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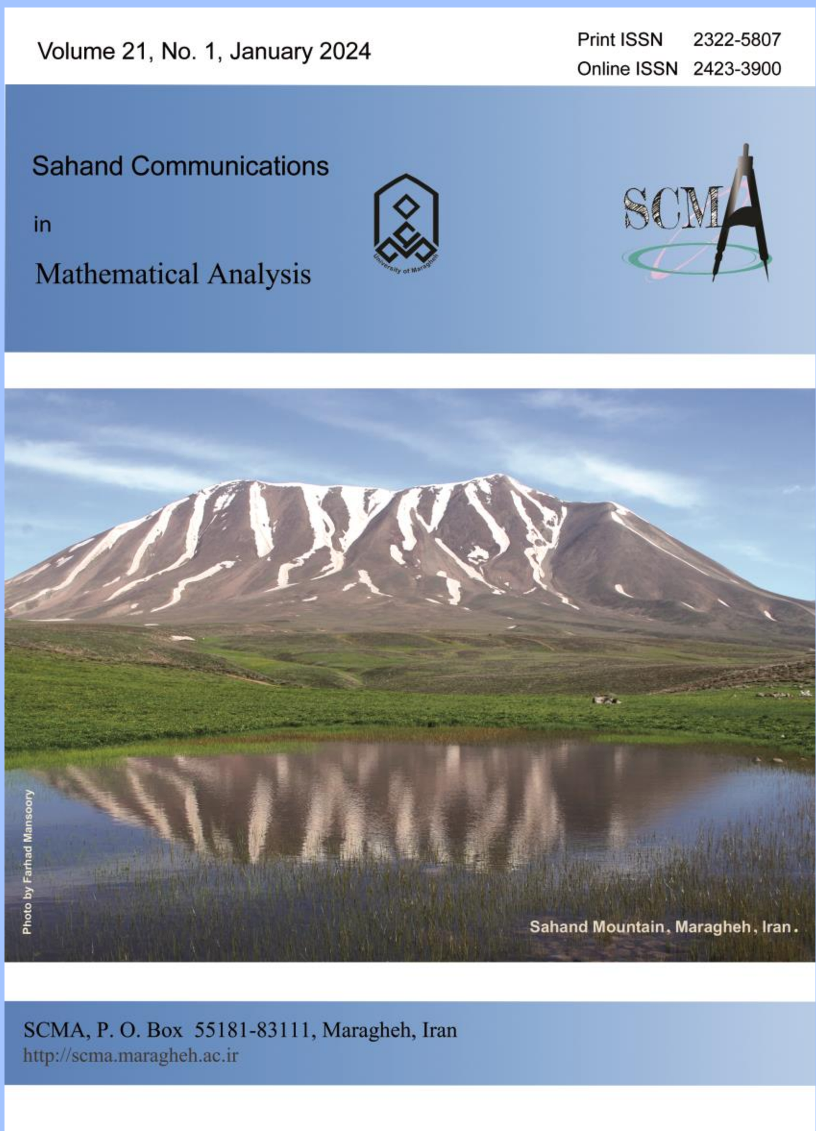

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Bi-Univalent Functions of Complex Order Defined by Hohlov Operator Associated with $(\mathcal{P}, \mathcal{Q})-$Lucas Polynomial

Elumalai Muthaiyan

Abstract. On this study, two new subclasses of the function class Ξ of bi-univalent functions of complex order defined in the open unit disc are introduced and investigated. These subclasses are connected to the Hohlov operator with $(\mathcal{P}, \mathcal{Q})-$Lucas polynomial and meet subordinate criteria. For functions in these new subclasses, we also get estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The results are also discussed as having a number of (old or new) repercussions.

1. Introduction and Definition

Let $\mathcal{A}$ denote the class of functions of the form:

$$s(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{\xi : \xi \in \mathbb{C} \quad \text{and} \quad |\xi| < 1\}.$$

Additionally, we'll use the symbol $\mathcal{S}$ to represent the class of all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$. The class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ are two notable and well-studied subclasses of the univalent function class $\mathcal{S}$. Every function $s \in \mathcal{S}$ has an inverse s^{-1}, which is defined by

$$s^{-1}(s(\xi)) = \xi, \quad (\xi \in \mathbb{U})$$

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and

$$s(s^{-1}(\omega)) = \omega, \quad \left(|\omega| < r_0(s);\ r_0(s) \geq \frac{1}{4}\right)$$

where

$$\begin{aligned}(1.2)\quad t(\omega) &= s^{-1}(\omega)\\ &= \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots.\end{aligned}$$

A function $s \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $s(\xi)$ and $s^{-1}(\xi)$ are univalent in $\mathbb{U}$. Let Ξ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Note that the functions

$$s_1(\xi) = \frac{\xi}{1-\xi}, \qquad s_2(\xi) = \frac{1}{2}\log\frac{1+\xi}{1-\xi}, \qquad s_3(\xi) = -\log(1-\xi)$$

with their corresponding inverses

$$s_1^{-1}(\omega) = \frac{\omega}{1-\omega}, \qquad s_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1}, \qquad s_3^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}}$$

are components of Ξ. This topic is covered in great detail in the ground-breaking work by Srivastava et al. [30], who recently resurrected the study of analytic and bi-univalent functions. Many successors to Srivastava et al. [30] were produced after it.

An analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$ supporting $s(\xi) = t(\omega(\xi))$, then s is subordinate to an analytic function t, written $s(\xi) \prec t(\xi)$. Recently, Ma and Minda combined different subclasses of starlike and convex functions for which the quantity $\frac{\xi s'(\xi)}{s(\xi)}$ or $1 + \frac{\xi s''(\xi)}{s'(\xi)}$ is subordinate to a more general superordinate function. They examined an analytic function ϕ with a positive real portion in the unit disc for this persistence $\mathbb{U}$, $\phi(0) = 1$. In addition, ϕ maps $\mathbb{U}$ onto an area that is symmetric with respect to the real axis and starlike with respect to 1. Functions meeting the subordination $\frac{\xi s'(\xi)}{s(\xi)} \prec \phi(\xi)$ fall within the category of Ma-Minda starlike functions.

The convolution or Hadamard product of two functions $s, h \in \mathcal{A}$ is denoted by $s * h$ and is defined as

$$(1.3) \qquad (s * h)(\xi) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n,$$

where (1.1) gives the value of $s(\xi)$ and $h(\xi) = \xi + \sum_{n=2}^{\infty} b_n\xi^n$. Dziok and Srivastava introduced and carefully explored the Dziok-Srivastava linear operator involving the generalised hypergeometric function in terms of the Hadamard product (or convolution) before being followed by numerous other authors. In this study, we recall the well-known convolution operator $\mathcal{J}_{a,b,c}$ attributed to Hohlov [16, 17], which is undoubtedly a highly specialized instance of the widely (and in-depthly) investigated Dziok-Srivastava operator as well as the much more general Srivastava Wright operator [31] (also see [19]).

For the complex parameters a, b and c with $c \neq 0, -1, -2, -3, \dots,$ the Gaussian hypergeometric function ${}_2F_1(a, b, c; \xi)$ is defined as

$$\begin{aligned} {}_2F_1(a, b, c; \xi) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\xi^n}{n!} \\ &= 1 + \sum_{n=0}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{\xi^{n-1}}{(n-1)!} \qquad (\xi \in \mathbb{U}) \end{aligned} \tag{1.4}$$

where $(a)_n$ is the Pochhammer symbol (or the shifted factorial) defined as follows:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.5}$$

For the positive real numbers a, b and c with $c \neq 0, -1, -2, -3, \dots.$ In order to present the well-known convolution operator $\mathcal{J}_{a,b,c}$ Hohlov introduced the Gaussian hypergeometric function provided by (1.4) as follows:

$$\begin{aligned} \mathcal{J}_{a,b,c} s(\xi) &= \xi \ {}_2F_1(a, b, c; \xi) * s(\xi) \\ &= \xi + \sum_{n=2}^{\infty} \kappa_n a_n \xi^n, \quad (\xi \in \mathbb{U}) \end{aligned} \tag{1.6}$$

where

$$\kappa_n = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!}. \tag{1.7}$$

Hohlov described some intriguing geometrical characteristics by the operator $\mathcal{J}_{a,b,c}$. The majority of the well-known linear integral or differential operators are included as special cases in the three-parameter family of operators $\mathcal{J}_{a,b,c}$. In instance, $\mathcal{J}_{a,b,c}$ reduces to the Carlson-Shaffer operator if $b = 1$ in (1.6). Similar to this, it is clear that the Bernardi-Libera-Livingston operator and the Ruscheweyh derivative operator are both generalizations of the Hohlov operator $\mathcal{J}_{a,b,c}$.

Interest in the study of the bi-univalent function class Ξ has recently increased, and non-sharp coefficient estimates have been found for the first two coefficients $|a_2|$ and $|a_3|$ of (1.1). However, the coefficient issue for each of the subsequent Taylor-Maclaurin coefficients is as follows:

$$|a_n|, \quad (n \in \mathbb{N} \ \{1, 2\}; \quad \mathbb{N} := \{1, 2, 3, \dots\}).$$

It's still an open issue. The first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ have non-sharp estimates, according to recent research that has introduced and examined a number of intriguing subclasses of the bi-univalent function class Ξ.

Definition 1.1. Let $\mathcal{P}(x)$ and $\mathcal{Q}(x)$ be polynomials with real coefficients. The $(\mathcal{P}, \mathcal{Q})-$ Lucas polynomials $L_{\mathcal{P},\mathcal{Q},n}(x)$ are defined by the reccurence relation

$$\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x) = \mathcal{P}(x)\mathcal{L}_{\mathcal{P},\mathcal{Q},n-1}(x) + \mathcal{Q}(x)\mathcal{L}_{\mathcal{P},\mathcal{Q},n-2}(x), \quad (n \geq 2) \tag{1.8}$$

from which the first few Lucas polynomials cab be found as follows:

$$\begin{aligned} \mathcal{L}_{\mathcal{P},\mathcal{Q},0}(x) &= 2 \\ \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x) &= \mathcal{P}(x) \\ \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x) &= \mathcal{P}^2(x) + 2\mathcal{Q}(x) \\ \mathcal{L}_{\mathcal{P},\mathcal{Q},3}(x) &= \mathcal{P}^3(x) + 3\mathcal{P}(x)\mathcal{Q}(x). \end{aligned} \tag{1.9}$$

