on $\mu \times \nu$. □

Theorem 3.20. *Let $(\mu, \mathcal{T}_\mu)$ and $(\nu, \mathcal{T}_\nu)$ be two fuzzy generalized topological spaces. Let $\mathcal{B}_\mu, \mathcal{B}_\nu$ be basis for the fuzzy generalized topologies $\mathcal{T}_\mu, \mathcal{T}_\nu$, respectively. define a mapping $\mathcal{B}_{\mu\times\nu} : \mathcal{J}_{\mu\times\nu} \to [0, 1]$ as*

- *Let $D \in \mathcal{J}_{\mu\times\nu}$. If D cannot be written as $A \times B$ for any $A \in \mathcal{J}_\mu$ and $B \in \mathcal{J}_\nu$, then $\mathcal{B}_{\mu\times\nu}(D) = 0$.*
- *Otherwise, $\mathcal{B}_{\mu\times\nu}(D) = \sup\limits_{\lambda\in\Lambda} \{\mathcal{B}_\mu(A_\lambda) \wedge \mathcal{B}_\nu(B_\lambda)\}$ where $(A_\lambda \times B_\Lambda)_{\lambda\in\Lambda}$ is the family of all possible ways of writing D as $D = A_\lambda \times B_\lambda$, where $A_\lambda \in \mathcal{J}_\mu$, $B_\lambda \in \mathcal{J}_\nu$.*

Then, $\mathcal{B}_{\mu\times\nu}$ is a basis for the product topology on $\mu \times \nu$.

Proof. Let $(x, y) \in X \times Y$ and $\epsilon > 0$. Since $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$ are basis for the fuzzy generalized topologies $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$, there exists $A \in \mathcal{J}_\mu$ and $B \in \mathcal{J}_\nu$ such that $A(x) = \mu(x), B(y) = \nu(y)$ with $\mathcal{B}_\mu(A) \geq 1-\epsilon$ and $\mathcal{B}_\nu(B) \geq 1-\epsilon$. We choose $D = A\times B$, By hypothesis $\mathcal{B}_{\mu\times\nu}(D) \geq \mathcal{B}_\mu(A)\wedge\mathcal{B}_\nu(B)$. Then, $\mathcal{B}_{\mu\times\nu}(D) \geq 1 - \epsilon$. Hence, $\mathcal{B}_{\mu\times\nu}$ is a basis for a fuzzy generalized topology on $\mu \times \nu$ denoted by $\mathcal{T}$.

Let $\mathcal{T}_{\mu\times\nu}$ be the product topology on $\mu\times\nu$ and $\mathcal{B}^p_{\mu\times\nu}$ be the basis for $\mathcal{T}_{\mu\times\nu}$ as described in Definition 3.18. We will show that $\mathcal{T}_{\mu\times\nu}=\mathcal{T}$.

Let $D\in\mathcal{J}_{\mu\times\nu}$, then by Definition 3.8 $\mathcal{T}_{\mu\times\nu}(D)=\sup\limits_{\Lambda\in\Gamma}\left\{\sup\limits_{D_\lambda\in C_\Lambda}\mathcal{B}^p_{\mu\times\nu}(D_\lambda)\right\}$, where $(C_\Lambda)_{\Lambda\in\Gamma}$ is the family of all possible inner covers $C_\Lambda=(D_\lambda)_{\lambda\in\Lambda}$ of D. We denote $\mathcal{C}=(C_\Lambda)_{\Lambda\in\Gamma}$. Let $\mathcal{C}'$ be the family all possible inner covers $(D_\lambda)_{\lambda\in\Lambda}$ of D, i.e., for all $\lambda\in\Lambda, D_\lambda$ is of the form $A_\lambda\times B_\lambda$ for at least one $A_\lambda\in\mathcal{J}_\mu$ and one $B_\lambda\in\mathcal{J}_\nu$. Let $\mathcal{C}''$ be the complement of $\mathcal{C}'$ in $\mathcal{C}$.