Definition 1.2. Let $\mathcal{G}_{\{\mathcal{L}_n(x)\}}(\xi)$ be the generating function of the $(\mathcal{P}, \mathcal{Q})-$ Lucas polynomial sequence $\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)$. Then

$$\begin{aligned} \mathcal{G}_{\{\mathcal{L}_n(x)\}}(\xi) &= \sum_{n=0}^{\infty} \mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\xi^n \\ &= \frac{2 - \mathcal{P}(x)\xi}{1 - \mathcal{P}(x)\xi - \mathcal{Q}(x)\xi^2}. \end{aligned} \tag{1.10}$$

Note that for particular values of $\mathcal{P}$ and $\mathcal{Q}$, the $(\mathcal{P}, \mathcal{Q})$-polynomial $\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)$ leads to various polynomials, among those, we list few cases here (see, [23] for more details, also [1, 18]):

- For $\mathcal{P}(x) = x$ and $\mathcal{Q}(x) = 1$, we obtain the Lucas polynomials $\mathcal{L}_n(x)$.
- For $\mathcal{P}(x) = 2x$ and $\mathcal{Q}(x) = 1$, we attain the Pell-Lucas polynomials $\mathcal{Q}_n(x)$.
- For $\mathcal{P}(x) = 1$ and $\mathcal{Q}(x) = 2x$, we attain the Jacobsthal-Lucas polynomials $j_n(x)$.
- For $\mathcal{P}(x) = 3x$ and $\mathcal{Q}(x) = -2$, we attain the Fermat-Lucas polynomials $f_n(x)$.
- For $\mathcal{P}(x) = 2x$ and $\mathcal{Q}(x) = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind.

A study on bi-univalent functions by [3, 11, 13, 15, 22] as well as numerous recent works on the Fekete-Szegö functional and other coefficient estimates (see [2, 5, 6, 8, 25]), served as the inspiration for the current paper. In this paper we introduce new subclasses of the function class Ξ of complex order $\beta \in \mathbb{C} \setminus \{0\}$ other relevant classes are taken into account and connections to previously reported results are made.

Definition 1.3. A function $s \in \Xi$ given by (1.1) is said to be in the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta, \delta, n; x)$ if the following conditions are satisfied:

$$1 + \frac{1}{\beta}\left(\frac{\xi(\mathcal{J}_{a,b,c}s(\xi))'}{(1-\delta)\xi + \delta\mathcal{J}_{a,b,c}s(\xi)} - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi) - 1 \tag{1.11}$$

and

$$1 + \frac{1}{\beta}\left(\frac{w(\mathcal{J}_{a,b,c}s(\omega))'}{(1-\delta)\xi + \delta\mathcal{J}_{a,b,c}s(\omega)} - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega) - 1 \tag{1.12}$$

where the function t is given by (1.2).

On specializing the parameters δ and a, b, c one can state the various new subclasses of Ξ as illustrated in the following examples.

Example 1.4. A function $s \in \Xi$, given by (1.1) is said to belong to the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta, n; x)$ if the following criteria are met for $\delta = 1$ and $\beta \in \mathbb{C} \setminus \{0\}$:

$$1 + \frac{1}{\beta}\left(\frac{\xi(\mathcal{J}_{a,b,c}s(\xi))'}{\mathcal{J}_{a,b,c}s(\xi)} - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi) - 1 \tag{1.13}$$

and

$$1 + \frac{1}{\beta}\left(\frac{w(\mathcal{J}_{a,b,c}s(\omega))'}{\mathcal{J}_{a,b,c}s(\omega)} - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega) - 1 \tag{1.14}$$

where $\xi, \omega \in \mathbb{U}$ and the function t is given by (1.2).

Example 1.5. A function $s \in \Xi$, given by (1.1) is said to belong to the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta, n; x)$ if the following criteria are met for $\delta = 0$ and $\beta \in \mathbb{C} \setminus \{0\}$:

$$1 + \frac{1}{\beta}\left((\mathcal{J}_{a,b,c}s(\xi))' - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi) - 1 \tag{1.15}$$

and

$$1 + \frac{1}{\beta}\left((\mathcal{J}_{a,b,c}s(\omega))' - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega) - 1 \tag{1.16}$$

where $\xi, \omega \in \mathbb{U}$ and the function t is given by (1.2).

Example 1.6. In the case of $\delta = 1$ and $\beta \in \mathbb{C} \setminus \{0\}$. If the following criteria are met, a function $s \in \Xi$, given by (1.1), is considered to belong to the class $\mathcal{M}_{\Xi}^{*}(\beta, n; x)$:

$$1 + \frac{1}{\beta}\left(\frac{\xi s'(\xi)}{s(\xi)} - 1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi) - 1 \tag{1.17}$$

and

$$1+\frac{1}{\beta}\left(\frac{\xi s'(\omega)}{s(\omega)}-1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega)-1 \tag{1.18}$$

where $\xi, \omega \in \mathbb{U}$ and the function t is given by (1.2).

Example 1.7. For $\delta = 0$ and $\beta \in \mathbb{C} \setminus \{0\}$. If the following criteria are met, a function $s \in \Xi$, given by (1.1), is considered to belong to the class $\mathcal{M}^{*}_{\Xi}(\beta, n; x)$:

$$1+\frac{1}{\beta}\left(s'(\xi)-1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi)-1 \tag{1.19}$$

and

$$1+\frac{1}{\beta}\left(s'(\omega)-1\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega)-1 \tag{1.20}$$

where $\xi, \omega \in \mathbb{U}$ and the function t is given by (1.2).

By using the methods previously employed by Deniz in [11], we derive estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above described subclasses $\mathcal{M}^{a,b;c}_{\Xi}(\beta, \delta, n; x)$ of the function class Ξ.

2. Coefficient Bounds for the Function Class $\mathcal{M}^{a,b;c}_{\Xi}(\beta, \delta, n; x)$

Theorem 2.1. *Let s be given by (1.1) and in the class $\mathcal{M}^{a,b;c}_{\Xi}(\beta, \delta, n; x)$. Then*

$$|a_2| \leq \frac{\beta^2|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{\begin{array}{r}|\{(2-\delta)[(1-2\beta)\delta-2]\kappa_2^2+2\beta(3-\delta)\kappa_3\}\mathcal{P}^2(x)\\ -2(2-\delta)^2\kappa_2^2\mathcal{Q}(x)|\end{array}}} \tag{2.1}$$

and

$$|a_3| \leq \frac{\beta^2\mathcal{P}^2(x)}{(2-\delta)^2\kappa_2^2}+\frac{\beta|\mathcal{P}(x)|}{2(3-\delta)\kappa_3}. \tag{2.2}$$

Proof. Let $s \in \mathcal{M}^{a,b;c}_{\Xi}(\beta, \delta, n; x)$. Then, from Definition 1.2, for some analytic functions, Ω, Λ such that $\Omega(0) = \Lambda(0) = 0$ and $|\Omega(\xi)| < 1$, $|\Lambda(\omega)| < 1$ for all $\xi, \omega \in \mathbb{U}$, we can write

$$1+\frac{1}{\beta}\left(\frac{\xi(\mathcal{J}_{a,b,c}s(\xi))'}{(1-\delta)\xi+\delta\mathcal{J}_{a,b,c}s(\xi)}-1\right) = \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\Omega(\xi))-1 \tag{2.3}$$

$$1+\frac{1}{\beta}\left(\frac{\omega(\mathcal{J}_{a,b,c}s(\omega))'}{(1-\delta)\omega+\delta\mathcal{J}_{a,b,c}s(\omega)}-1\right) = \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\Lambda(\omega))-1 \tag{2.4}$$

or equivalently

$$1+\frac{1}{\beta}\left(\frac{\xi(\mathcal{J}_{a,b,c}s(\xi))'}{(1-\delta)\xi+\delta\mathcal{J}_{a,b,c}s(\xi)}-1\right) \tag{2.5}$$

$$= -1 + \mathcal{L}_{\mathcal{P},\mathcal{Q},0}(x) + \mathcal{L}_{\mathcal{P},\mathcal{Q},1}\Omega(\xi) + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}\Omega^2(\xi) + \cdots$$

$$1 + \frac{1}{\beta}\left(\frac{\omega(\mathcal{J}_{a,b,c}s(\omega))'}{(1-\delta)w + \delta\mathcal{J}_{a,b,c}s(\omega)} - 1\right) \tag{2.6}$$

$$= -1 + \mathcal{L}_{\mathcal{P},\mathcal{Q},0}(x) + \mathcal{L}_{\mathcal{P},\mathcal{Q},1}\Lambda(\omega) + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}\Lambda^2(\omega) + \cdots.$$

From equalities (2.5) and (2.6)

$$1 + \frac{1}{\beta}\left(\frac{\xi(\mathcal{J}_{a,b,c}s(\xi))'}{(1-\delta)\xi + \delta\mathcal{J}_{a,b,c}s(\xi)} - 1\right) \tag{2.7}$$

$$= 1 + \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_1\xi + [\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_2 + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)l_1^2]\xi^2 + \cdots$$

$$1 + \frac{1}{\beta}\left(\frac{\omega(\mathcal{J}_{a,b,c}s(\omega))'}{(1-\delta)\omega + \delta\mathcal{J}_{a,b,c}s(\omega)} - 1\right) \tag{2.8}$$

$$= 1 + \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_1\omega + [\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_2 + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)r_1^2]\omega^2 + \cdots.$$

It is already known that if for $\xi, \omega \in \mathbb{U}$,

$$\Omega(\xi) = \left|\sum_{i=1}^{n} l_i\xi^i\right| < 1$$

and

$$\Lambda(\omega) = \left|\sum_{i=1}^{n} r_i\omega^i\right| < 1$$

then

$$\Omega(\xi) = |l_i| < 1$$

and

$$\Lambda(\omega) = |r_i| < 1$$

where $i \in \mathbb{N}$. Thus, comparing the corresponding coefficients in (2.7) and (2.8), we get

$$\frac{2-\delta}{\beta}\kappa_2 a_2 = \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_1 \tag{2.9}$$

$$\frac{\delta^2 - 2\delta}{\beta}\kappa_2^2 a_2^2 + \frac{3-\delta}{\beta}\kappa_3 a_3 = \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_2 + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)l_1^2 \tag{2.10}$$

$$-\frac{2-\delta}{\beta}\kappa_2 a_2 = \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_1 \tag{2.11}$$

$$\frac{\delta^2 - 2\delta}{\beta}\kappa_2^2 a_2^2 + \frac{3-\delta}{\beta}\kappa_3(2a_2^2 - a_3) = \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_2 + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)r_1^2. \tag{2.12}$$