If $\mathcal{C}'=\emptyset$, then for at least one $\lambda_0\in\lambda$, D_{λ_0} is not of the form $A\times B$ for any $A\in\mathcal{J}_\mu$ and $B\in\mathcal{J}_\nu$, hence $\mathcal{B}^p_{\mu\times\nu}(D_{\lambda_0})=0$ and therefore, $\sup\limits_{D_\lambda\in C_\Lambda}\left\{\mathcal{B}^p_{\mu\times\nu}(D_\lambda)\right\}=0=\sup\limits_{D_\lambda\in C_\Lambda}\{\mathcal{B}_{\mu\times\nu}(D_\lambda)\}$, hence $\mathcal{T}_{\mu\times\nu}(D)=\mathcal{T}(D)=0$. Now, we consider the case $\mathcal{C}'\neq\emptyset$,

$$\begin{aligned}
\mathcal{T}_{\mu\times\nu}(D)&=\sup_{\Lambda\in\mathcal{C}}\left\{\sup_{D_\lambda\in C_\Lambda}\mathcal{B}^p_{\mu\times\nu}(D_\lambda)\right\}\\
&=\sup_{\Lambda\in\mathcal{C}'}\left\{\sup_{D_\lambda\in C_\Lambda}\mathcal{B}^p_{\mu\times\nu}(D_\lambda)\right\}\\
&=\sup_{\Lambda\in\mathcal{C}'}\left\{\sup_{D_\lambda\in C_\Lambda}\left\{\sup_{D_\lambda=A_\lambda\times B_\Lambda}\{\sup\{\mathcal{T}_\mu(A_\lambda),\mathcal{T}_\nu(B_\lambda)\}\right\}\right\}\\
&\quad\text{(By Definition 3.18)}\\
&\geq\sup_{\Lambda\in\mathcal{C}'}\left\{\sup_{D_\lambda\in C_\Lambda}\left\{\sup_{D_\lambda=A_\lambda\times B_\Lambda}\{\sup\{\mathcal{B}_\mu(A_\lambda),\mathcal{B}_\nu(B_\lambda)\}\right\}\right\}\\
&\quad\text{(By Theorem 3.12)}\\
&=\sup_{\Lambda\in\mathcal{C}'}\left\{\sup_{D_\lambda\in C_\Lambda}\{\mathcal{B}_{\mu\times\nu}(D_\lambda)\}\right\}\quad\text{(By Hypthesis)}\\
&=\mathcal{T}(D)\quad\text{(By Definition 3.8).}
\end{aligned}$$

Therefore, $\mathcal{T}_{\mu\times\nu}\geq\mathcal{T}$. Conversely, let $D\in\mathcal{J}_{\mu\times\nu}$ and $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be as above. Let $C_\Lambda=(D_\lambda)_{\lambda\in\Lambda}$ be an inner cover for D. As above just take the case $\mathcal{C}'\neq\emptyset$. Now, let $C_\Lambda\in\mathcal{C}'$. Then, for all $\lambda\in\Lambda$, $D_\lambda=A\times B$ for at least one $A\in\mathcal{J}_\mu$ and one $\mathcal{B}\in\mathcal{J}_\nu$. Fix a $\lambda\in\Lambda$. Let $\mathcal{B}_\lambda$ denote the set of all pairs (A,B) such that $D_\lambda=A\times B$.

Let $(A,B)\in\mathcal{B}_\lambda$ and $\epsilon>0$. Since $\mathcal{B}_\mu,\mathcal{B}_\nu$ are basis for $\mathcal{T}_\mu,\mathcal{T}_\nu$ and by Theorem 3.12, for any $x\in X$ and $y\in Y$, there exists $A_x\in\mathcal{J}_\mu$ and $B_y\in\mathcal{J}_\nu$ such that $A_x(x)=A(x), A_x\leq A$ and $B_y(y)=B(y)$, $B_y\leq B$ with $\mathcal{B}_\mu(A_x)+\epsilon\geq\mathcal{T}_\mu(A)$ and $\mathcal{B}_\nu(B_y)+\epsilon\geq\mathcal{T}_\nu(B)$. Clearly $(A_x)_{x\in X}$ is an inner cover for A and $(B_y)_{y\in Y}$ is an inner cover for B. Then, the family$(A_x \times B_y)_{x\in X, y\in Y}$ is an inner cover for $A \times B = D_\lambda$. Thus, for any pair $(A, B) \in \mathcal{B}_\lambda$, we have an inner cover $(A_x \times B_y)_{x\in X, y\in Y}$ for D_λ such that $\mathcal{B}_\mu(A_x) + \epsilon \geq \mathcal{T}_\mu(A)$ and $\mathcal{B}_\nu(B_y) + \epsilon \geq \mathcal{T}_\nu(B)$, for all $x \in X, y \in X$. Therefore