From (2.9) and (2.11),

$$a_2 = \frac{\beta l_1}{(2-\delta)\kappa_2} = \frac{-\beta r_1}{(2-\delta)\kappa_2} \tag{2.13}$$

$$l_1 = -r_1 \tag{2.14}$$

$$(2-\delta)^2\kappa_2^2 a_2^2 = \beta^2 \mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)(l_1^2 + r_1^2) \tag{2.15}$$

adding (2.10) and (2.12),

$$2\left(\frac{\delta^2-2\delta}{\beta}\kappa_2^2 + \frac{3-\delta}{\beta}\kappa_3\right)a_2^2 \tag{2.16}$$
$$= \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)(l_2+r_2) + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(l_1^2+r_1^2).$$

By using (2.15) and (2.16), we have

$$[2\beta\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)[(\delta^2-2\delta)\kappa_2^2 + (3-\delta)\kappa_3] - \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(2-\delta)^2\kappa_2^2]a_2^2 \tag{2.17}$$
$$= \beta^2\mathcal{L}^3_{\mathcal{P},\mathcal{Q},1}(x)(l_2+r_2)$$

$$a_2^2 = \frac{\beta^2\mathcal{L}^3_{\mathcal{P},\mathcal{Q},1}(x)(l_2+r_2)}{[2\beta\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)[(\delta^2-2\delta)\kappa_2^2 + (3-\delta)\kappa_3] - \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(2-\delta)^2\kappa_2^2]}$$

which gives

$$|a_2| \le \frac{\beta^2|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{\begin{array}{r}|\{(2-\delta)[(1-2\beta)\delta-2]\kappa_2^2 + 2\beta(3-\delta)\kappa_3\}\mathcal{P}^2(x)\\ -2(2-\delta)^2\kappa_2^2\mathcal{Q}(x)|\end{array}}}$$

also, by subtracting (2.12) from (2.10), we get

$$\frac{2(3-\delta)}{\beta}\kappa_3 a_3 - \frac{2(3-\delta)}{\beta}\kappa_3 a_2^2 = \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)(l_2-r_2). \tag{2.18}$$

Then, by using (2.14) and (2.15) in (2.18), we have

$$a_3 = \frac{\beta^2\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)(l_1^2+r_1)^2}{(2-\delta)^2\kappa_2^2} + \frac{\beta\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)(l_2-r_2)}{2(3-\delta)\kappa_3},$$

and by the help of (1.9), we conclude that

$$|a_3| \le \frac{\beta^2\mathcal{P}^2(x)}{(2-\delta)^2\kappa_2^2} + \frac{\beta|\mathcal{P}(x)|}{2(3-\delta)\kappa_3}. \qquad \square$$

Fixing $\delta = 1$ in Theorem 2.1, we have the following:

Corollary 2.2. *Let s be given by (1.1) and in the class $\mathcal{M}^{a,b;c}_{\Xi}(\beta, n; x)$. Then*

$$|a_2| \le \frac{\beta^2|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{|\{[(1-2\beta)-2]\kappa_2^2 + 4\beta\kappa_3\}\mathcal{P}^2(x) - 2\kappa_2^2\mathcal{Q}(x)|}} \tag{2.19}$$

and

$$|a_3| \le \frac{\beta^2\mathcal{P}^2(x)}{\kappa_2^2} + \frac{\beta|\mathcal{P}(x)|}{4\kappa_3}. \tag{2.20}$$

Assuming that $a = c$ and $b = 1$, in Corollary 2.2 we obtain the following:

Corollary 2.3. *Let s be given by (1.1) and in the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta,n;x)$. Then*

$$|a_2| \leq \frac{\beta^2|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{|\{[(1-2\beta)-2]+4\beta\}\mathcal{P}^2(x)-2\mathcal{Q}(x)|}} \tag{2.21}$$

and

$$|a_3| \leq \beta^2\mathcal{P}^2(x)+\frac{\beta|\mathcal{P}(x)|}{4}. \tag{2.22}$$

Fixing $\delta=0$ in Theorem 2.1, we have the following:

Corollary 2.4. *Let s be given by (1.1) and in the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta,n;x)$. Then*

$$|a_2| \leq \frac{\beta^2|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{|\{-4\kappa_2^2+6\beta\kappa_3\}\mathcal{P}^2(x)-8\kappa_2^2\mathcal{Q}(x)|}} \tag{2.23}$$

and

$$|a_3| \leq \frac{\beta^2\mathcal{P}^2(x)}{4\kappa_2^2}+\frac{\beta|\mathcal{P}(x)|}{6\kappa_3}. \tag{2.24}$$

Using the above corollaries $a=c$ and $b=1$, we obtain the following:

Corollary 2.5. *Let s be given by (1.1) and in the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta,n;x)$. Then*

$$|a_2| \leq \frac{\beta^2|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{|\{6\beta-4\}\mathcal{P}^2(x)-8\mathcal{Q}(x)|}} \tag{2.25}$$

and

$$|a_3| \leq \frac{\beta^2\mathcal{P}^2(x)}{4}+\frac{\beta|\mathcal{P}(x)|}{6}. \tag{2.26}$$

We demonstrate Fekete-Szegö inequalities for the functions
$s\in\mathcal{M}_{\Xi}^{a,b;c}(\beta,\delta,n;x)$ to Zaprawa [37].

Theorem 2.6. *Let s given by (1.1) belongs to the class $\mathcal{M}_{\Xi}^{a,b;c}(\beta,\delta,n;x)$. Then,*

$$|a_3-\tau a_2^2| \leq \begin{cases} \dfrac{|\mathcal{P}(x)|}{(3-\delta)\kappa_3}, & 0\leq|\mathcal{T}(\tau;x)|<\frac{\beta}{2(3-\delta)\kappa_3} \\ 2|\mathcal{P}(x)||\mathcal{T}(\tau;x)|, & |\mathcal{T}(\tau;x)|\geq\frac{\beta}{2(3-\delta)\kappa_3} \end{cases} \tag{2.27}$$

where

$$\mathcal{T}(\tau;x)=\frac{(1-\tau)\beta^2\mathcal{L}_{\mathcal{P},\mathcal{Q},1}^2(x)}{[2\beta\mathcal{L}_{\mathcal{P},\mathcal{Q},1}^2(x)[(\delta^2-2\delta)\kappa_2^2+(3-\delta)\kappa_3]-\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(2-\delta)^2\kappa_2^2]}.$$

Proof. From equations (2.17) and (2.18), we get

$$\begin{aligned} a_3 - \tau a_2^2 &= \frac{\beta \mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)(l_2 - r_2)}{2(3-\delta)\kappa_3} + (1-\tau) a_2^2 \\ &= \frac{\beta \mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)(l_2 - r_2)}{2(3-\delta)\kappa_3} \\ &\quad + \frac{(1-\tau)\beta^2 \mathcal{L}^3_{\mathcal{P},\mathcal{Q},1}(x)(l_2 + r_2)}{[2\beta \mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)[(\delta^2 - 2\delta)\kappa_2^2 + (3-\delta)\kappa_3] - \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(2-\delta)^2\kappa_2^2]} \\ &= \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)\left[\left(\mathcal{T}(\tau;x) + \frac{\beta}{2(3-\delta)\kappa_3}\right) l_2 + \left(\mathcal{T}(\tau;x) - \frac{\beta}{2(3-\delta)\kappa_3}\right) l_2\right] \end{aligned}$$

where

$$\mathcal{T}(\tau;x) = \frac{(1-\tau)\beta^2 \mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)}{[2\beta \mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)[(\delta^2 - 2\delta)\kappa_2^2 + (3-\delta)\kappa_3] - \mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(2-\delta)^2\kappa_2^2]}.$$

□

One may easily assert the following by assuming that $\tau = 1$ in above Theorem 2.1.

Remark 2.7. Let the function s be assumed by $s \in \mathcal{M}_{\Xi}^{a,b;c}(\beta, \delta, n; x)$. Then

$$|a_3 - a_2^2| \leq \frac{|\mathcal{P}(x)|}{(3-\delta)\kappa_3}$$

3. Subclass of Bi-Univalent Function $\mathcal{V}_{\Xi}^{a,b;c}(\varsigma, n; x)$

Obradovic et al. provided several requirements for univalence in the cited work, expressing them mathematically as $\mathcal{R}(s'(\xi)) > 0$, for the linear combinations

$$\varsigma\left(1 + \frac{\xi s''(\xi)}{s'(\xi)}\right) + (1-\varsigma)\frac{1}{s'(\xi)} > 0, \quad (\varsigma \geq 1,\ \xi \in \mathbb{U}).$$

Recently, Lashin in [20] introduced and explored the new subclasses of bi-univalent function based on the aforementioned definitions.

Definition 3.1. If a function $s \in \Xi$ given by (1.1) satisfies the following criteria, it is said to belong to the class $\mathcal{V}_{\Xi}^{a,b;c}(\varsigma, n; x)$:

$$\varsigma\left(1 + \frac{\xi(\mathcal{J}_{a,b,c}s(\xi))''}{(\mathcal{J}_{a,b,c}s(\xi))'}\right) + (1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\xi))'} \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi) - 1 \tag{3.1}$$

and

$$\varsigma\left(1 + \frac{\omega(\mathcal{J}_{a,b,c}s(\omega))''}{(\mathcal{J}_{a,b,c}s(\omega))'}\right) + (1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\omega))'} \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega) - 1 \tag{3.2}$$

where $\varsigma \geq 1, \xi, \omega \in \mathbb{U}$ and the function t is given by (1.2).

Remark 3.2. If a function $s \in \Xi$ provided by (1.1) satisfies the following criteria, it is said to belong to the class $\mathcal{V}_{\Xi}^{a,b;c}(1,n;x) = \mathcal{R}_{\Xi}^{a,b;c}(n;x)$:

$$\left(1 + \frac{\xi(\mathcal{J}_{a,b,c}s(\xi))''}{(\mathcal{J}_{a,b,c}s(\xi))'}\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\xi) - 1$$

and

$$\left(1 + \frac{\omega(\mathcal{J}_{a,b,c}s(\omega))''}{(\mathcal{J}_{a,b,c}s(\omega))'}\right) \prec \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\omega) - 1$$

where $\xi, \omega \in \mathbb{U}$ and the function t is given by (1.2).