$$\sup_{(x,y)\in X\times Y} \{\mathcal{B}_\mu(A_x) \wedge \mathcal{B}_\nu(B_y) + \epsilon\} \geq \mathcal{T}_\mu(A) \wedge \mathcal{T}_\nu(B), \tag{3.2}$$

Now,

$$\begin{aligned}
\mathcal{T}_{\mu\times\nu}(D) &= \sup_{\Lambda\in\mathcal{C}} \left\{ \sup_{D_\lambda\in C_\Lambda} \mathcal{B}^p_{\mu\times\nu}(D_\lambda) \right\} \\
&= \sup_{\Lambda\in\mathcal{C}'} \left\{ \sup_{D_\lambda\in C_\Lambda} \left\{ \sup_{(A,B)\in\mathcal{B}_\Lambda} \{\mathcal{T}_\mu(A) \wedge \mathcal{T}_\nu(B)\} \right\} \right\} \\
&\text{(By Definition 3.18)} \\
&\leq \sup_{\Lambda\in\mathcal{C}'} \left\{ \sup_{E_\lambda\in C_\Lambda} \left\{ \sup_{(A,B)\in\mathcal{B}_\Lambda} \left\{ \sup_{(x,y)\in X\times Y} \{\mathcal{B}_\mu(A_x) \wedge \mathcal{B}_\nu(B_y) + \epsilon\} \right\} \right\} \right\} \\
&\text{(By (3.2))} \\
&\leq \sup_{\Lambda\in\mathcal{C}'} \left\{ \sup_{D_\lambda\in C_\Lambda} \left\{ \sup_{(A,B)\in\mathcal{B}_\Lambda} \left\{ \sup_{(x,y)\in X\times Y} \{\mathcal{B}_{\mu\times\nu}(A_x \times B_y)\} \right\} \right\} \right\} + \epsilon \\
&\text{(By Hypthesis)} \\
&= \sup_{\Lambda\in\mathcal{C}'} \left\{ \sup_{D_\lambda\in C_\Lambda} \{\mathcal{T}(D_\lambda)\} \right\} + \epsilon \quad \text{(By Definition 3.8)} \\
&\leq \sup_{\Lambda\in\mathcal{C}} \left\{ \sup_{D_\lambda\in C_\Lambda} \{\mathcal{T}(D_\lambda)\} \right\} + \epsilon \\
&\leq \mathcal{T}(D) + \epsilon. \quad \text{(By Definition 3.8 and Theorem 3.10)}
\end{aligned}$$

Therefore, $\mathcal{T}_{\mu\times\nu}(E) \leq \mathcal{T}(E)$ and hence, $\mathcal{T}_{\mu\times\nu} \leq \mathcal{T}$. Thus, $\mathcal{T}_{\mu\times\nu} = \mathcal{T}$. □

4. Conclusion

In this paper, we have introduced the concepts of the basis of fuzzy generalized topological space in view of the definition of fuzzy generalized topology [10]. Our basis is defined as some functions from the family of all fuzzy subsets of a set X to $[0, 1]$ as opposed to a classical definition. Based on the proposed approach for basis, this research has developed new characterizations of fuzzy generalized topology, fuzzy generalized cotopology, fuzzy generalized continuity, weakly fuzzy generalized continuity and the product of fuzzy generalized topological spaces. We hope that their basis can be used in intuitionistic fuzzy topological space by changing hypothsis of their basis to characterize several results of the notion of intuitionistic fuzzy $eT_{1/2}$-spaces, intuitionistic fuzzy GEO-connectedness, intuitionistic fuzzy GEO-compactness cited in [21]. The concept of the categorical structure of the collection of soft topological spaces and the soft T_0-reflection in the category $SOFTOP$ has been discussed in [9]. Moreover, we hope to develop fuzzy generalized continuity on a well-chosen basis so that we can study their concepts.

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