Theorem 3.3. *Let s be given by (1.1) and $s \in \mathcal{V}_{\Xi}^{a,b;c}(\varsigma,n;x)$, $\varsigma \geq 1$. Then*

$$|a_2| \leq \min\begin{cases} \frac{|\mathcal{P}(x)|}{2(2\varsigma-1)\kappa_2}, \\ \frac{\sqrt{2}|\mathcal{P}(x)|\sqrt{|\mathcal{P}(x)|}}{\sqrt{|([2(1+\varsigma)-8(2\varsigma-1)^2]\mathcal{P}(x)^2-16\mathcal{Q}(x)(2\varsigma-1)^2)\kappa_2^2|}} \end{cases} \tag{3.3}$$

and

$$|a_3| \leq \min\begin{cases} \frac{|\mathcal{P}(x)|}{3(3\varsigma-1)\kappa_3} + \frac{|\mathcal{P}^2(x)|}{4(2\varsigma-1)^2\kappa_2^2}, \\ \frac{|\mathcal{P}(x)|}{3(3\varsigma-1)\kappa_3} + \frac{2\mathcal{P}^3(x)}{|([2(1+\varsigma)-8(2\varsigma-1)^2]\mathcal{P}(x)^2-16\mathcal{Q}(x)(2\varsigma-1)^2)\kappa_2^2|}. \end{cases} \tag{3.4}$$

Proof. From (3.1) and (3.2), it is evident that

$$\varsigma\left(1 + \frac{\xi(\mathcal{J}_{a,b,c}s(\xi))''}{(\mathcal{J}_{a,b,c}s(\xi))'}\right) + (1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\xi))'} = \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\Omega(\xi)) - 1 \tag{3.5}$$

and

$$\varsigma\left(1 + \frac{\omega(\mathcal{J}_{a,b,c}s(\omega))''}{(\mathcal{J}_{a,b,c}s(\omega))'}\right) + (1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\omega))'} = \mathcal{G}_{\{\mathcal{L}_{\mathcal{P},\mathcal{Q},n}(x)\}}(\Lambda(\omega)) - 1 \tag{3.6}$$

or equivalently

$$\varsigma\left(1 + \frac{\xi(\mathcal{J}_{a,b,c}s(\xi))''}{(\mathcal{J}_{a,b,c}s(\xi))'}\right) + (1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\xi))'} = -1 + \mathcal{L}_{\mathcal{P},\mathcal{Q},0}(x) + \mathcal{L}_{\mathcal{P},\mathcal{Q},1}\Omega(\xi) + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}\Omega^2(\xi) + \cdots \tag{3.7}$$

$$\varsigma\left(1 + \frac{\omega(\mathcal{J}_{a,b,c}s(\omega))''}{(\mathcal{J}_{a,b,c}s(\omega))'}\right) + (1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\omega))'} = -1 + \mathcal{L}_{\mathcal{P},\mathcal{Q},0}(x) + \mathcal{L}_{\mathcal{P},\mathcal{Q},1}\Lambda(\omega) + \mathcal{L}_{\mathcal{P},\mathcal{Q},2}\Lambda^2(\omega) + \cdots \tag{3.8}$$

from equalities (3.7) and (3.8)

$$(3.9)\qquad \varsigma\left(1+\frac{\omega(\mathcal{J}_{a,b,c}s(\omega))''}{(\mathcal{J}_{a,b,c}s(\omega))'}\right)+(1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\omega))'} = 1+\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_1\xi+[\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_2+\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)l_1^2]\xi^2+\cdots$$

$$(3.10)\qquad \varsigma\left(1+\frac{\omega(\mathcal{J}_{a,b,c}s(\omega))''}{(\mathcal{J}_{a,b,c}s(\omega))'}\right)+(1-\varsigma)\frac{1}{(\mathcal{J}_{a,b,c}s(\omega))'} = 1+\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_1\omega+[\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_2+\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)r_1^2]\omega^2+\cdots$$

it is already known that if for $\xi,\omega\in U$.

Consequently, we obtain by comparing the equivalent coefficients in (3.9) and (3.10)

$$(3.11)\qquad 2(2\varsigma-1)\kappa_2a_2=\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_1$$

$$(3.12)\qquad 3(3\varsigma-1)\kappa_3a_3+4(1-2\varsigma)\kappa_2^2a_2^2=\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)l_2+\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)l_1^2$$

$$(3.13)\qquad -2(2\varsigma-1)\kappa_2a_2=\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_1$$

$$(3.14)\qquad 2(5\varsigma-1)\kappa_2a_2-3(3\varsigma-1)\kappa_3a_3=\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)r_2+\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)r_1^2.$$

From (3.11) and (3.13),

$$(3.15)\qquad l_1=-r_1$$

from (3.11) by using (1.9)

$$(3.16)\qquad |a_2|\le\frac{\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)}{2(2\varsigma-1)\kappa_2}\le\frac{|\mathcal{P}(x)|}{2(2\varsigma-1)\kappa_2}.$$

Also

$$8(2\varsigma-1)^2\kappa_2^2a_2^2=\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)(l_1^2+r_1^2)$$

and

$$(3.17)\qquad a_2^2=\frac{\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)(l_1^2+r_1^2)}{8(2\varsigma-1)^2\kappa_2^2}.$$

Thus by (1.9), we get

$$(3.18)\qquad |a_2|\le\frac{\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)}{2(2\varsigma-1)\kappa_2}=\frac{|\mathcal{P}(x)|}{2(2\varsigma-1)\kappa_2}.$$

Now from (3.12), (3.14) and using (3.17), we obtain

$$(3.19)\qquad [4(1-2\varsigma)\kappa_2^2+2(5\varsigma-1)\kappa_2^2]a_2^2 = \mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)(l_2+r_2)+\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)(l_1^2+r_1^2).$$

Thus, by (3.19) we obtain

$$a_2^2=\frac{\mathcal{L}^3_{\mathcal{P},\mathcal{Q},1}(x)(l_2+r_2)}{[\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)2(1+\varsigma)-\mathcal{L}_{\mathcal{P},\mathcal{Q},2}(x)8(2\varsigma-1)^2]\kappa_2^2}$$

$$|a_2^2| = \frac{2|\mathcal{P}^3(x)|}{|[\mathcal{P}^2(x)2(1+\varsigma) - (\mathcal{P}^2(x) + 2\mathcal{Q}(x))8(2\varsigma-1)^2]\kappa_2^2|}$$

$$|a_2| \leq \frac{\sqrt{2}|\mathcal{P}(x)||\sqrt{\mathcal{P}(x)}|}{\sqrt{|[\mathcal{P}^2(x)2(1+\varsigma) - (\mathcal{P}^2(x) + 2\mathcal{Q}(x))8(2\varsigma-1)^2]\kappa_2^2|}}.$$

From (3.12),(3.14) and using (3.15), we get

$$a_3 = \frac{\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)(l_2 - r_2)}{6(3\varsigma-1)\kappa_3} + a_2^2. \tag{3.20}$$

Then taking modulus, we get

$$|a_3| \leq \frac{\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)}{3(3\varsigma-1)\kappa_3} + |a_2^2| \tag{3.21}$$

using (3.16) and (3.18), we get

$$\begin{aligned} |a_3| &\leq \frac{\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)}{3(3\varsigma-1)\kappa_3} + \frac{\mathcal{L}^2_{\mathcal{P},\mathcal{Q},1}(x)}{4(2\varsigma-1)^2\kappa_2^2} \\ &= \frac{|\mathcal{P}(x)|}{3(3\varsigma-1)\kappa_3} + \frac{\mathcal{P}^2(x)}{4(2\varsigma-1)^2\kappa_2^2}. \end{aligned} \tag{3.22}$$

Now by using (3.19) in (3.21)

$$\begin{aligned} |a_3| &\leq \frac{\mathcal{L}_{\mathcal{P},\mathcal{Q},1}(x)}{3(3\varsigma-1)\kappa_3} + |a_2^2| \\ &= \frac{|\mathcal{P}(x)|}{3(3\varsigma-1)\kappa_3} \\ &\quad + \frac{2\mathcal{P}^3(x)}{|\left([2(1+\varsigma) - 8(2\varsigma-1)^2]\mathcal{P}(x)^2 - 16\mathcal{Q}(x)(2\varsigma-1)^2\right)\kappa_2^2|}. \end{aligned}$$

□

Due to Zaprawa [37], we prove Fekete-Szegö inequalities [12] for functions $s \in \mathcal{V}_{\Xi}^{a,b;c}(\varsigma, n; x)$.

Theorem 3.4. *Let s given by (1.1) belongs to the class $\mathcal{V}_{\Xi}^{a,b;c}(\varsigma, n; x)$. Then,*

$$|a_3 - \tau a_2^2| \leq \begin{cases} \dfrac{|\mathcal{P}(x)|}{3(3\varsigma-1)\kappa_3}, & 0 \leq |\mathcal{T}(\tau;x)| < \frac{1}{6(3\varsigma-1)\kappa_3} \\ 2|\mathcal{P}(x)||\mathcal{T}(\tau;x)|, & |\mathcal{T}(\tau;x)| \geq \frac{1}{6(3\varsigma-1)\kappa_3} \end{cases} \tag{3.23}$$

where

$$\mathcal{T}(\tau;x) = \frac{(1-\tau)\mathcal{P}^2(x)}{[(2(1+\varsigma) - 8(2\varsigma-1)^2)\mathcal{P}^2(x) - 16\mathcal{Q}(x)(2\varsigma-1)^2]\kappa_2^2}.$$

Proof. From equations (3.20), we get

$$\begin{aligned} a_3 - \tau a_2^2 &= \frac{\mathcal{P}(x)(l_2 - r_2)}{6(3\varsigma - 1)\kappa_3} + (1-\tau)a_2^2 \\ &= \frac{\mathcal{P}(x)(l_2 - r_2)}{6(3\varsigma - 1)\kappa_3} \\ &\quad + \frac{(1-\tau)\mathcal{P}^3(x)(l_2 + r_2)}{[(2(1+\varsigma) - 8(2\varsigma - 1)^2)\mathcal{P}^2(x) - 16\mathcal{Q}(x)(2\varsigma - 1)^2]\kappa_2^2} \\ &= \mathcal{P}(x)\left[\left(\mathcal{T}(\tau; x) + \frac{1}{6(3\varsigma - 1)\kappa_3}\right) l_2 + \left(\mathcal{T}(\tau; x) - \frac{1}{6(3\varsigma - 1)\kappa_3}\right) l_2\right] \end{aligned}$$

where

$$\mathcal{T}(\tau; x) = \frac{(1-\tau)\mathcal{P}^2(x)}{[(2(1+\varsigma) - 8(2\varsigma - 1)^2)\mathcal{P}^2(x) - 16\mathcal{Q}(x)(2\varsigma - 1)^2]\kappa_2^2}. \quad \square$$

One can easily assert the following by using the above Theorem 3.4 and taking $\tau = 1$.

Remark 3.5. Let $s \in \mathcal{V}_{\Xi}^{a,b;c}(\varsigma, n; x)$ represent the function s. Then

$$|a_3 - a_2^2| \leq \frac{|\mathcal{P}(x)|}{3(3\varsigma - 1)\kappa_3}.$$

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